## Counts of bitangents of tropical plane

 quarticsHannah Markwig

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## Counts of bitangents of tropical plane quartics

## Plan for the talk

Showcase tropical geometry as a tool for simultaneous geometric counts over various fields.

Tropical curves
Tropical bitangents

Lifting

- Introduce the counting problem, bitangents of quartics
- Introduce tropical geometry
- Bitangents to tropical quartics
- Lifting results
- Outlook: Arithmetic counts


## Bitangents to quartics

Counting bitangents

- A smooth plane quartic defined over an algebraically closed field has 28 bitangents (Plücker, 1834).
- A real plane quartic can have $4,8,16$ or 28 real bitangents (depending on the topology).
- A real bitangent is called totally real if the tangency points are also real.


## Examples of 28 real bitangents

## Counting

bitangents


Tropical curves
Tropical bitangents

Lifting
Arithmetic
(Wikipedia)

## Examples of 28 real bitangents

Counting
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Tropical curves
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Lifting
Arithmetic

## Examples of 28 real bitangents



## Counting

bitangents

## Tropical curves

Tropical bitangents

## Lifting

Arithmetic

## Tropical geometry

For a complex algebraic curve $C$, consider the limit of the amoeba $\log (C)$, where

$$
\log :\left(\mathbb{C}^{*}\right)^{2} \rightarrow \mathbb{R}^{2}:(x, y) \mapsto(\log |x|, \log |y|)
$$

## Example

Line $L$ in $\mathbb{P}_{\mathbb{C}}^{2}$, pick chart with coordinates $\left(\frac{x}{z}, \frac{y}{z}\right)$.
Intersects $\{x=0\}$ at $\left(0, y_{0}\right)$. Image tends to $\left(-\infty, \log \left|y_{0}\right|\right)$.
$\log (L)$ is complex 1-dim, real 2-dim.



## Counting

 bitangentsTropical curves
Tropical bitangents

## Tropical geometry

Counting bitangents

Tropical curves

$$
f_{t}=\sum a_{i, j}(t) x^{i} y^{j}, \quad \lim _{t \rightarrow \infty} \log _{t}\left(V\left(f_{t}\right)\right)
$$

- View the coefficients $a_{i, j}(t)$ as (Puiseux) series in $t$, then $V\left(f_{t}\right)$ is defined over the field $K$ of Puiseux series.
- Tropicalization can be viewed as coordinatewise valuation of a field with non-Archimedean valuation.


## Tropicalized plane curves

## Counting

 bitangentsField: $K=k\{\{t\}\}$, i.e., Puiseux series over a field $k$ with characteristic not 2 . The tropicalization map

$$
(x, y) \mapsto(-\operatorname{val}(x),-\operatorname{val}(y))
$$

The plane quartic $V(f)$ for

$$
\begin{array}{r}
f(x, y)=\quad t^{36} x^{4}+t^{18} x^{3} y+t^{2} x^{2} y^{2}+t^{18} x y^{3}+t^{36} y^{4}+t^{23} x^{3} \\
+t^{6} x^{2} y+t^{6} x y^{2}+t^{23} y^{3}+t^{12} x^{2}+x y+t^{12} y^{2}+t^{2} x \\
+t^{2} y+1
\end{array}
$$

## Tropicalization of a plane quartic

The tropicalization of $V(f)$ :

## Counting

 bitangentsTropical curves
Tropical bitangents

Lifting
Arithmetic

## Some tropical enumerative geometry

Counting bitangents

- Mikhalkin's correspondence theorem for numbers of complex and real plane curves of fixed genus and degree satisfying point conditions, 2003
- Asymptotic statements about Welschinger invariants (Itenberg-Kharlamov-Shustin, 2005)
- Aspects in mirror symmetry (Gross-Siebert, ...)
- Correspondence theorem for Hurwitz numbers (Cavalieri-Johnson-M, Bertrand-Brugallé-Mikhalkin, 2008)
- Wall-crossing formulas for double Hurwitz numbers (Cavalieri-Johnson-M, 2010)


## Complex and real count of rational cubics through 8 points



Counting bitangents

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## Bitangents to tropical quartics

- A plane quartic has 28 bitangents (Plücker, 1834).
- A tropical plane quartic may have infinitely many bitangents.
- We identify: $L_{1} \sim L_{2}$ if we can continously move $L_{1}$ to $L_{2}$ while maintaining bitangency.
bitangents
Tropical curves
Tropical bitangents


## Example

## Bitangents

## Counting

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## Lifting

Arithmetic

## Example

## Bitangents

## Counting

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Tropical curves


Tropical bitangents

## Lifting

Arithmetic

## Example



Counting bitangents

For $q \in \mathbb{C}\{\{t\}\}[x, y]$ a (generic) quartic polynomial with $\operatorname{Trop}(V(q))=C$, exactly 2 of the 28 bitangent lines to $V(q)$ tropicalize to the tropical line with vertex the upper red point, exactly 2 to the one with vertex the lower red point, and none to a point in the interior of the red segment.

## Bitangents to tropical quartics

- A plane quartic has 28 bitangents (Plücker, 1834).
- A tropical plane quartic may have infinitely many bitangents.
- We identify: $L_{1} \sim L_{2}$ if we can continously move $L_{1}$ to $L_{2}$ while maintaining bitangency.
- Then: A tropical quartic in $\mathbb{R}^{2}$ has 7 bitangent classes (Baker-Len-Morrison-Pflueger-Ren, 2014).
- If the skeleton of the tropical quartic is a $K_{4}$, then each bitangent class has 4 lifts (Chan-Jiradilok, 2015).
- For any generic smooth tropical quartic in $\mathbb{R}^{2}$, each bitangent class has 4 lifts (Len-M, 2017).


## Combinatorics: Example

## Counting

bitangents
Tropical curves
Tropical bitangents

## Lifting

Arithmetic

## Combinatorial Classification



## Counting bitangents

41 shapes for bitangent classes, up to symmetry.
The black cells of each bitangent class miss the curve, whereas the red ones lie on it. The unfilled vertices indicate points that must be vertices. (Cueto-M, 2020)

## Lifting

## Theorem (Len-M, 2017)

A tropical bitangent class of a generic smooth tropical quartic in $\mathbb{R}^{2}$ has 4 complex lifts.

## Theorem (Cueto-M, 2020)

A tropical bitangent class of a generic smooth tropical quartic in $\mathbb{R}^{2}$ has either 0 or 4 real lifts.
Any lift which is real is also totally real.
Techniques of proof: Combinatorial classification and local lifting computations.

## Other fields

## Theorem (Payne-Shaw-M, 2022)

A tropical bitangent class of a generic smooth tropical quartic in $\mathbb{R}^{2}$ has 4 lifts over an algebraically closed field.

## Theorem (Payne-Shaw-M, 2022)

A tropical bitangent class of a generic smooth tropical quartic in $\mathbb{R}^{2}$ has either 0 or 4 lifts over any field $K$ of characteristic $\neq 2$. All lifts live in a quadratic field extension. If $\sqrt{2}$ and $\sqrt{3}$ exist, then all lifts which exist over $K$ also have their tangency points over $K$.

## Questions

Counting bitangents

- What are the tropicalizations of real quartics which have real, but not totally real, bitangents? (Lee-Len)
- How can we show that altogether, there are $4,8,16$ or 28 real lifts? (Geiger-Panizzut)
- What about bitangents of tropical quartics which are not in $\mathbb{R}^{2}$, but in a different model of the tropical plane?


## Avoidance loci



## Theorem (Kummer, Vinnikov,...)

Every connected component of the avoidance locus of a smooth real quartic contains precisely 4 bitangents in its closure.

## Theorem (Payne-Shaw-M, 2023)

A tropical bitangent class which is liftable to the reals is (roughly) the tropicalization of a connected component of the avoidance locus.

## Tropicalizations of avoidance loci

## Corollary

A tropical bitangent class is tropically convex.
Lifting
Arithmetic

## Corollary

Tropical grouping into 4 and real grouping into 4 coincides.
Tropical grouping into 4 exists independently of the field.

## Arithmetic counts

## Definition

Let $k$ be a field. The Grothendieck-Witt ring GW $(k)$ contains all formal sums of isomorphism classes of quadratic forms $V \times V \rightarrow k$ over $k$.

## Example

For $k=\mathbb{C}$,

$$
\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \sim\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

since

$$
\left[\begin{array}{cc}
1 & 0 \\
0 & i
\end{array}\right] \cdot\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \cdot\left[\begin{array}{ll}
1 & 0 \\
0 & i
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

but not for $k=\mathbb{R}$.

## Arithmetic counts

Associates an element in $\mathrm{GW}(k)$ to a geometric object to be counted.
There exist arithmetic counts of

- ... lines in cubic surfaces (Kass-Wickelgren),
- ... plane curves satisfying point conditions (Levine),
- ... bitangents of a quartic w.r.t. infinite line (Larson-Vogt).

Insert

- $k=\mathbb{C} \rightsquigarrow \operatorname{dim} \equiv$ "number"
- $k=\mathbb{R} \rightsquigarrow$ other meaningful real invariants (e.g. Welschinger invariants)

Tropical geometry plays intermediary role, e.g. quantum counts ${ }^{-}$ of plane curves.

## Bitangent to quartics

## Theorem (Payne-Shaw-M, 2022)

The element in $G W(k)$ that belongs to the 4 bitangents in a tropical equivalence class can be determined with tropical methods.

For each lift, it is (a root of) a Laurent monomial in the coefficients of the quartic.

For many tropical bitangent classes, the sum of the four lifts viewed in $G W(k)$ equals $2 \cdot \mathbb{H}$, where $\mathbb{H}$ is the hyperbolic form.

## Arithmetic count

Counting bitangents

Tropical curves
Tropical bitangents

## Relation to real count


$\langle 1\rangle$

$\langle-1\rangle$

## Conjecture (Larson-Vogt, 19)

The arithmetic count of bitangent lines to a quartic, when specialized to the reals, is in $\{0,2,4,6,8\}$.

## $2 \mathbb{H}$ and avoidance loci


$2 \mathbb{H}$ specializes to 4 over $\mathbb{C}$, and to 0 over the reals.
Remember tropical grouping into 4 and avoidance grouping into 4 is the same.

## Real signed count of bitangents

## Conjecture (Larson-Vogt, 19)

The arithmetic count of bitangent lines to a quartic, when specialized to the reals, is in $\{0,2,4,6,8\}$.

## Theorem (Payne-Shaw-M, 22)

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The arithmetic count of tropical bitangents to a quartic, when specialized to the reals, is in $\{0,2,4\}$.
Consequently, the conjecture holds for quartics near the tropical limit.

## Theorem (Kummer-McKean, 23)

Proof of conjecture.

