Block-diagonal reduction of matrices over commutative rings.
Decomposition of (sheaves of) modules vs decomposition of their support.

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Prologue

Let $\mathbb{k}$ be a field, $A \in Mat_{n\times n}(\mathbb{k})$. The characteristic polynomial $p_A(t) = det[t\mathbb{1} - A]$.

1. Suppose $p_A(t) = \prod p_j(t)$, where $\{p_j(t)\}$ are coprime polynomials. Then $A \overset{conjug.}{\sim} \oplus A_j$, with $p_{A_j}(t) = p_j(t)$. i.e. $UAU^{-1} = \oplus A_j$.
2. If $\mathbb{k} = \bar{\mathbb{k}}$ then $p_A(t)$ splits into (powers of) linear factors. And A admits the Jordan form (by conjugation).

This is the case of "constant" matrices. What about the "matrices of functions"?

$A \in Mat_{m\times n}(R)$, $m \leq n$, for a (commutative, unital) ring R.
e.g. $R = \mathbb{k}[x_1, \dots, x_p]$, $R = \mathbb{k}[[x_1, \dots, x_p]]$, $R = C^\infty(\mathcal{U})$, for an open $\mathcal{U} \subseteq \mathbb{R}^p$, ...
The left-right equivalence: $A \overset{l.r.}{\sim} UAV^{-1}$, with $U \in GL(m, R)$, $V \in GL(n, R)$.
(This is much weaker than the conjugation.)
When is A equivalent to a (block-) diagonal matrix?

Smith normal form

Theorem: Any matrix over a principal ideal domain (PID) admits the diagonal reduction.

i.e. if R is PID and $A \in Mat_{m\times n}(R)$ then $A \overset{l.r.}{\sim} \begin{bmatrix} \lambda_1 & 0 & \dots & \\ 0 & \lambda_2 & 0 & \dots \\ 0 & \dots & \dots & \dots \end{bmatrix}$, over R.

Examples of PID: $\Bbbk[x]$, $\Bbbk[[x]]$, $\Bbbk\{x\}$, $\Bbbk[x, \frac{1}{x}]$.
(For $R = \mathbb{C}\{x\}$ the Smith normal form is known also as Birkhoff's theorem.)

Geometrically: $Spec(R) = \mathbb{A}^1_{\Bbbk}$, or an open subset of $\mathbb{A}^1_{\Bbbk}$, or the germ (C, o) of a smooth curve.

Interpretation (modules and sheaves): Consider A as a morphism of free modules,

$$R^n \xrightarrow{A} R^m \to Coker(A) \to 0.$$

Thus $Coker(A) \in mod(R)$. Or $Coker(A) \in Coh(Spec(R))$.
Thus the reformulation: any module (coh.sheaf) over a PID is the direct sum of “principal modules" (each generated by one element).

Corollary: Let $\mathcal{F} \in Coh(\mathbb{P}^1_{\Bbbk})$. Then $\mathcal{F} \cong \oplus \mathcal{O}_{\mathbb{P}^1}(d_j) \oplus (Torsion)$.

“Most" rings are non-PID. e.g. $\Bbbk[x_1,\dots,x_p]$, $\Bbbk[[x_1,\dots,x_p]]$, for $p\geq 2$, $C^\infty(\mathcal{U})\dots$.
There is no Smith normal form if $dim(R)>1$.

Example: $A=\begin{bmatrix} x & y\\ z & w\end{bmatrix}$ for $R=\Bbbk[[x,y,z,w]]$. Here $det(A)=xw-yz$ is irreducible in R. Thus A is not l.r. equivalent to a diagonal matrix.
(Not even to a triangular one.)

In 1950's-70's there was serious activity to characterize rings over which every matrix admits the diagonal reduction. (i.e every module decomposes) Such rings are called “elementary divisor rings". They are all close relatives of PID's.
In particular $dim(R)=1$, and $Spec(R)$ is close to $\mathbb{A}^1$ or to a germ of $\mathbb{A}^1$.

What to do when $dim(R) > 1$?

Note: if $A \sim A_1 \oplus A_2$ then $det(A) = det(A_1) \cdot det(A_2)$.
Thus "most matrices" are not equivalent to block-diagonal, as $det(A) \in R$ is irreducible.

For $A \in Mat_{m\times n}(R)$, $m \leq n$, take the ideal of maximal minors, $I_m(A) \subseteq R$.
If $A \sim \oplus A_i$ then $I_m(A) = \prod I_{m_i}(A_i)$. Therefore a necessary condition for block-diagonalization is the factorization of $I_m(A)$.

Def. Suppose $I_m(A) = \prod J_i$. A is called $\{J_i\}$-decomposable if $A \overset{l.r.}{\sim} \oplus A_i$, $I_{m_i}(A_i) = J_i$.

Recall: $A \rightsquigarrow Coker(A) \in mod(R)$. Then $I_m(A) = Fitt_0(Coker(A))$, the support of the module.
Def. Let $M \in mod(R)$ and $Fitt_0(M) = \prod J_i$. M is called $\{J_i\}$-decomposable if $M \sim \oplus M_i$, where $Fitt_0(M_i) = J_i$.

Facts: • $\big(A$ is $\{J_i\}$ decomposable$\big)$ $\Rrightarrow$ $\big($ $Coker(A)$ is $\{J_i\}$-decomposable.$\big)$
• If R is a local ring then also $\Lleftarrow$. (Uniqueness of the minimal resolution)
• Over an arbitrary ring:
$\big($ $Coker(A)$ is $\{J_i\}$-decomposable.$\big)$ $\Rrightarrow$ $\big(A$ is "stably"-$\{J_i\}$ decomposable$\big)$.
i.e. $A \oplus \mathbb{1} \sim \oplus A_i \oplus \mathbb{1}$.

Example 1. $A = \begin{bmatrix} y & x^k \\ x^l & y \end{bmatrix}$, $det(A) = y^2 - x^{k+l} \in \Bbbk[[x, y]]$. Thus $Coker(A)$ is a module over $V(y^2 - x^{k+l}) \subset (\Bbbk^2, o)$. Assume $k + l \in 2\mathbb{N}$ (reducibility). Then $A \overset{l.r.}{\sim} \begin{bmatrix} y - x^{\frac{k+l}{2}} & 0 \\ 0 & y + x^{\frac{k+l}{2}} \end{bmatrix}$? $I_1 \begin{bmatrix} y - x^{\frac{k+l}{2}} & 0 \\ 0 & y + x^{\frac{k+l}{2}} \end{bmatrix} = (y, x^{\frac{k+l}{2}})$. (Assume $2 \in \Bbbk^\times$)

$I_1(A) = (y, x^k, x^l) \overset{??}{=} (y, x^{\frac{k+l}{2}})$. If $k \neq l$ then A is indecomposable.

Def. $I_j(A) \subseteq R$ is the ideal of all $j \times j$ minors of A.
$R = I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_m(A) \supseteq I_{m+1}(A) = 0$.
Fact: $I_j(A)$ is invariant under $GL(m, R) \times GL(n, R)$-equivalence.

Theorem

Let $det(A) = f_1 \cdot f_2 \in R$. Suppose f_1, f_2 are coprime, not zero divisors.
1. *A is stably-$\{f_i\}$-decomposable iff $I_{n-1}(A) \subseteq (f_1, f_2) \subseteq R$.*
2. *(R local) A is $\{f_i\}$-decomposable iff $I_{n-1}(A) \subseteq (f_1, f_2) \subseteq R$.*

Thus in Example 1: A is decomposable iff $k = l$.
Example 2. Let $A \in Mat_{2\times 2}(R)$, with R a local ring. Suppose $det(A) = f_1 f_2$, coprime, not zero divisors. Then A is $\{f_i\}$-decomposable iff $(a_{11}, a_{12}, a_{21}, a_{22}) \subseteq (f_1, f_2)$.
Note: the condition $I_{n-1}(A) \subseteq (f_1, f_2)$ is simple to verify.

Theorem

Let $det(A) = f_1 \cdot f_2 \in R$*. Suppose* f_1, f_2 *are coprime, not zero divisors.*
1. A is stably-$\{f_i\}$*-decomposable iff* $I_{n-1}(A) \subseteq (f_{1,2}) \subseteq R$*.*
2. (R local) A is $\{f_i\}$*-decomposable iff* $I_{n-1}(A) \subseteq (f_1, f_2) \subseteq R$*.*

Geometry: Let $M \in mod(R)$, of projective dimension one. or $M \in Coh(Spec(R))$ Suppose $Supp(M) = V(f_1) \cup V(f_2) \subset Spec(R)$, hypersurfaces with no common component. If $Fitt_1(M) \subseteq (f_1, f_2)$ then $M \cong M|_{V(f_1)} \oplus M|_{V(f_2)}$.

Corollary. Let $R = \Bbbk[[x, y]]$, $\mathfrak{m} = (x, y)$, $A \in Mat_{n\times n}(\mathfrak{m})$. (Thus $ord[det(A)] \geq n$.) Suppose $det(A) = f_1 \cdot f_2$, such that the curve germs $V(f_1), V(f_2) \subset (\Bbbk^2, o)$ have no common tangents. Suppose $ord[det(A)] = n$. Then

$$Coker(A) \cong Coker(A)|_{V(f_1)} \oplus Coker(A)|_{V(f_2)}.$$

The condition $ord[det(A)] = n$. means: A is a “locally maximal determinantal representation". Also “$Coker(A)$ is an Ulrich-maximal module".

The graded version. Let $R = \oplus_{d\in\mathbb{N}} R_d$. Suppose A is graded, $ord(a_{ij}) = d_i + d_j$. Then $Coker(A) \in Coh(Proj(R))$:

$$0 \to \oplus_j \mathcal{O}_{Proj(R)}(-d_j) \xrightarrow{A} \oplus_j \mathcal{O}_{Proj(R)}(d_j) \to Coker(A) \to 0.$$

For each point $x \in Proj(R)$ take an affine chart $x \in \mathcal{U} \subset Proj(R)$, and some local coordinates. Get the local version, $A^{(x)}$ over $R^{(x)}$. Compare the global decomposability, over $Proj(R)$, to the local one, at each point of $Proj(R)$.

Theorem

Suppose $Supp(Coker(A)) = \mathbb{P}V(f_1) \cup \mathbb{P}V(f_2) \subset Proj(R)$, hypersurfaces, no common components. Suppose there exists a hypersurface $V(g)$ intersecting properly $V(f_1)$, $V(f_2)$, $V(f_1, f_2)$ in $Spec(R)$. TFAE:

- $Coker(A) \cong Coker(A)|_{\mathbb{P}V(f_1)} \oplus Coker(A)|_{\mathbb{P}V(f_2)}$
- $Coker(A^{(x)})$ *is locally decomposable for each* $x \in \mathbb{P}V(f_1) \cap \mathbb{P}V(f_2)$
- $I_{n-1}(A^{(x)}) \subseteq (f_1, f_2)^{(x)} \subseteq \mathcal{O}_{(Proj(R),x)}$ *for each* $x \in \mathbb{P}V(f_1) \cap \mathbb{P}V(f_2)$.

Thus a graded question in $dim(R)$, i.e. a global question in $dim(R) - 1$, is reduced to many local questions in $dim(R) - 1$.

Example 4. Let $R = \mathbb{k}[x_0, x_1, x_2]$ and A homogeneous. Thus $Coker(A)$ is a sheaf on the curve $V(det(A)) \subset \mathbb{P}^2$. Suppose $V(det(A)) = C_1 \cup C_2$, no common components. Then $Coker(A) \in Coh(\mathbb{P}^2)$ decomposes iff its stalks at all the points of $C_1 \cap C_2$ decompose. (A bit surprising, as one could expect monodromies.)

Remarks and Applications

- Until now: $A\in Mat_{n\times n}(R)$. The results extend to the rectangular case, with many technicalities.

- Until now the equivalence was: $A\overset{l.r.}{\sim}UAV^{-1}$. For square matrices one wants the conjugation, $A\overset{conjug}{\sim}UAU^{-1}$. The conjug.decomposition problem is "embedded" into the l.r.decomposition, by the ring extension. $R\rightsquigarrow R[[t]]$. Then $A\overset{conjug}{\sim}B$ iff $(t\mathbb{1}-A)\overset{l.r.}{\sim}(t\mathbb{1}-B)$.

Corollary

Suppose the characteristic polynomial factorizes, $det[t\mathbb{1}-A]=f_1\cdot f_2\in R[[t]]$, *with* f_1,f_2 *co-prime. Then* $A\overset{conjug}{\sim}A_1\oplus A_2$ *iff* $I_{n-1}(t\mathbb{1}-A)\subseteq(f_1,f_2)\subset R[[t]]$.

Example. Is $A=\begin{bmatrix}y & x^k\\ x^l & y\end{bmatrix}$ diagonalizable? $det[t\mathbb{1}-A]=(t-y)^2-x^{k+l}$.
Assume $k+l\in 2\mathbb{N}$. Compare $I_1(t\mathbb{1}-A)=(t-y,x^k,x^l)$ to $(t-y,x^{\frac{k+l}{2}})$.
Then A is diagonalizable (by conjugation) iff $k=l$.

- (*An application to operator theory*) Given a set of matrices over a field $\{A^{(\nu)}\}_\nu$ (the set can be infinite/uncountable). When can these matrices be simultaneously (whatever)? e.g. simultaneous block-diagonalization. This was long studied through 20'th century, with numerous partial criteria.

  Define $A := \sum_\nu x_\nu A^{(\nu)} \in Mat_{n\times n}(R)$, $R = \Bbbk[\{x_\nu\}]$. The necessary condition: $det(A) = f_1 \cdot f_2 \in R$. Assume f_1, f_2 coprime. Then

  $$\{A^{(\nu)}\}_\nu \overset{simult.\ l.r.}{\sim} \{A_1^{(\nu)} \oplus A_2^{(\nu)}\}_\nu \quad \text{iff} \quad I_{n-1}(A) \subseteq (f_1, f_2).$$

  For block-diagonalization by congjugation take $A = t\mathbb{1} + \sum_\nu x_\nu A^{(\nu)}$. Get a similar criterion.

- (*An application to representation theory*) Take a group G, resp. an algebra $\mathfrak{g}$. Take a finite-dimensional representation, $G \xrightarrow{\rho} GL_n(\Bbbk)$, $\mathfrak{g} \xrightarrow{\rho} Mat_{n\times n}(\Bbbk)$. Thus $\rho(G)$, resp. $\rho(\mathfrak{g})$ is a set of matrices. (possibly uncountabe)
  Then ρ is decomposable iff this set is simultaneously block-diagonalizable, by conjugation. Now this is easy to verify.

- Similarly one can treat the equivalence $A \rightsquigarrow UAU^t$, for (skew-)symmetric matrices. More generally, we get the decomposition criterion for quiver representations.

Theorem

Let $det(A) = f_1 \cdot f_2 \in R$*. Suppose* f_1, f_2 *are coprime, not zero divisors.*
1. A *is stably-*$\{f_i\}$*-decomposable iff* $I_{n-1}(A) \subseteq (f_{1,2}) \subseteq R$*.*
2. (R *local)* A *is* $\{f_i\}$*-decomposable iff* $I_{n-1}(A) \subseteq (f_1, f_2) \subseteq R$*.*

- (*The meaning of assumptions*)
  $V(det(A)) \subset Spec(R)$ is the locus of points where $rank(A) < n$.
  $V(I_{n-1}(A)) \subset Spec(R)$ is the locus of points where $rank(A) < n - 1$.
  The condition "$I_{n-1}(A) \subseteq (f_1, f_2)$" means: "$corank[A] \geq 2$ at the points of $V(f_1) \cap V(f_2)$".
- *The proof.*
  Step 1. Reduction to the case: R is local and henselian. (Using some commutative algebra, *Tor*, *Ext*, . . .)
  Step 2. For R local and henselian one uses "linear algebra over a ring".

Thanks for your attention!