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Classification of Algebraic Curves and Surfaces: Topological Methods and Invariants

The moduli space:

In algebraic geometry, a moduli space is a geometric space whose points represent algebraic-geometric objects (curves, surfaces) of some fixed type, or isomorphism classes of such objects.

If we can show that a collection of interesting surfaces induces the structure of a geometric space, then one can parameterize such surfaces by introducing coordinates on the resulting space.

In higher dimensions, moduli of algebraic surfaces are more difficult to construct and study.

The talk: We give a comprehensive scientific background and list of methods to ease the classification.

***Classification of surfaces*: Catanese, Kulikov, Manetti, Moishezon-Teicher.**

Let $X \hookrightarrow \mathbb{CP}^N$ be an embedded algebraic surface, $f : X \to \mathbb{CP}^2$ be a generic projection of degree n. The branch curve of X in the plane $\mathbb{CP}^2$ is S.

We compute the fundamental group $\pi_1(\mathbb{CP}^2 - S)$, this group induces connected components in the moduli space: surfaces with the same group $\pi_1(\mathbb{CP}^2 - S)$ are in the same connected component (Moishezon-Teicher).

The ultimate goal of the classification:
The group $\pi_1(\mathbb{CP}^2 - S)$ is sometimes complicated $\Rrightarrow$ finding new invariants which distinguish connected components of the moduli space of surfaces (of general type).

The set-up

$$
\begin{array}{ccc}
X \subset \mathbb{CP}^{N_1} & \xrightarrow{\text{degeneration}} & X_0 \subset \mathbb{CP}^{N_0} \\
\downarrow f & & \downarrow \\
S \subset \mathbb{CP}^2 & \xleftarrow{\text{regeneration}} & S_0 \subset \mathbb{CP}^2
\end{array}
$$

Steps of work:

(1) Degeneration of X to X_0,

(2) Projection of X_0 onto $\mathbb{CP}^2$ to get S_0.

(3) Regeneration of S_0 to S.

(4) Braid monodromy technique of Moishezon-Teicher.

(5) Fundamental group of the complement of S.

(6) Fundamental group of the Galois cover of X.

Step 1.

Take a surface X embedded in $\mathbb{CP}^{N_1}$, we degenerate X to a union of planes X_0.

The surface $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ (Moishezon, Teicher).

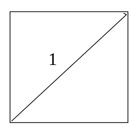


FIGURE 1. $X_0 = \bigcup^2 \mathbb{CP}^2$.

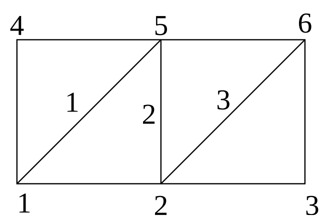


FIGURE 2. $X_0 = \bigcup^4 \mathbb{CP}^2$.

Explanation:

The surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ is defined by

$$z_1 z_2 - 1 z_0 z_3 = 0 \hookrightarrow \mathbb{CP}^3.$$

When $t = 0$ in $z_1 z_2 - t z_0 z_3 = 0$, we get $z_1 z_2 = 0$, which is $\mathbb{CP}^2 \cup \mathbb{CP}^2$.

Therefore $X \xrightarrow{\text{degeneration}} X_0 = \bigcup^{number of planes} \mathbb{CP}^2.$

The Veronese surface V_n (A., Lehman, Shwartz, Teicher).

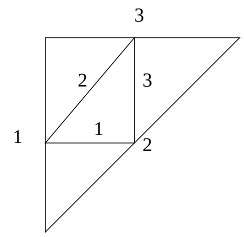


FIGURE 3. Degeneration X_0 of the Veronese V_2.

The Hirzebruch surface $F_k(a, b)$ (A., Ogata).

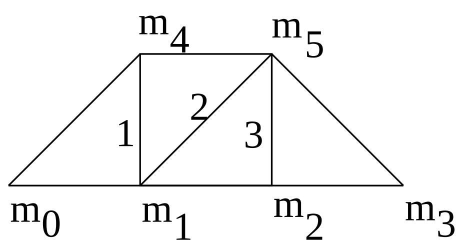


FIGURE 4. Degeneration of the Hirzebruch $F_2(1, 1)$.

The surface $T \times T$ (A., Teicher, Vishne).

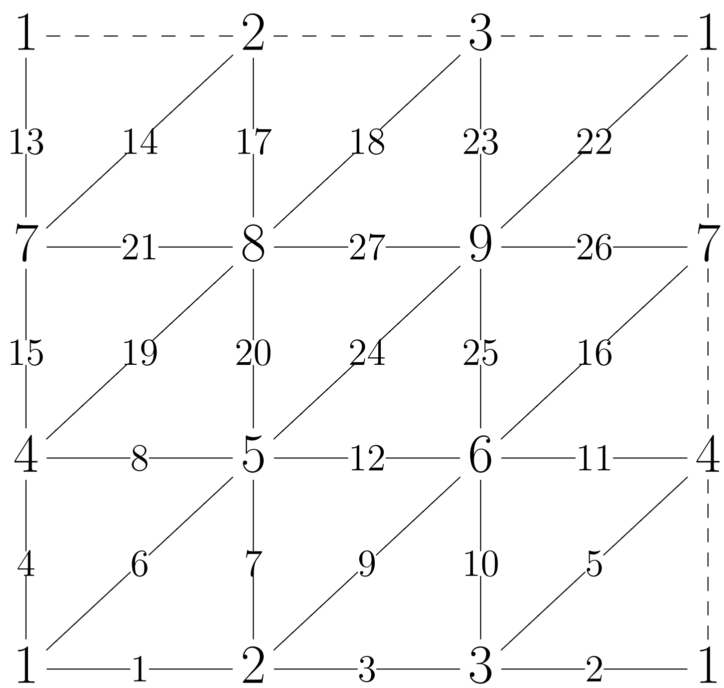


FIGURE 5. Degeneration of $T \times T$.

The $K3$ **surfaces** (A., Ciliberto, Miranda, Teicher).

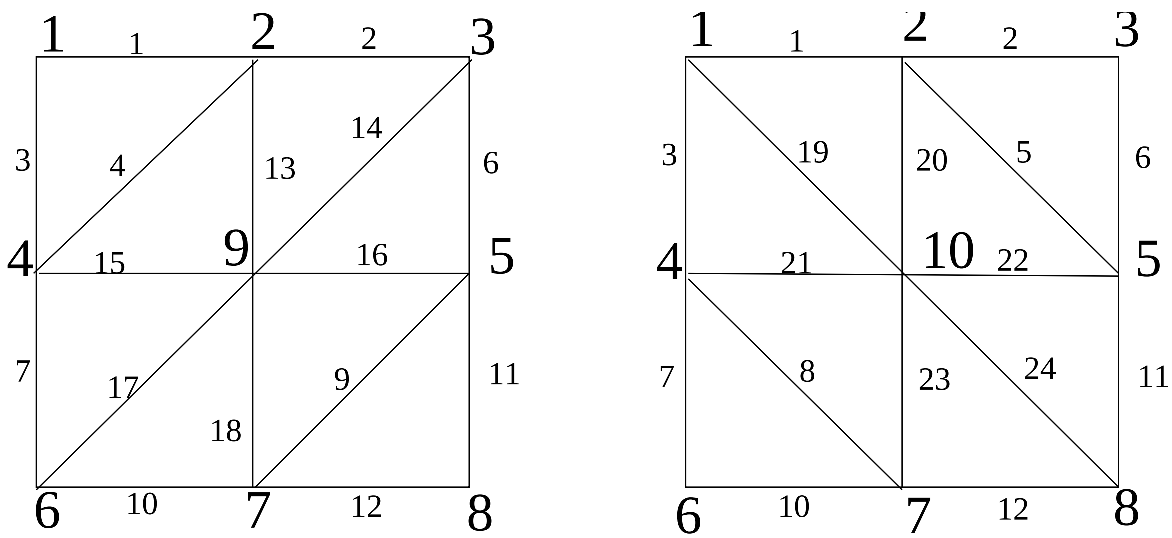


FIGURE 6. The $(2, 2)$-pillow degeneration.

Step 2.

Projecting X_0 to the projective plane $\mathbb{CP}^2$ to get the branch curve S_0.

Example:

$X_0 =$ the degenerated Hirzebruch surface $F_1(2,2)$.

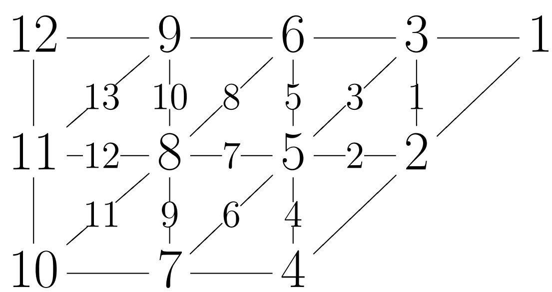


FIGURE 7. The degenerated Hirzebruch X_0.

A generic projection $f_0 : X_0 \to \mathbb{CP}^2$ is the degeneration of f. Under f_0, each of the 12 planes is mapped isomorphically to $\mathbb{CP}^2$. The ramification locus R_0 of f_0 is composed of points in which f_0 is not isomorphism locally. Thus R_0 is the union of the 13 intersection lines. Let $S_0 = f_0(R_0)$ be the degenerated branch curve; it is a line arrangement, composed of the images of the 13 lines.

Very interesting papers.

Before we continue.....

Moishezon-Teicher 1986-1996:

(1) B. Moishezon, M. Teicher,
”*Braid group techniques in complex geometry III, Projective degeneration of* V_3”,
Contemporary Math. 162 (1994), p. 313-332.

(2) B. Moishezon, M. Teicher,
”*Braid group techniques in complex geometry IV, Braid monodromy of* S_3*, the branch curve of* $f_3 : V_3 \to \mathbb{CP}^2$ *and applications to* $\pi_1(\mathbb{CP}^2 - S_3)$”,
Contemporary Math. 162 (1994), p. 333-358.

Step 3.

We regenerate the curve S_0 and recover the branch curve S.

Regeneration Rules of Moishezon-Teicher on k-points:

2-points.

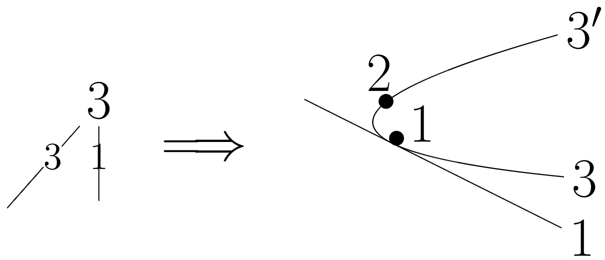


FIGURE 8. Regeneration of the 2-point 3.

3-points.

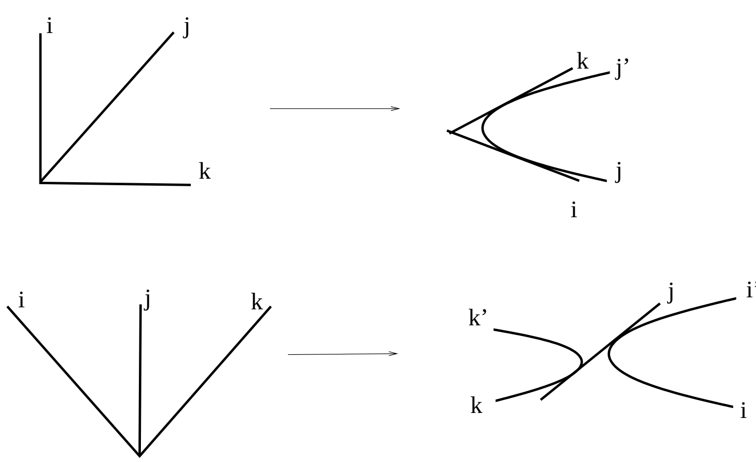


FIGURE 9. Regeneration of the two types of 3-point.

6-points.

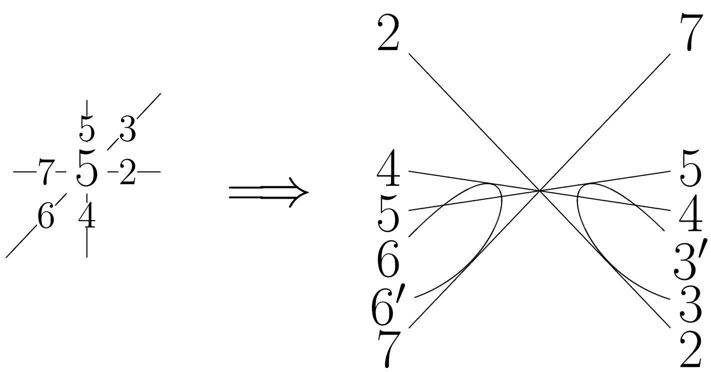


FIGURE 10. Regeneration of the 6-point 5.

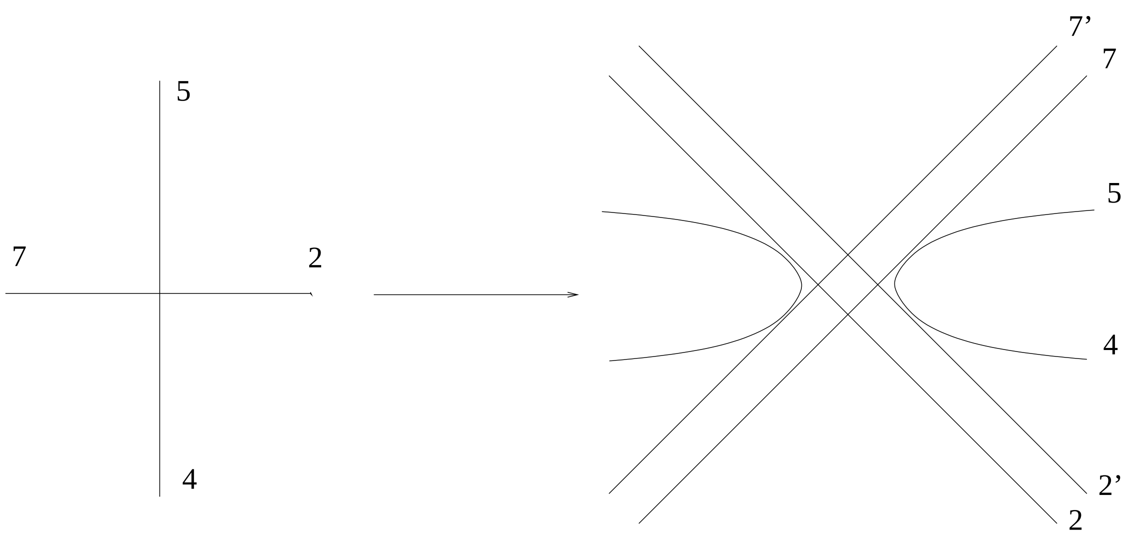


FIGURE 11. Regeneration of a 4-point.

In the next regeneration step, each tangency point regenerates to 3 cusps. Therefore S is a cuspidal curve.

Step 4.

The braid monodromy of Moishezon-Teicher:

1. Find singularities in a curve S and take their x-coordinates x_i,
2. take "good" points M_i next to these x_i,
3. take loops around the x_i at M_i,
4. lift them and project to the fiber above M_i,
5. get a motion of the intersection points of S with the fiber over M_i.

Example: The curve $y^2 = x^2$.

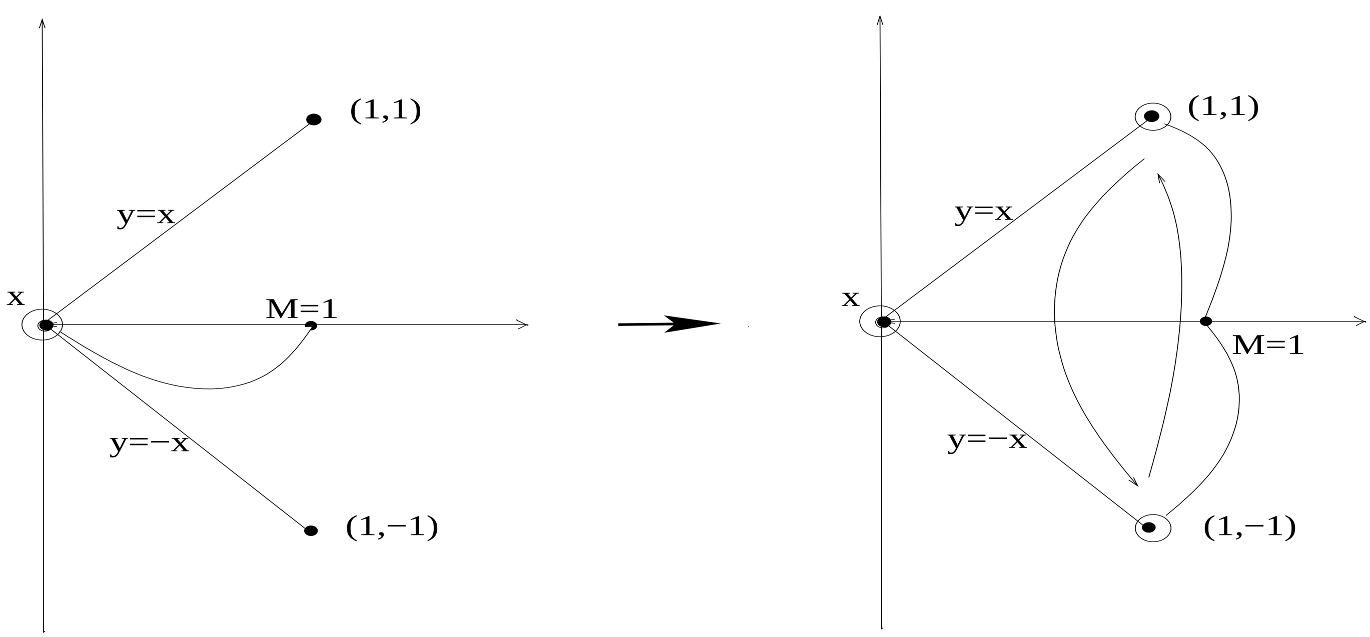


Figure 12. The braid monodromy.

Proposition (Moishezon-Teicher).

Take the curve S defined by $y^2 = x^m$. Then the braid monodromy is h^m, where h is a positive half-twist.

Proof.

Take a tiny loop coming from the "good" point 1 to x_1, denoted by $x = e^{2\pi it}$, $t \in [0,1]$. We lift this loop to the curve S and get two paths:

$$(e^{2\pi it}, e^{2\pi imt/2})$$
$$(e^{2\pi it}, -e^{2\pi imt/2}).$$

We project them onto the fiber above 1 and get two paths:

$$e^{\pi imt}$$
$$-e^{\pi imt}.$$

This gives the m-th power of the motion corresponding to $[-1,1]$:

$$e^{\pi it}$$
$$-e^{\pi it}.$$

Step 5.

We compute the group $\pi_1(\mathbb{CP}^2 - S) = \langle \Gamma_j \mid \{R\} \rangle$, it is the fundamental group of the complement of the branch curve S in $\mathbb{CP}^2$.

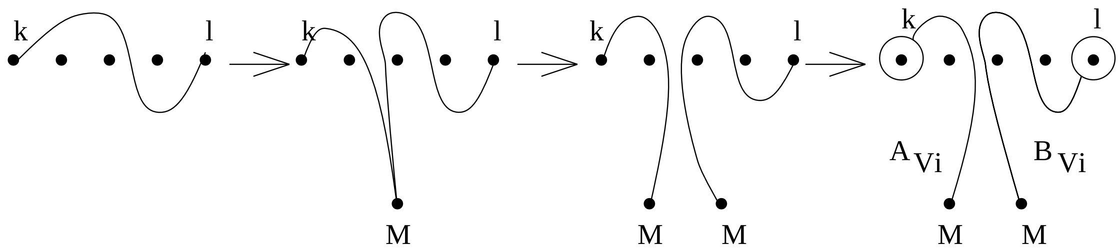


FIGURE 13. The elements of the fundamental group.

Example: Elements in $\pi_1(\mathbb{CP}^2 - S)$.

$\Gamma_i = \Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1}$ $\qquad$ $\Gamma_{i+1} = \Gamma_5^{-1}\Gamma_6^{-1}\Gamma_8\Gamma_6\Gamma_5$

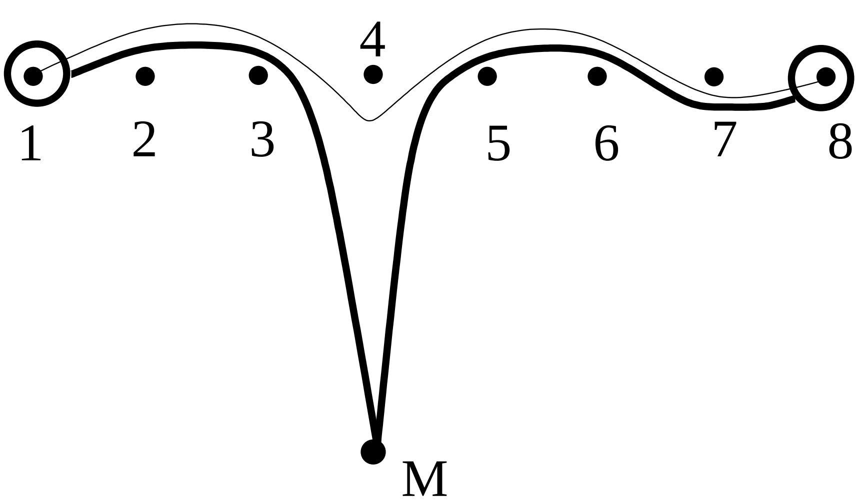


FIGURE 14. Example of elements in the fundamental group.

We give the Theorem of Zariski-van Kampen for cuspidal curves:

Theorem (Zariski-van Kampen).

Let S be a cuspidal curve of degree n, then the fundamental group is generated by n generators and admits the following relations:

(1) $\Gamma_i = \Gamma_{i+1}$ for a branch point,
(2) $[\Gamma_i, \Gamma_{i+1}] = \Gamma_i \Gamma_{i+1} \Gamma_i^{-1} \Gamma_{i+1}^{-1} = e$ for a node,
(3) $\langle \Gamma_i, \Gamma_{i+1} \rangle = \Gamma_i \Gamma_{i+1} \Gamma_i \Gamma_{i+1}^{-1} \Gamma_i^{-1} \Gamma_{i+1}^{-1} = e$ for a cusp.

Example:

For the surface $T \times T$, the group $\pi_1(\mathbb{CP}^2 - S)$ has 54 generators and admits around 2000 relations.
COMPLICATED!!!

$$
\begin{array}{ccccccc}
1 & \text{- -} & 2 & \text{- -} & 3 & \text{- -} & 1 \\
| & / & | & / & | & / & | \\
7 & — & 8 & — & 9 & — & 7 \\
| & / & | & / & | & / & | \\
4 & — & 5 & — & 6 & — & 4 \\
| & / & | & / & | & / & | \\
1 & — & 2 & — & 3 & — & 1
\end{array}
$$

Step 6.

The group $\pi_1(\mathbb{CP}^2 - S)$ is relatively complicated, therefore there is a new invariant, the fundamental group of the Galois cover of a surface X.

Very interesting paper.

B. Moishezon, M. Teicher,
"*Simply connected algebraic surfaces with positive index*",
Inventiones Math. 89 (1987), p. 601-643.

Definition. We define the fibred product $(1 \le k \le n)$

$$\underbrace{X \times_f \cdots \times_f X}_{k} = \{(x_1, \ldots, x_k) \in X^k \mid f(x_1) = \cdots = f(x_k)\}$$

and the extended diagonal

$$\Delta = \{(x_1, \ldots, x_k) \in X^k \mid x_i = x_j \quad \text{for some} \quad i \neq j\}.$$

The closure $X^k_{\mathrm{Gal}} = \overline{X \times_f \cdots \times_f X - \Delta}$ is called the Galois cover w.r.t. the symmetric group S_k.

Remark. There are many Galois covers w.r.t. S_k, $k < n$, but only for $k = n$ we identify the $\pi_1(X_{\text{Gal}})$ as the needed group. The Galois cover is a minimal smooth surface of general type.

Finding $\pi_1(X_{\mathbf{Gal}})$:

Take a canonical surjection ψ from $\pi_1(\mathbb{CP}^2 - S)$ to the symmetric group S_n:

$$1 \to \pi_1(X_{\text{Gal}}) \to \pi_1(\mathbb{CP}^2 - S)/\langle \Gamma_j^2 \rangle \to S_n \to 1$$

The fundamental group $\pi_1(X_{\text{Gal}})$ is the kernel of this surjection (Moishezon-Teicher).

Various Results:

<u>$X = \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$</u> :

$\pi_1(X_{\text{Gal}})$ is finite and commutative, $a \geq 3$, $b \geq 2$.

$\pi_1(X_{\text{Gal}}) = 0$, a, b relatively prime ($gcd(a, b) = 1$).

<u>$X = \mathbb{CP}^1 \times T \hookrightarrow \mathbb{CP}^5$</u> :

$\pi_1(X_{\text{Gal}}) \cong \mathbb{Z}^{10}$, and in general, $\pi_1(X_{\text{Gal}}) \cong \mathbb{Z}^{4n-2}$.

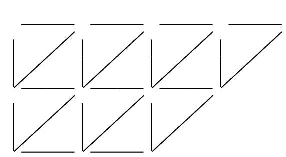
<u>$X = T \times T \hookrightarrow \mathbb{CP}^8$</u> :

$\pi_1(X_{\text{Gal}})$ is nilpotent of nilpotency class 3 (there is a central series $G = H_1 \geq H_2 \geq ... \geq H_n = e$ such that each H_i is a normal subgroup of G and H_i/H_{i+1} is in the center of G/H_{i+1} and in our case $n = 3$).

<u>$X = F_1(2, 2)$ (Hirzebruch)</u>:

$\pi_1(X_{\text{Gal}}) \cong \mathbb{Z}_2^{10}$.

In general, if $c = gcd(a, b)$ and $n = 2ab + kb^2$, then $\pi_1(X_{\text{Gal}}) \cong \mathbb{Z}_c^{n-2}$.



The breakthrough to Coxeter and Artin groups:

"Coxeter covers of symmetric groups"

[Rowen, Teicher, Vishne]

$C(T)$ is a Coxeter group defined by a graph T:

- generators s_i are the edges of T
- relations

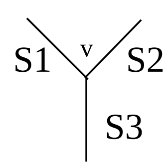
$$s_i^2 = e \ \forall i$$

$$(s_i s_j)^2 = e \text{ if } s_i, s_j \text{ are disjoint}$$

$$(s_i s_j)^3 = e \text{ if } s_i \text{ meets } s_j \text{ in a vertex}$$

$C(T)$ has a natural map onto S_n, where n is the number of vertices of T.

$C_Y(T)$ is a quotient of $C(T)$ and includes such cases:



The "fork" relation is $(s_1 s_2 s_3 s_2)^2 = e, \quad \forall v \in T$.

Main Theorem [RTV].

$$C_Y(T) \cong A_{t,n} \rtimes S_n,$$

where

$A_{t,n}$ is a group which contains t copies of $\mathbb{Z}^{n-1}$,

n is the number of vertices of T,

t is the number of cycles of T.

Generators:

$x^r_{ij}, \quad 1 \leq r \leq t, \; 1 \leq i, j \leq n.$

Relations:

- $x^r_{ii} = e,$ $\quad 1 \leq r \leq t; \quad 1 \leq i \leq n$
- $(x^r_{ij})^{-1} = x^r_{ji},$ $\quad 1 \leq r \leq t; \quad 1 \leq i, j \leq n$
- $x^r_{ij} x^r_{jk} = x^r_{jk} x^r_{ij} = x^r_{ik},$ $\quad 1 \leq r \leq t; \; 1 \leq i, j, k \leq n$ (not necessarily distinct)
- $x^r_{ij} x^s_{k\ell} = x^s_{k\ell} x^r_{ij},$ $\quad 1 \leq r, s \leq t; \quad 1 \leq i, j, k, \ell \leq n$ (distinct)

Notation: $G = N \rtimes H$ is a semidirect product, where N is a normal subgroup of G and $H = G/N$ is a quotient group.

The connection to algebraic geometry.

Example: Take the surface $T \times T$, where T is the complex torus.

Projecting $T \times T$ to $\mathbb{CP}^2$, we get the branch curve S.
The group $\pi_1(\mathbb{CP}^2 - S)$ has 54 generators and admits around 2000 relations.
We compute the quotient $\pi_1(\mathbb{CP}^2 - S)/\langle \Gamma_j^2 \rangle$.
The following exact sequence holds:

$$1 \to \pi_1(X_{\mathrm{Gal}}) \to \pi_1(\mathbb{CP}^2 - S)/\langle \Gamma_j^2 \rangle \to S_{18} \to 1.$$

And we get

$$\pi_1(\mathbb{CP}^2 - S)/\langle \Gamma_j^2 \rangle \cong \pi_1(X_{\mathrm{Gal}}) \rtimes S_{18},$$

where $\pi_1(X_{\mathrm{Gal}})$ is nilpotent of class 3.

- In general, there is a projection of the group $C_Y(T) \cong A_{t,n} \rtimes S_n$ on the group $\pi_1(\mathbb{CP}^2 - S)/\langle \Gamma_j^2 \rangle \cong \pi_1(X_{\mathrm{Gal}}) \rtimes S_n$, so it is possible to calculate $\pi_1(X_{\mathrm{Gal}})$ explicitly.

Line arrangements and conic-line arrangements: Yau, Fan, Rybnikov, Nazir-Yoshinaga, Ye.

1: *Line arrangements.*

Goal: Classification by Zariski pairs.

Two line arrangements are a Zariski pair if both have the same topology but the fundamental group of their complements are different.

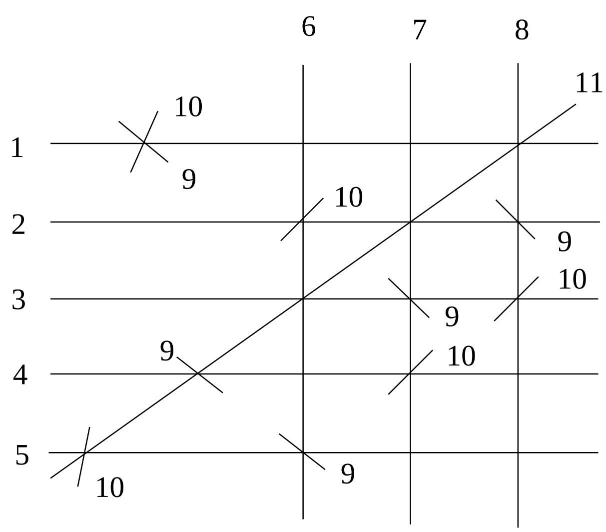


FIGURE 15. Example of construction of a line arrangement with 11 lines with a quintic singularity, the singularity is a point at infinity.

Results: There is a partial classification up to 11 lines. The last work done is with Gong, Teicher, Xu on 11 lines with quintic singularity.

2: *Conic-line arrangements.*

A., Teicher, Uludağ:

Fundamental groups are easy to compute: one generator which corresponds to the conic, commutes with the other generators which correspond to the lines.

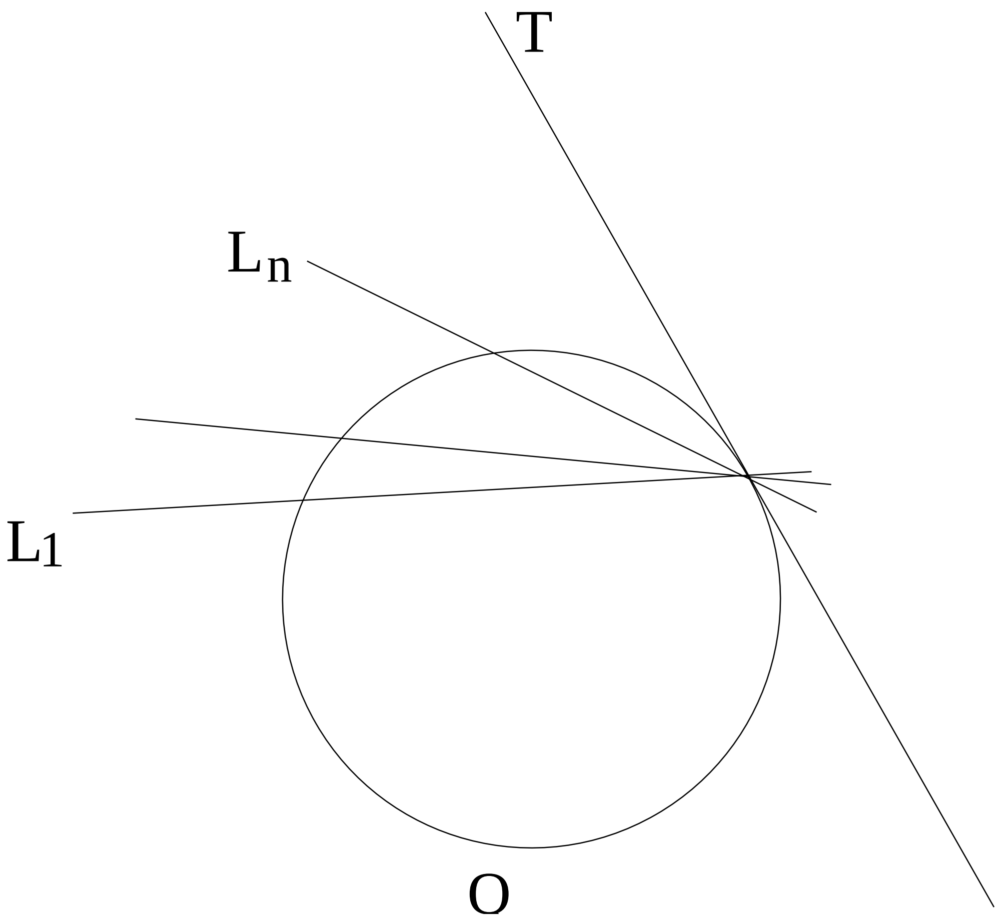


FIGURE 16. One conic, one tangent line, n intersecting lines.

A., Garber, Teicher:

Complicated group, but the result is a "big" group (it contains a free subgroup, generated by two or more generators).

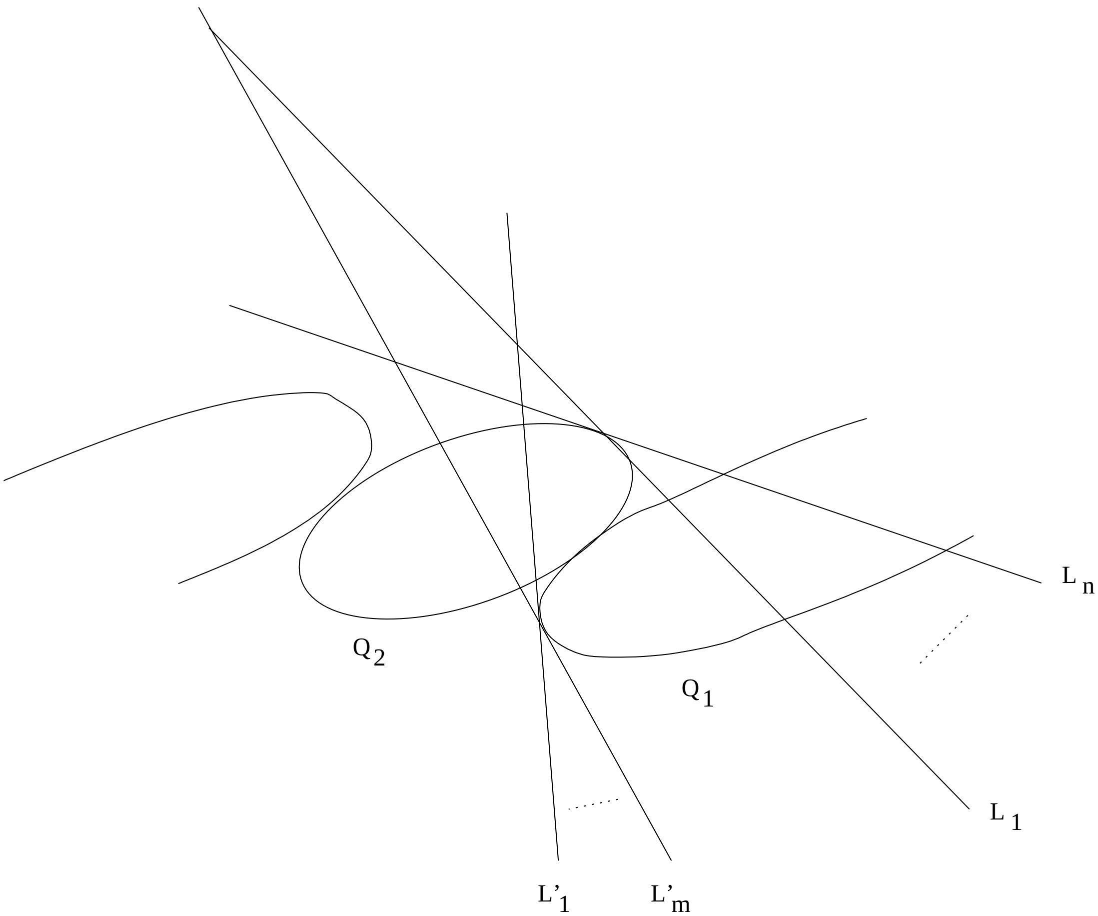
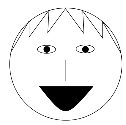


FIGURE 17. Two conics, $n + m$ tangent lines.

Thank you !!!



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