Enumerative Invariants of Calabi-Yau Threefolds with Torsion and Noncommutative Resolutions

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Outline

Determinantal Octic Double Solid

- $A(x) \in M_8(H^0(\mathbb{P}^3, \mathcal{O}(1)))$ $x \in \mathbb{P}^3$, generic symmetric 8×8 matrix
- $\det(A) = 0$ deg 8 hypersurface $S \subset \mathbb{P}^3$, 84 A_1 singularities p_i

$$S = \left\{ x \in \mathbb{P}^3 \middle| \mathrm{rank}(A(x)) \leq 7 \right\}, \qquad \{p_i\} = \left\{ x \in \mathbb{P}^3 \middle| \mathrm{rank}(A(x)) = 6 \right\}$$

- $Y \to \mathbb{P}^3$ 2-1 cover branched along S, nodal singularities over p_i

$$y^2 = \det(A(x)), \ x \in \mathbb{P}^3$$

- Y *determinantal octic double solid*
- $K_Y \simeq \left(K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}\left(\frac{8}{2}\right)\right)\Big|_Y$ is trivial
- $X \to Y$ small resolution
- Exceptional $\mathbb{P}^1$'s are 2-torsion classes in $H_2(X, \mathbb{Z})$
- X is *not Kähler*

Quadrics in $\mathbb{P}^7$

- Write $A = \sum_{i=1}^{4} A_i x_i$, $y \in \mathbb{C}^8 = V$
- $Q_i(y) = y^t A_i y$, four quadrics in $\mathbb{P}^7$
- $Z = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^7$ CY 3fold, $h^{1,1}(Z) = 1$
- X is a moduli space of spinor sheaves on quadrics in $\mathbb{P}^7$, restricted to Z
- No universal sheaf on $X \times Z$
- Universal α-twisted spinor sheaf $\mathcal{E}$ on $X \times Z$, $\alpha \in \mathrm{Br}(X)$ Addington 0904.1764

Noncommutative resolution

- $\mathcal{Q} \to \mathbb{P}^3$ bundle of quadric hypersurfaces in $\mathbb{P}^7$
- $\mathcal{B}_0$ sheaf of even Clifford algebras on $\mathbb{P}^3$ Kuznetsov 0510670

$$\mathcal{B}_0 = \mathcal{O}_{\mathbb{P}^3} \oplus \left(\Lambda^2(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\right) \oplus \left(\Lambda^4(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-2)\right) \oplus$$

$$\left(\Lambda^6(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-3)\right) \oplus \left(\Lambda^8(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-4)\right)$$

- X is a moduli space of $\mathcal{B}_0$-modules supported on points of $\mathbb{P}^3$
- Points of any small resolution of Y represented by $\mathcal{B}_0$-modules supported on points $p \in \mathbb{P}^3$
- In other words, a $\mathcal{B}_0 \otimes \mathcal{O}_p = \mathrm{Cl}^0(A(p))$ representation

$\mathcal{B}_0$-modules and points of Y

- Cl_k (complex) Clifford algebra of nondegenerate quadratic form in k variables

$$\mathrm{Cl}(V, Q) = \bigotimes V / (v_1 v_2 + v_2 v_1 = 2Q(v_1, v_2) \cdot 1)$$

- M_r algebra of $r \times r$ matrices
- $p \in \mathbb{P}^3 - S$:
  - $\mathcal{B}_0 \otimes \mathcal{O}_p \simeq \mathrm{Cl}_8^0 \simeq \mathrm{Cl}_7 \simeq M_8 \times M_8$
  - Two 8-dimensional reps, corresponding to two points of X over p
- p smooth point of S:
  - $\mathcal{B}_0 \otimes \mathcal{O}_p \simeq \mathrm{Cl}_6 \otimes \mathrm{Cl}(\mathbf{C}, 0) \simeq M_8 \otimes \mathbb{C}[\epsilon]/\epsilon^2$
  - Unique 8-d rep, corresponding to unique point of X over p

$\mathcal{B}_0$-modules and points of small resolutions

- p node of S
- W 2-dimensional complex vector space
- $\mathcal{B}_0 \otimes \mathcal{O}_{p_i} \simeq \mathrm{Cl}_5 \otimes \mathrm{Cl}(W,0)$
- $\mathrm{Cl}_5 \simeq M_4 \times M_4$
- 4-dimensional reps S' and S'' of Cl_5
- For $[w] \in \mathrm{P}(W) \simeq \mathbb{P}^1$ have 2-dim'l rep $R_{[w]} := w\Lambda^* W$ of $\Lambda^* W$
- Get two $\mathbb{P}^1$'s worth of 8-d reps of $\mathcal{B}_0 \otimes \mathcal{O}_{p_i}$: $S' \otimes R_{[w]}$, $S'' \otimes R_{[w]}$ of $\mathcal{B}_0$
- Identified with the points of the two small resolutions of p_i

Derived equivalences

$$D^b(X, \alpha) \overset{\text{Addington}}{=} D^b(Z) \overset{\text{Kuznetsov}}{=} D^b(\mathbb{P}^3, \mathcal{B}_0)$$

B-model

- Mirror $Z^\circ = \tilde{Z}^\circ_\psi / G$, $|G| < \infty$ of Z has $h^{2,1}(Z) = 1$

$$x_0^2+x_1^2=2\psi x_0x_1,\ x_2^2+x_3^2=2\psi x_2x_3,\ x_4^2+x_5^2=2\psi x_4x_5,\ x_6^2+x_7^2=2\psi x_6x_7$$

- Two points of maximal unipotent monodromy
  - Large complex structure limit $\psi = \infty$ has fundamental period

$$\varpi(\psi) = \sum_{n=0}^{\infty} \frac{((2n)!)^4}{(n!)^8} \psi^{-8n}$$

    and GW invariants of Z found by Picard-Fuchs system, mirror symmetry, holomorphic anomaly, conifold gap condition...
  - Picard-Fuchs near MUM point at $\psi = 0$ used in calculations
  - Together with the mirror of the smooth octic double solid Clemens this determines enumerative invariants

- *What is the B-model computing?*

Conjecture

- Let $\mathcal{S}$ be the set of all small resolutions of Y, $|\mathcal{S}| = 2^{84}$
- $X^{\text{smooth}} = X'^{\text{smooth}}$ for any $X, X' \in \mathcal{S}$
- $\Gamma = \Gamma_{XX'} \subset X \times X'$ graph closure
- $\psi_{XX'} : D^b(X) \to D^b(X')$ equivalence from FM with kernel $\mathcal{O}_\Gamma$

$$\psi_{XX'}(F) = R\pi_{X'*}\left(L\pi_X^*(F) \overset{L}{\otimes} \mathcal{O}_\Gamma\right)$$

- $\mathrm{Coh}_{\leq 1}(X)$ coherent sheaves on X of dimension at most 1
- **Conjecture**
  B-model computes Gopakumar-Vafa invariants of

$$\left(\bigcup_{X \in \mathcal{S}} \mathrm{Coh}_{\leq 1}(X)\right) \Big/ E^\bullet \sim \psi_{XX'}(E^\bullet),\ E^\bullet \in D^b(X)$$

- Conjecture is more general

General setting

- $\mathbb{P}^3$ can be replaced with a more general Fano threefold B of even index
- A can be replaced with a symmetric matrix whose entries are sections of line bundles on B
- For $B = \mathbb{P}^3$, let $\vec{d} = (d_1, \ldots, d_k), \sum d_i = 8$, all d_i have same parity
- A symmetric matrix, $A_{ij} \in H^0(\mathbb{P}^3, \mathcal{O}((d_i + d_j)/2))$
- Main example: $\vec{d} = 1^8$
- Y double cover of $\mathbb{P}^3$ branched over $\det(A) = 0$, X small resolution

Outline

Flat B-fields

- X compact manifold
- Flat B-field $B \in H^2(X, U(1))$

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

$$\begin{array}{ccc} 0 \to H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}) \to H^2(X,U(1)) & \to & H^3(X,\mathbb{Z}) \to H^3(X,\mathbb{R}) \\ & \searrow & \cup \\ & & H^3(X,\mathbb{Z})_{\mathrm{tors}} \end{array}$$

- $c : H^2(X, U(1)) \to H^3(X, \mathbb{Z})_{\mathrm{tors}}$
- $\gamma = c(B)$ is a characteristic class of B

Kähler Moduli space

- X Calabi-Yau threefold
- (Complexified) Kähler moduli space

$$\mathcal{K} = \left\{ B + iJ \mid B \in H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}),\ J \,\text{Kahler},\ J >> 0 \right\}$$

- Naturally generalizes to connected components in the presence of torsion.
- Fix a characteristic class $\gamma \in H^3(X, \mathbb{Z})_{\text{tors}}$

$$\mathcal{K}_\gamma := \left\{ B + iJ \mid B \in H^2(X, U(1)),\ c(B) = \gamma,\ J \,\text{Kahler},\ J >> 0 \right\}$$

- All isomorphic to $(\Delta^*)^{b_2(X)}$
- Multiple large radius limits, different topological string partition functions

- To determine a mirror, a characteristic class $\gamma \in H^3(X, \mathbb{Z})_{\text{tors}}$ must be fixed in addition to X
- The $\psi = 0$ MUM point of the mirror of the $(2, 2, 2, 2)$ complete intersection is mirror to X and B-fields with nontrivial characteristic class
- Physicists refer to B-fields with nonzero characteristic class as a *fractional B-field*

Gromov-Witten theory

- Usual GW expansion variables

$$q^\beta = \exp\left(2\pi i \int_\beta B + iJ\right), \qquad \beta \in H_2(X,\mathbb{Z})$$

- Have cap product pairing

$$H^2(X, U(1)) \times H_2(X,\mathbb{Z}) \to U(1)$$

- On any $\mathcal{K}_\gamma$ have expansion variables

$$q_\gamma^\beta = (B \cap \beta) \exp\left(-2\pi \int_\beta J\right)$$

$$F_\gamma = \sum N_\beta^g \lambda^{2g-2} q_\gamma^\beta, \qquad N_\beta^g \text{ GW invariants}$$

- See also Aspinwall-Gross-Morrison hep-th/9503208

More on expansion variables

- $\mathcal{K}_\gamma \simeq \mathcal{K}$, not canonically.
- Up to choices, can relate q_γ^β to usual expansion variables
- For exposition, assume $H^3(X,\mathbb{Z})_{\mathrm{tors}} \simeq \mathbb{Z}_k$
- Choose B_0 with $c(B_0) = \gamma$ and $kB_0 = 0$ ($k^{b_2(X)}$ choices)

$$c^{-1}(\gamma) = \left\{ B_0 + B \mid B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}) \right\}$$

$$q_\gamma^\beta = (B_0 \cap \beta)\, q^\beta$$

- Since $kB_0 = 0$, $B_0 \cap \beta$ kth root of $1 \in U(1)$

$$F_\gamma = \sum N_\beta^g \,(B_0 \cap \beta)\, \lambda^{2g-2} q^\beta$$

- This structure facilitates the determination of the F_γ by B-model techniques

Torsion in $H_2(X, \mathbb{Z})$

- $H_2(X, \mathbb{Z})_{\mathrm{tors}} \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ universal coefficient theorem
- $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{b_2(X)} \oplus H_2(X, \mathbb{Z})_{\mathrm{tors}}$, noncanonical isomorphism
- Choice of splitting ($H_2(X, \mathbb{Z})_{\mathrm{tors}} \simeq \mathbb{Z}_k$)

$$H_2(X, \mathbb{Z}) \to \mu_k \simeq \mathbb{Z}_k, \qquad \beta \mapsto B_0 \cap \beta$$

- $\deg(\beta) \in \mathbb{Z}^{b_2(X)}$ degree
- $t(\beta) \in \mathbb{Z}_k$ *torsion class* (M-theory $\mathbb{Z}_k$ charge)

Organization of invariants

- For $\beta \in H_2(X,\mathbb{Z})$ have Gopakumar-Vafa invariants $n^g_\beta \in \mathbb{Z}$
- Sometimes write $n^g_{\mathrm{deg}(\beta),t(\beta)}$ in place of n^g_β
- For all $\gamma \in H^3(X,\mathbb{Z})_{\mathrm{tors}}$

$$F_\gamma = \sum_\beta \frac{n^g_\beta}{m}\left(2\sin\left(\frac{m\lambda}{2}\right)\right)^{2g-2} q_\gamma^{m\beta}$$

- Have k times as many GV invariants as in the torsion-free case
- Have k times as many generating functions to compensate, along with k different mirrors

Outline

Identifying the torsion

- $X \to Y$ small resolution of determinantal octic double solid Y
- $\pi : X \to \mathbb{P}^3$
- $H \in H^2(\mathbb{P}^3, \mathbb{Z})$ hyperplane class
- $H^2(X, \mathbb{Z})$ generated by $\pi^*(H)$
- $H_2(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$
- $\beta \in H_2(X, \mathbb{Z})$, $d = d(\beta) = \pi^*(H) \cap \beta \in \mathbb{Z}$, torsion class $j \in \mathbb{Z}_2$
- $[C_i] \in H_2(X, \mathbb{Z})$ nontrivial 2-torsion class ($d = 0, j = 1$)
- $\pi^{-1}(\mathrm{line}) = \pi^*(H^2)$ has class $d = 2, j = 1$, $\pi : X \to \mathbb{P}^3$

$$d = \pi^*(H) \cdot \pi^*(H^2) = \pi^*(H^3) = 2$$

Proof of claims about torsion

- Degenerate Y to Y', requiring lower right 3×3 block of A vanishes

$$A = \begin{pmatrix} B_{5\times 5} & C^{\mathrm{t}} \\ C & 0_{3\times 3} \end{pmatrix}$$

- Nodes of Y'
  - 84 nodes p_i where $\mathrm{rank}(A(p_i)) = 6$
  - nodes q_j where $\mathrm{rank}(C(q_j)) = 2$
- Y' has an explicit projective small resolution X': a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^4$

$$\sum_{j=0}^{4} C_{ij}(x) y_j = 0, \sum_{i,j=0}^{4} B_{ij} y_i y_j = 0$$

- $H_2(X', \mathbb{Z}) \simeq \mathbb{Z}^2$

Proof of claims about torsion

$$\sum_{j=0}^{4} C_{ij}(x)y_j = 0, \sum_{i,j=0}^{4} B_{ij}y_iy_j = 0$$

- Exceptional curves over p_i are lines in $\mathbb{P}^4$; exceptional curves over q_j are conics in $\mathbb{P}^4$
- C_1 class of exceptional curve over p_i; C_2 class of exceptional curve over q_j
- $[C_1] = (0,1), [C_2] = (0,2)$
- $D := [\pi'^{-1}(\text{line})] = \pi_1^*(H_1^2)$
- D has class $((\pi_1^*(H_1) \cdot D), (\pi_2^*(H_2) \cdot D)) = (2,7)$

$$(H_2 \cdot D)_{X'} = (H_2H_1^2 \cdot (H_1 + H_2)^3(H_1 + 2H_2))_{\mathbb{P}^3 \times \mathbb{P}^4} = 7$$

- X from X' by contracting C_2 curves to Y' and smoothing
- $H_2(X,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$
- $[C_1] \in H_2(X,\mathbb{Z})$ nontrivial 2-torsion class ($d = 0, j = 1$)
- $\pi^{-1}(\text{line})$ has class $d = 2, j = 1$.

Computing GV invariants

- GV invariants denoted $n^g_{d,j}$
- From conifold transition $X \to Y_t$ Ruan

$$n^g_{d,0}(X) + n^g_{d,1}(X) = n^g_d(Y_t)$$

- Y_t smooth octic double solid, GV invariants computed long ago
- Determinantal double solid provides torsion refinement of these GV invariants

Torsion class zero

$n^g_{d,0}$	$d=1$	2	3	4
$g=0$	14752	64415616	711860273440	11596528004344320
1	0	20160	10732175296	902646044328864
2	0	504	-8275872	6249833130944
3	0	0	-88512	-87429839184
4	0	0	0	198065872
5	0	0	0	157306
6	0	0	0	1632
7	0	0	0	24

Torsion class one

$n^g_{d,1}$	$\beta = 1$	2	3	4
$g = 0$	14752	64419296	711860273440	11596528020448992
1	0	21152	10732175296	902646048376992
2	0	360	-8275872	6249834146800
3	0	6	-88512	-87429664640
4	0	0	0	198149928
5	0	0	0	144144
6	0	0	0	2520
7	0	0	0	0

Geometric calculations

- $n^0_{1,0} = n^0_{1,1} = 14752$
- $n^3_{2,0} = 0$; $n^3_{2,1} = 6$; $n^2_{2,0} = 504$; $n^2_{2,1} = 360$
- $n^3_{3,0} = n^3_{3,1} = 14752 \cdot (-6) = -88512$
- $n^{d^2+d+1}_{2d,d} = (-1)^{(d^2+3d+6)/2}\left(2d^2+6d+4\right)$, $n^{d^2+d+1}_{2d,d+1} = 0$ $(d \geq 2)$
- $n^{d^2+d+1}_{2d,d+1} = 252(d^2+3d)$ $(d \geq 2)$
- $n^{d^2+2d}_{2d+1,0} = n^{d^2+2d}_{2d+1,1} = (-1)^{(d^2+3d+2)/2}\,14752\left(d^2+3d+2\right)$ $(d \geq 2)$
- $n^{d^2+d}_{2d,d} = (-1)^{(d^2+3d+8)/2}\left(4d^4+16d^3-172d^2-568d\right)$ $(d \geq 2)$

Torsion class zero

$n^g_{d,0}$	$d=1$	2	3	4
$g=0$	14752	64415616	711860273440	1159652800434432
1	0	20160	10732175296	902646044328864
2	0	504	−8275872	6249833130944
3	0	0	−88512	−87429839184
4	0	0	0	198065872
5	0	0	0	157306
6	0	0	0	1632
7	0	0	0	24

Torsion class one

$n^g_{\beta,1}$	$\beta = 1$	2	3	4
$g=0$	14752	64419296	711860273440	11596528020448992
1	0	21152	10732175296	902646048376992
2	0	360	-8275872	6249834146800
3	0	6	-88512	-87429664640
4	0	0	0	198149928
5	0	0	0	144144
6	0	0	0	2520
7	0	0	0	0

Four-tangent lines

- $L \subset \mathbb{P}^3$ meeting S in 4 points of tangency.
- 14,752 such lines H. Schubert, 1879.0001
- $\pi^{-1}(L) = L_1 \cup L_2 \subset Y$, rational curves meeting at the 4 points over the tangencies
- Identify with $L_i \subset X$
- $d(L_i) = \pi^*(H) \cdot L_i = H \cdot \pi_*(L_i) = 1$
- L_2 has class $(1, j)$ for some $j \in \mathbb{Z}_2$
- $L_1 - L_2 = (L_1 + L_2) - 2L_2$ has class $(2, 1) - 2(1, j) = (0, 1)$
- L_1, L_2 have different torsion classes

Degree 1

- $C \subset X, \; d = 1 = \pi^*(H) \cdot C = H \cdot \pi_*(C)$
- $\pi(C) \subset \mathbb{P}^3$ line L
- $\pi^{-1}(L) \subset X$ curve of degree 2 containing C
- $\pi^{-1}(L)$ has two degree 1 components, hence L 4-tangent line, $C = L_1$ (say), $g = 0$
- 14,752 such lines.
- If L_1 has torsion class j, then L_2 has torsion class $j + 1$.

$$n^g_{1,j} = \begin{cases} 14752 & g = 0 \\ 0 & g > 0 \end{cases}$$

- **Remark.** The isomorphism $H_2(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ not canonical, so no intrinsic meaning for torsion class of the L_i
- $\mathbb{Z} \oplus \mathbb{Z}_2$ has automorphism $(d, j) \mapsto (d, d + j)$
- j intrinsically defined for even d

KKV methods

- $\beta \in H^2(X, \mathbb{Z})$
- $g(\beta)$ maximal (arithmetic) genus of curve of class $g(\beta)$ (Castelnuovo bound)
- $\mathcal{M}_\beta$ Chow variety of curves of class β
- $\mathcal{M}'_\beta \subset \mathcal{M}_\beta$ parametrizing curves of arithmetic genus $g(\beta)$
- $n_\beta^{g(\beta)}$ Behrend-weighted Euler characteristic of $\mathcal{M}'_\beta$
  Gopakumar-Vafa; K-Klemm-Vafa

KKV methods, continued

- $\mathcal{C} \to \mathcal{M}'_\beta$ universal curve of (arithmetic) genus $g(\beta)$
- Suppose $\mathcal{C}, \mathcal{M}'_\beta$ smooth (+...)

$$n_\beta^{g(\beta)-1} = (-1)^{\dim(\mathcal{C})} \left(\chi_{\rm top}\left(\mathcal{C}\right) + \left(2g\left(\beta\right) - 2\right)\chi_{\rm top}\left(\mathcal{M}'_\beta\right)\right)$$

- Proven using Maulik-Toda definition of GV invariants 1610.07303 and the decomposition theorem for perverse sheaves L. Zhao

Degree 2

- $C \subset X, \ d = 2$
- $\pi_*(C)$ degree 2 cycle in $\mathbb{P}^3$
- Either
  - $\pi(C)$ is a line L, C degree 2 over L with 8 branch points $(g(C) = 3)$
  - $\pi(C)$ is a smooth conic $(g(C) = 0)$
- Case 1: class of C is $(2, 1)$
- Moduli space of such curves is $G(2, 4)$
- Second case does not contribute to genus 3 invariants
- $n^3_{2,1} = \chi(G(2, 4)) = 6$
- $n^3_{2,0} = 0$
- Generalizes to double covers of plane curves of degree $d \geq 2$

$$n^g_{2d,d} = \begin{cases} (-1)^{(d^2+3d)/2+1}\left(2d^2 + 6d + 4\right) & g = d^2 + d + 1 \\ 0 & g > d^2 + d + 1 \end{cases}$$

- $d = 2 : \ n^7_{4,0} = 24$ (double cover of plane conic branched at $2 \cdot 8 = 16$ points has genus 7)

Degree 2, continued

- $\beta = (2, 1)$
- $\mathcal{M}_{2,1} = G(2, 4)$
- $\mathcal{C} \xrightarrow{(p \in \pi^{-1}(L)) \mapsto p} X$
- Fiber over p: $\mathbb{P}^2$ of lines in $\mathbb{P}^3$ through $\pi(p)$
- $\chi_{\text{top}}(\mathcal{C}) = 3\chi_{\text{top}}(X) = -384$
- $n^2_{2,1} = -(-384 + 4 \cdot 6) = 360$

$n^2_{2,0}$

- Class $(2,0)$ for $g > 0$: necessarily 2-1 $\pi : C \to L$
- $\pi^{-1}(L) = C + C_i$
- L passes through p_i
- Both small resolutions contribute
- $n^2_{2,0} = 2 \cdot 84 \cdot 3 = 504$

Outline

Brauer group

- $Br(X)$ generated by projective bundles or Azumaya algebras
- $P \to X$ rank $r-1$ projective bundle
- Gluing of trivializations determines class in $H^1(X, \mathrm{PGL}(r))$, hence $H^2(X, \mathcal{O}_X^*)_{\mathrm{tors}}$ via

$$0 \to \mu_r \to \mathrm{GL}(r) \to \mathrm{PGL}(r) \to 0$$

$$H^1(X, \mathrm{PGL}(r)) \to H^2(X, \mu_r) \to H^2(X, \mathcal{O}_X^*)$$

- $Br(X) \simeq H^2(X, \mathcal{O}_X^*)_{\mathrm{tors}}$ Gabber-de Jong
- For a CY 3fold, $Br(X) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ via

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0, \qquad H^2(X, \mathcal{O}_X^*) \hookrightarrow H^3(X, \mathbb{Z})$$

Twisted sheaves

- Represent $\alpha \in H^2(X, \mathcal{O}_X^*)$ by cocycle $c_{\eta\theta\iota} \in Z^2(\{U_\eta\}, \mathcal{O}_X^*)$
- Twisted sheaf determined by sheaves F_η on U_η and isomorphisms

$$\phi_{\eta\theta} : F_\theta|_{U_\eta \cap U_\theta} \simeq F_\eta|_{U_\eta \cap U_\theta}$$

  satifying $\phi_{\iota\eta} \circ \phi_{\eta\theta} \circ \phi_{\theta\iota} = c_{\eta\theta\iota}$
- Derived category $D^b(X, \alpha)$ independent of choice of cocycle up to equivalence
- $\alpha \in Br(X)$ determines derived category of twisted sheaves $D^b(X, \alpha)$

Bridgeland stability

- Conjecturally $D^b(X)$ admits Bridgeland stability conditions (proven in many cases)
- Space of stability conditions “extended Kähler moduli space"
- Kähler moduli is a slice of this

$$Z(E^\bullet) = -\int_X \exp\left(-2\pi i\right)\left(B + iJ\right) \mathrm{ch}(E^\bullet)\hat{\Gamma}(X)$$

Twisted derived category $D^b(X, \alpha)$

- $Br(X) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ suggests $\mathcal{K}_\alpha$ parametrizes Bridgeland stability conditions on $D^b(X, \alpha)$
- $H^2(C, \mathcal{O}_C^*) = 0$ for C a curve
- Twisting can be ignored in DT computations
- $Br(X) \simeq H^3(X, \mathbb{Z})_{\mathrm{tors}}$ suggests $\mathcal{K}_\alpha$ parametrizes Bridgeland stability conditions on $D^b(X, \alpha)$

Conclusions

- Torsion provides additional structure which can be exploited in many ways
  - A feature, not a bug
- GV invariants of non-Kähler resolutions and their flops can be computed by B-model techniques
- A general description in terms of noncommutative resolutions is anticipated

THANK YOU!