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Tropical curves

Tropical bitangents

Lifting

Arithmetic

Counts of bitangents of tropical plane quartics

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Counts of bitangents of tropical plane quartics

Plan for the talk

Showcase tropical geometry as a tool for simultaneous geometric counts over various fields.

- Introduce the counting problem, bitangents of quartics
- Introduce tropical geometry
- Bitangents to tropical quartics
- Lifting results
- Outlook: Arithmetic counts

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Bitangents to quartics

- A smooth plane quartic defined over an algebraically closed field has 28 bitangents (Plücker, 1834).
- A real plane quartic can have 4, 8, 16 or 28 real bitangents (depending on the topology).
- A real bitangent is called totally real if the tangency points are also real.

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Examples of 28 real bitangents



(Wikipedia)



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Examples of 28 real bitangents



(Wikipedia)



Examples of 28 real bitangents



(Plaumann, Sturmfels, Vinzant 2011)



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Tropical geometry

For a complex algebraic curve C, consider the limit of the amoeba Log(C), where

$$\operatorname{Log}: (\mathbb{C}^*)^2 \to \mathbb{R}^2: (x, y) \mapsto (\log |x|, \log |y|).$$

Example

Line L in $\mathbb{P}^2_{\mathbb{C}}$, pick chart with coordinates $(\frac{x}{z}, \frac{y}{z})$. Intersects $\{x = 0\}$ at $(0, y_0)$. Image tends to $(-\infty, \log |y_0|)$. Log(L) is complex 1-dim, real 2-dim.





Tropical geometry

• Consider limit of amoebas of families of curves:

$$f_t = \sum a_{i,j}(t) x^i y^j, \quad \lim_{t \to \infty} \operatorname{Log}_t(V(f_t))$$

- View the coefficients $a_{i,j}(t)$ as (Puiseux) series in t, then $V(f_t)$ is defined over the field K of Puiseux series.
- Tropicalization can be viewed as coordinatewise valuation of a field with non-Archimedean valuation.

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Tropicalized plane curves

Field: $K = k\{\{t\}\}$, i.e., Puiseux series over a field k with characteristic not 2. The tropicalization map

 $(x, y) \mapsto (-\operatorname{val}(x), -\operatorname{val}(y)).$

The plane quartic V(f) for

$$\begin{split} f(x,y) &= \quad t^{36}x^4 + t^{18}x^3y + t^2x^2y^2 + t^{18}xy^3 + t^{36}y^4 + t^{23}x^3 \\ &+ t^6x^2y + t^6xy^2 + t^{23}y^3 + t^{12}x^2 + xy + t^{12}y^2 + t^2x \\ &+ t^2y + 1. \end{split}$$

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Tropicalization of a plane quartic

The tropicalization of V(f):



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Some tropical enumerative geometry

- Mikhalkin's correspondence theorem for numbers of **complex and real** plane curves of fixed genus and degree satisfying point conditions, 2003
- Asymptotic statements about Welschinger invariants (Itenberg-Kharlamov-Shustin, 2005)
- Aspects in mirror symmetry (Gross-Siebert, ...)
- Correspondence theorem for Hurwitz numbers (Cavalieri-Johnson-M, Bertrand-Brugallé-Mikhalkin, 2008)
- Wall-crossing formulas for double Hurwitz numbers (Cavalieri-Johnson-M, 2010)

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Complex and real count of rational cubics through 8 points



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Bitangents to tropical quartics

- A plane quartic has 28 bitangents (Plücker, 1834).
- A tropical plane quartic may have infinitely many bitangents.
- We identify: $L_1 \sim L_2$ if we can continuously move L_1 to L_2 while maintaining bitangency.

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Example





Example



Example



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For $q \in \mathbb{C}\{\{t\}\}[x, y]$ a (generic) quartic polynomial with $\operatorname{Trop}(V(q)) = C$, exactly 2 of the 28 bitangent lines to V(q)tropicalize to the tropical line with vertex the upper red point, exactly 2 to the one with vertex the lower red point, and none to a point in the interior of the red segment.

Bitangents to tropical quartics

- A plane quartic has 28 bitangents (Plücker, 1834).
- A tropical plane quartic may have infinitely many bitangents.
- We identify: L₁ ~ L₂ if we can continuously move L₁ to L₂ while maintaining bitangency.
- Then: A tropical quartic in \mathbb{R}^2 has 7 bitangent classes (Baker-Len-Morrison-Pflueger-Ren, 2014).
- If the skeleton of the tropical quartic is a K_4 , then each bitangent class has 4 lifts (Chan-Jiradilok, 2015).
- For any generic smooth tropical quartic in ℝ², each bitangent class has 4 lifts (Len-M, 2017).

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Combinatorics: Example



Combinatorial Classification



41 shapes for bitangent classes, up to symmetry.

The black cells of each bitangent class miss the curve, whereas the red ones lie on it. The unfilled vertices indicate points that $\frac{2}{3}$ must be vertices. (Cueto-M, 2020)

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Theorem (Len-M, 2017)

A tropical bitangent class of a generic smooth tropical quartic in \mathbb{R}^2 has 4 complex lifts.

Theorem (Cueto-M, 2020)

A tropical bitangent class of a generic smooth tropical quartic in \mathbb{R}^2 has either 0 or 4 real lifts.

Any lift which is real is also totally real.

Techniques of proof: Combinatorial classification and local lifting computations.

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Other fields

Theorem (Payne-Shaw-M, 2022)

A tropical bitangent class of a generic smooth tropical quartic in \mathbb{R}^2 has 4 lifts over an algebraically closed field.

Theorem (Payne-Shaw-M, 2022)

A tropical bitangent class of a generic smooth tropical quartic in \mathbb{R}^2 has either 0 or 4 lifts over any field K of characteristic $\neq 2$. All lifts live in a quadratic field extension. If $\sqrt{2}$ and $\sqrt{3}$ exist, then all lifts which exist over K also have their tangency points over K.

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Questions

- What are the tropicalizations of real quartics which have real, but not totally real, bitangents? (Lee-Len)
- How can we show that altogether, there are 4, 8, 16 or 28 real lifts? (Geiger-Panizzut)
- What about bitangents of tropical quartics which are not in \mathbb{R}^2 , but in a different model of the tropical plane?

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Avoidance loci



Theorem (Kummer, Vinnikov,...)

Every connected component of the avoidance locus of a smooth real quartic contains precisely 4 bitangents in its closure.

Theorem (Payne-Shaw-M, 2023)

A tropical bitangent class which is liftable to the reals is (roughly) the tropicalization of a connected component of the avoidance locus.

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Tropicalizations of avoidance loci

Corollary

A tropical bitangent class is tropically convex.

Corollary

Tropical grouping into 4 and real grouping into 4 coincides.

Tropical grouping into 4 exists independently of the field.

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Arithmetic counts

Definition

Let k be a field. The Grothendieck-Witt ring GW(k) contains all formal sums of isomorphism classes of quadratic forms $V \times V \rightarrow k$ over k.

Example

For $k = \mathbb{C}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but not for $k = \mathbb{R}$.

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Arithmetic counts

Associates an element in $\mathrm{GW}(k)$ to a geometric object to be counted.

There exist arithmetic counts of

- ... lines in cubic surfaces (Kass-Wickelgren),
- ... plane curves satisfying point conditions (Levine),

 $\bullet\,$... bit angents of a quartic w.r.t. infinite line (Larson-Vogt).

Insert

- $k = \mathbb{C} \rightsquigarrow \dim \equiv$ "number"
- k = ℝ → other meaningful real invariants (e.g. Welschinger invariants)

Tropical geometry plays intermediary role, e.g. quantum counts of plane curves.

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Bitangent to quartics

Theorem (Payne-Shaw-M, 2022)

The element in GW(k) that belongs to the 4 bitangents in a tropical equivalence class can be determined with tropical methods.

For each lift, it is (a root of) a Laurent monomial in the coefficients of the quartic.

For many tropical bitangent classes, the sum of the four lifts viewed in GW(k) equals $2 \cdot \mathbb{H}$, where \mathbb{H} is the hyperbolic form.

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Arithmetic count



The arithmetic count of bit angents is $14 \cdot \mathbb{H}.$







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$2\mathbb{H}$ and avoidance loci



 $2\mathbb{H}$ specializes to 4 over \mathbb{C} , and to 0 over the reals. Remember tropical grouping into 4 and avoidance grouping into 4 is the same.



Real signed count of bitangents

Conjecture (Larson-Vogt, 19)

The arithmetic count of bitangent lines to a quartic, when specialized to the reals, is in $\{0, 2, 4, 6, 8\}$.

Theorem (Payne-Shaw-M, 22)

The arithmetic count of tropical bitangents to a quartic, when specialized to the reals, is in $\{0, 2, 4\}$.

Consequently, the conjecture holds for quartics near the tropical limit.

Theorem (Kummer-McKean, 23)

Proof of conjecture.

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