Enumerative Invariants of Calabi-Yau Threefolds with Torsion and Noncommutative Resolutions

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Seminar in Real and Complex Geometry

Joint work with A. Klemm, T. Schimannek, and E. Sharpe

arXiv:2212.08655[hep-th], and
work in progress with T. Schimannek
1 Overview

2 Torsion

3 Determinantal octic double solid

4 Brauer group and twisted sheaves
Determinantal Octic Double Solid

- \( A(x) \in M_8(H^0(\mathbb{P}^3, \mathcal{O}(1))) \ x \in \mathbb{P}^3 \), generic symmetric 8 \times 8 matrix
- \( \det(A) = 0 \) deg 8 hypersurface \( S \subset \mathbb{P}^3 \), 84 \( A_1 \) singularities \( p_i \)

\[
S = \left\{ x \in \mathbb{P}^3 \left| \text{rank}(A(x)) \leq 7 \right. \right\}, \quad \{ p_i \} = \left\{ x \in \mathbb{P}^3 \left| \text{rank}(A(x)) = 6 \right. \right\}
\]

- \( Y \rightarrow \mathbb{P}^3 \) 2-1 cover branched along \( S \), nodal singularities over \( p_i \)

\[
y^2 = \det(A(x)), \ x \in \mathbb{P}^3
\]

- \( Y \) determinantal octic double solid
- \( K_Y \cong (K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(8/2)) \ |_Y \) is trivial
- \( X \rightarrow Y \) small resolution
- Exceptional \( \mathbb{P}^1 \)'s are 2-torsion classes in \( H_2(X, \mathbb{Z}) \)
- \( X \) is not Kähler
Write $A = \sum_{i=1}^{4} A_i x_i$, $y \in \mathbb{C}^8 = V$

$Q_i(y) = y^t A_i y$, four quadrics in $\mathbb{P}^7$

$Z = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^7$ CY 3fold, $h^{1,1}(Z) = 1$

$X$ is a moduli space of spinor sheaves on quadrics in $\mathbb{P}^7$, restricted to $Z$

No universal sheaf on $X \times Z$

Universal $\alpha$-twisted spinor sheaf $\mathcal{E}$ on $X \times Z$, $\alpha \in Br(X)$ Addington 0904.1764
$\mathcal{Q} \to \mathbb{P}^3$ bundle of quadric hypersurfaces in $\mathbb{P}^7$

$\mathcal{B}_0$ sheaf of even Clifford algebras on $\mathbb{P}^3$ Kuznetsov 0510670

$$\mathcal{B}_0 = \mathcal{O}_{\mathbb{P}^3} \oplus \left( \Lambda^2(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \right) \oplus \left( \Lambda^4(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \right) \oplus \left( \Lambda^6(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \right) \oplus \left( \Lambda^8(V) \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \right)$$

$X$ is a moduli space of $\mathcal{B}_0$-modules supported on points of $\mathbb{P}^3$

Points of any small resolution of $Y$ represented by $\mathcal{B}_0$-modules supported on points $p \in \mathbb{P}^3$

In other words, a $\mathcal{B}_0 \otimes \mathcal{O}_p = \text{Cl}^0(A(p))$ representation
Cl}_k \) (complex) Clifford algebra of nondegenerate quadratic form in \( k \) variables

\[
\text{Cl}(V, Q) = \bigotimes V / (v_1 v_2 + v_2 v_1 = 2Q(v_1, v_2) \cdot 1)
\]

- \( \mathbb{M}_r \) algebra of \( r \times r \) matrices
- \( p \in \mathbb{P}^3 - S: \)
  - \( \mathcal{B}_0 \otimes \mathcal{O}_p \simeq \text{Cl}^0_8 \simeq \text{Cl}_7 \simeq \mathbb{M}_8 \times \mathbb{M}_8 \)
  - Two 8-dimensional reps, corresponding to two points of \( X \) over \( p \)
- \( p \) smooth point of \( S: \)
  - \( \mathcal{B}_0 \otimes \mathcal{O}_p \simeq \text{Cl}_6 \otimes \text{Cl}(\mathbb{C}, 0) \simeq \mathbb{M}_8 \otimes \mathbb{C}[\epsilon]/\epsilon^2 \)
  - Unique 8-d rep, corresponding to unique point of \( X \) over \( p \)
- $p$ node of $S$
- $W$ 2-dimensional complex vector space
- $B_0 \otimes O_{p_i} \simeq \text{Cl}_5 \otimes \text{Cl}(W, 0)$
- $\text{Cl}_5 \simeq M_4 \times M_4$
- 4-dimensional reps $S'$ and $S''$ of $\text{Cl}_5$
- For $[w] \in \mathbb{P}(W) \simeq \mathbb{P}^1$ have 2-dim'l rep $R[w] := w \wedge^* W$ of $\wedge^* W$
- Get two $\mathbb{P}^1$'s worth of 8-d reps of $B_0 \otimes O_{p_i} : S' \otimes R[w], S'' \otimes R[w]$ of $B_0$
- Identified with the points of the two small resolutions of $p_i$
\[ D^b(X, \alpha) \overset{\text{Addington}}{=} D^b(Z) \overset{\text{Kuznetsov}}{=} D^b(\mathbb{P}^3, \mathcal{B}_0) \]
Mirror $Z^\circ = \tilde{Z}_{\psi}/G$, $|G| < \infty$ of $Z$ has $h^{2,1}(Z) = 1$

$$x_0^2 + x_1^2 = 2\psi x_0 x_1, \ x_2^2 + x_3^2 = 2\psi x_2 x_3, \ x_4^2 + x_5^2 = 2\psi x_4 x_5, \ x_6^2 + x_7^2 = 2\psi x_6 x_7$$

Two points of maximal unipotent monodromy

- Large complex structure limit $\psi = \infty$ has fundamental period

$$\varpi(\psi) = \sum_{n=0}^{\infty} \frac{((2n)!)^4}{(n!)^8} \psi^{-8n}$$

and GW invariants of $Z$ found by Picard-Fuchs system, mirror symmetry, holomorphic anomaly, conifold gap condition. . .

- Picard-Fuchs near MUM point at $\psi = 0$ used in calculations
- Together with the mirror of the smooth octic double solid Clemens this determines enumerative invariants

What is the B-model computing?
Conjecture

Let $S$ be the set of all small resolutions of $Y$, $|S| = 2^{84}$

$X^{\text{smooth}} = X'^{\text{smooth}}$ for any $X, X' \in S$

$\Gamma = \Gamma_{XX'} \subset X \times X'$ graph closure

$\psi_{XX'} : D^b(X) \rightarrow D^b(X')$ equivalence from FM with kernel $O_\Gamma$

$$\psi_{XX'}(F) = R\pi_{X'}^* \left( L\pi^*_X(F) \otimes O_\Gamma \right)$$

$\text{Coh}_{\leq 1}(X)$ coherent sheaves on $X$ of dimension at most 1

**Conjecture**

B-model computes Gopakumar-Vafa invariants of

$$\left( \bigcup_{X \in S} \text{Coh}_{\leq 1}(X) \right) / E^\bullet \sim \psi_{XX'}(E^\bullet), \ E^\bullet \in D^b(X)$$

Conjecture is more general
\[ P^3 \] can be replaced with a more general Fano threefold \( B \) of even index

\( \mathbf{A} \) can be replaced with a symmetric matrix whose entries are sections of line bundles on \( B \)

For \( B = \mathbb{P}^3 \), let \( \vec{d} = (d_1, \ldots, d_k), \sum d_i = 8 \), all \( d_i \) have same parity

\( A \) symmetric matrix, \( A_{ij} \in H^0(\mathbb{P}^3, \mathcal{O}((d_i + d_j)/2)) \)

Main example: \( \vec{d} = 1^8 \)

\( Y \) double cover of \( \mathbb{P}^3 \) branched over \( \det(A) = 0 \), \( X \) small resolution
1. Overview

2. Torsion

3. Determinantal octic double solid

4. Brauer group and twisted sheaves
Flat B-fields

- $X$ compact manifold
- Flat B-field $B \in H^2(X, U(1))$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0$$

$$0 \rightarrow H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) \rightarrow H^2(X, U(1)) \rightarrow H^3(X, \mathbb{Z}) \rightarrow H^3(X, \mathbb{R})$$

$c : H^2(X, U(1)) \rightarrow H^3(X, \mathbb{Z})_{\text{tors}}$

$\gamma = c(B)$ is a characteristic class of $B$
Kähler Moduli space

- $X$ Calabi-Yau threefold
- (Complexified) Kähler moduli space

$$\mathcal{K} = \left\{ B + iJ \mid B \in H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}), \ J \text{ Kahler, } J >> 0 \right\}$$

- Naturally generalizes to connected components in the presence of torsion.
- Fix a characteristic class $\gamma \in H^3(X, \mathbb{Z})_{tors}$

$$\mathcal{K}_\gamma := \left\{ B + iJ \mid B \in H^2(X, U(1)), \ c(B) = \gamma, \ J \text{ Kahler, } J >> 0 \right\}$$

- All isomorphic to $(\Delta^*)^{b_2(X)}$
- Multiple large radius limits, different topological string partition functions
To determine a mirror, a characteristic class $\gamma \in H^3(X, \mathbb{Z})_{\text{tors}}$ must be fixed in addition to $X$.

The $\psi = 0$ MUM point of the mirror of the $(2, 2, 2, 2)$ complete intersection is mirror to $X$ and $B$-fields with nontrivial characteristic class.

Physicists refer to $B$-fields with nonzero characteristic class as a *fractional $B$-field*. 
Gromov-Witten theory

- Usual GW expansion variables

\[ q^\beta = \exp \left( 2\pi i \int_{\beta} B + iJ \right), \quad \beta \in H_2(X, \mathbb{Z}) \]

- Have cap product pairing

\[ H^2(X, U(1)) \times H_2(X, \mathbb{Z}) \to U(1) \]

- On any \( \mathcal{K}_\gamma \) have expansion variables

\[ q^\beta_\gamma = (B \cap \beta) \exp \left( -2\pi \int_{\beta} J \right) \]

\[ F_\gamma = \sum N^g_{\beta} \lambda^{2g-2} q^\beta_\gamma, \quad N^g_{\beta} \text{ GW invariants} \]

- See also Aspinwall-Gross-Morrison hep-th/9503208
More on expansion variables

- \( \mathcal{K}_\gamma \cong \mathcal{K} \), not canonically.
- Up to choices, can relate \( q^\beta_\gamma \) to usual expansion variables
- For exposition, assume \( H^3(X, \mathbb{Z})_{\text{tors}} \cong \mathbb{Z}_k \)
- Choose \( B_0 \) with \( c(B_0) = \gamma \) and \( kB_0 = 0 \) (\( k^{b_2(X)} \) choices)

\[
c^{-1}(\gamma) = \left\{ B_0 + B \mid B \in H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) \right\}
\]

\[
q^\beta_\gamma = (B_0 \cap \beta) q^\beta
\]

- Since \( kB_0 = 0 \), \( B_0 \cap \beta \) kth root of 1 \( \in U(1) \)

\[
F_\gamma = \sum N^g_\beta (B_0 \cap \beta) \lambda^{2g-2} q^\beta
\]

- This structure facilitates the determination of the \( F_\gamma \) by B-model techniques
Torsion in $H_2(X, \mathbb{Z})$

- $H_2(X, \mathbb{Z})_{\text{tors}} \cong H^3(X, \mathbb{Z})_{\text{tors}}$ universal coefficient theorem
- $H_2(X, \mathbb{Z}) \cong \mathbb{Z}^{b_2(X)} \oplus H_2(X, \mathbb{Z})_{\text{tors}}$, noncanonical isomorphism
- Choice of splitting $(H_2(X, \mathbb{Z})_{\text{tors}} \cong \mathbb{Z}_k)$

$$H_2(X, \mathbb{Z}) \rightarrow \mu_k \cong \mathbb{Z}_k, \quad \beta \mapsto B_0 \cap \beta$$

- $\deg(\beta) \in \mathbb{Z}^{b_2(X)}$ degree
- $t(\beta) \in \mathbb{Z}_k$ torsion class (M-theory $\mathbb{Z}_k$ charge)
For $\beta \in H_2(X, \mathbb{Z})$ have Gopakumar-Vafa invariants $n^g_\beta \in \mathbb{Z}$

Sometimes write $n^g_{\deg(\beta), t(\beta)}$ in place of $n^g_\beta$

For all $\gamma \in H^3(X, \mathbb{Z})_{\text{tors}}$

$$F_\gamma = \sum_{\beta} \frac{n^g_\beta}{m} \left( 2 \sin \left( \frac{m\lambda}{2} \right) \right)^{2g-2} q^{m_\beta}$$

Have $k$ times as many GV invariants as in the torsion-free case

Have $k$ times as many generating functions to compensate, along with $k$ different mirrors
Outline

1. Overview
2. Torsion
3. Determinantal octic double solid
4. Brauer group and twisted sheaves
Identifying the torsion

- $X \rightarrow Y$ small resolution of determinantal octic double solid $Y$
- $\pi : X \rightarrow \mathbb{P}^3$
- $H \in H^2(\mathbb{P}^3, \mathbb{Z})$ hyperplane class
- $H^2(X, \mathbb{Z})$ generated by $\pi^*(H)$
- $H_2(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$
- $\beta \in H_2(X, \mathbb{Z})$, $d = d(\beta) = \pi^*(H) \cap \beta \in \mathbb{Z}$, torsion class $j \in \mathbb{Z}_2$
- $[C_i] \in H_2(X, \mathbb{Z})$ nontrivial 2-torsion class ($d = 0, j = 1$)
- $\pi^{-1}(\text{line}) = \pi^*(H^2)$ has class $d = 2, j = 1$, $\pi : X \rightarrow \mathbb{P}^3$

\[d = \pi^*(H) \cdot \pi^*(H^2) = \pi^*(H^3) = 2\]
Proof of claims about torsion

- Degenerate $Y$ to $Y'$, requiring lower right $3 \times 3$ block of $A$ vanishes

$$A = \begin{pmatrix} B_{5 \times 5} & C^t \\ C & 0_{3 \times 3} \end{pmatrix}$$

- Nodes of $Y'$
  - 84 nodes $p_i$ where $\text{rank}(A(p_i)) = 6$
  - Nodes $q_j$ where $\text{rank}(C(q_j)) = 2$

- $Y'$ has an explicit projective small resolution $X'$: a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^4$

$$\sum_{j=0}^{4} C_{ij}(x)y_j = 0, \sum_{i,j=0}^{4} B_{ij}y_iy_j = 0$$

- $H_2(X', \mathbb{Z}) \cong \mathbb{Z}^2$
Proof of claims about torsion

\[ \sum_{j=0}^{4} C_{ij}(x)y_j = 0, \sum_{i,j=0}^{4} B_{ij}y_i y_j = 0 \]

- Exceptional curves over \( p_i \) are lines in \( \mathbb{P}^4 \); exceptional curves over \( q_j \) are conics in \( \mathbb{P}^4 \)
- \( C_1 \) class of exceptional curve over \( p_i \); \( C_2 \) class of exceptional curve over \( q_j \)
- \([C_1] = (0, 1), [C_2] = (0, 2)\)
- \( D := [\pi'^{-1}(\text{line})] = \pi_1^* (H_1^2) \)
- \( D \) has class \( ((\pi_1^*(H_1) \cdot D), (\pi_2^*(H_2) \cdot D)) = (2, 7) \)
  \[ (H_2 \cdot D)_{X'} = (H_2 H_1^2 \cdot (H_1 + H_2)^3 (H_1 + 2H_2))_{\mathbb{P}^3 \times \mathbb{P}^4} = 7 \]
- \( X \) from \( X' \) by contracting \( C_2 \) curves to \( Y' \) and smoothing
- \( H_2(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \)
- \([C_1] \in H_2(X, \mathbb{Z}) \) nontrivial 2-torsion class \( (d = 0, j = 1) \)
- \( \pi^{-1}(\text{line}) \) has class \( d = 2, j = 1 \).
Computing GV invariants

- GV invariants denoted $n_{d,j}^g$
- From conifold transition $X \rightarrow Y_t$ Ruan

\[ n_{d,0}^g(X) + n_{d,1}^g(X) = n_d^g(Y_t) \]

- $Y_t$ smooth octic double solid, GV invariants computed long ago
- Determinantal double solid provides torsion refinement of these GV invariants
### Torsion class zero

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Geometric calculations

- \( n_{1,0}^0 = n_{1,1}^0 = 14752 \)
- \( n_{2,0}^3 = 0; n_{2,1}^3 = 6; n_{2,0}^2 = 504; n_{2,1}^2 = 360 \)
- \( n_{3,0}^3 = n_{3,1}^3 = 14752 \cdot (-6) = -88512 \)
- \( n_{2d,d}^{d^2+d+1} = (-1)^{(d^2+3d+6)/2} (2d^2 + 6d + 4), n_{2d,d+1}^{d^2+d+1} = 0 (d \geq 2) \)
- \( n_{2d,d+1}^{d^2+d+1} = 252(d^2 + 3d) (d \geq 2) \)
- \( n_{2d+1,0}^{d^2+2d} = n_{2d+1,1}^{d^2+2d} = (-1)^{(d^2+3d+2)/2} 14752 (d^2 + 3d + 2) (d \geq 2) \)
- \( n_{2d,d}^{d^2+d} = (-1)^{(d^2+3d+8)/2} (4d^4 + 16d^3 - 172d^2 - 568d) (d \geq 2) \)
**Torsion class zero**

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Four-tangent lines

- $L \subset \mathbb{P}^3$ meeting $S$ in 4 points of tangency.
- 14,752 such lines H. Schubert, 1879.0001
- $\pi^{-1}(L) = L_1 \cup L_2 \subset Y$, rational curves meeting at the 4 points over the tangencies
- Identify with $L_i \subset X$
- $d(L_i) = \pi^*(H) \cdot L_i = H \cdot \pi^*(L_i) = 1$
- $L_2$ has class $(1, j)$ for some $j \in \mathbb{Z}_2$
- $L_1 - L_2 = (L_1 + L_2) - 2L_2$ has class $(2, 1) - 2(1, j) = (0, 1)$
- $L_1, L_2$ have different torsion classes
\[ C \subset X, \ d = 1 = \pi^*(H) \cdot C = H \cdot \pi_*(C) \]
\[ \pi(C) \subset \mathbb{P}^3 \text{ line } L \]
\[ \pi^{-1}(L) \subset X \text{ curve of degree 2 containing } C \]
\[ \pi^{-1}(L) \text{ has two degree 1 components, hence } L \text{ 4-tangent line, } C = L_1 \text{ (say), } g = 0 \]
\[ 14,752 \text{ such lines.} \]
\[ \text{If } L_1 \text{ has torsion class } j, \text{ then } L_2 \text{ has torsion class } j + 1. \]

\[ n_{1,j}^g = \begin{cases} 
14752 & g = 0 \\
0 & g > 0 
\end{cases} \]

**Remark.** The isomorphism \( H_2(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \) not canonical, so no intrinsic meaning for torsion class of the \( L_i \)
\[ \mathbb{Z} \oplus \mathbb{Z}_2 \text{ has automorphism } (d, j) \mapsto (d, d + j) \]
\( j \) intrinsically defined for even \( d \)
KKV methods

- $\beta \in H^2(X, \mathbb{Z})$
- $g(\beta)$ maximal (arithmetic) genus of curve of class $g(\beta)$ (Castelnuovo bound)
- $\mathcal{M}_\beta$ Chow variety of curves of class $\beta$
- $\mathcal{M}'_\beta \subset \mathcal{M}_\beta$ parametrizing curves of arithmetic genus $g(\beta)$
- $n^{g(\beta)}_\beta$ Behrend-weighted Euler characteristic of $\mathcal{M}'_\beta$

Gopakumar-Vafa; K-Klemm-Vafa
KKV methods, continued

- $\mathcal{C} \rightarrow \mathcal{M}^\prime_{\beta}$ universal curve of (arithmetic) genus $g(\beta)$
- Suppose $\mathcal{C}, \mathcal{M}^\prime_{\beta}$ smooth (+...)

$$n_{\beta}^{g(\beta)-1} = (-1)^{\dim(C)} \left( \chi_{\text{top}}(\mathcal{C}) + (2g(\beta) - 2) \chi_{\text{top}}(\mathcal{M}^\prime_{\beta}) \right)$$

- Proven using Maulik-Toda definition of GV invariants \texttt{1610.07303} and the decomposition theorem for perverse sheaves L. Zhao
Degree 2

- $C \subset X, \; d = 2$
- $\pi_*(C)$ degree 2 cycle in $\mathbb{P}^3$
- Either
  - $\pi(C)$ is a line $L$, $C$ degree 2 over $L$ with 8 branch points ($g(C) = 3$)
  - $\pi(C)$ is a smooth conic ($g(C) = 0$)
- Case 1: class of $C$ is $(2, 1)$
- Moduli space of such curves is $G(2, 4)$
- Second case does not contribute to genus 3 invariants
- $n^{3}_{2,1} = \chi(G(2, 4)) = 6$
- $n^{3}_{2,0} = 0$
- Generalizes to double covers of plane curves of degree $d \geq 2$
  $$n^{g}_{2d,d} = \begin{cases} (-1)^{(d^2+3d)/2+1} (2d^2 + 6d + 4) & g = d^2 + d + 1 \\ 0 & g > d^2 + d + 1 \end{cases}$$
- $d = 2 : n^{7}_{4,0} = 24$ (double cover of plane conic branched at $2 \cdot 8 = 16$ points has genus 7)
\begin{itemize}
\item $\beta = (2, 1)$
\item $\mathcal{M}_{2,1} = G(2, 4)$
\item $C \xrightarrow{(p \in \pi^{-1}(L)) \mapsto p} X$
\item Fiber over $p$: $\mathbb{P}^2$ of lines in $\mathbb{P}^3$ through $\pi(p)$
\item $\chi_{\text{top}}(C) = 3\chi_{\text{top}}(X) = -384$
\item $n_{2,1}^2 = -(-384 + 4 \cdot 6) = 360$
\end{itemize}
Class $(2, 0)$ for $g > 0$: necessarily $2$-1 $\pi : C \rightarrow L$

$\pi^{-1}(L) = C + C_i$

$L$ passes through $p_i$

Both small resolutions contribute

$n_{2,0}^2 = 2 \cdot 84 \cdot 3 = 504$
Outline

1. Overview
2. Torsion
3. Determinantal octic double solid
4. Brauer group and twisted sheaves
- $Br(X)$ generated by projective bundles or Azumaya algebras
- $P \to X$ rank $r - 1$ projective bundle
- Gluing of trivializations determines class in $H^1(X, \text{PGL}(r))$, hence $H^2(X, \mathcal{O}_X^*)_{\text{tors}}$ via

$$0 \to \mu_r \to \text{GL}(r) \to \text{PGL}(r) \to 0$$

$$H^1(X, \text{PGL}(r)) \to H^2(X, \mu_r) \to H^2(X, \mathcal{O}_X^*)$$

- $Br(X) \simeq H^2(X, \mathcal{O}_X^*)_{\text{tors}}$ Gabber-de Jong
- For a CY 3fold, $Br(X) \simeq H^3(X, \mathbb{Z})_{\text{tors}}$ via

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$  

$$H^2(X, \mathcal{O}_X^*) \hookrightarrow H^3(X, \mathbb{Z})$$
Twisted sheaves

- Represent $\alpha \in H^2(X, \mathcal{O}_X^*)$ by cocycle $c_{\eta\theta\iota} \in Z^2(\{U_\eta\}, \mathcal{O}_X^*)$
- Twisted sheaf determined by sheaves $F_\eta$ on $U_\eta$ and isomorphisms

$$\phi_{\eta\theta} : F_\theta|_{U_\eta \cap U_\theta} \simeq F_\eta|_{U_\eta \cap U_\theta}$$

satisfying $\phi_{\iota\eta} \circ \phi_{\eta\theta} \circ \phi_{\theta\iota} = c_{\eta\theta\iota}$
- Derived category $D^b(X, \alpha)$ independent of choice of cocycle up to equivalence
- $\alpha \in Br(X)$ determines derived category of twisted sheaves $D^b(X, \alpha)$
- Conjecturally $D^b(X)$ admits Bridgeland stability conditions (proven in many cases)
- Space of stability conditions “extended Kähler moduli space"
- Kähler moduli is a slice of this

\[
Z(E^\bullet) = - \int_X \exp(-2\pi i)(B + iJ) \operatorname{ch}(E^\bullet) \hat{\Gamma}(X)
\]
Twisted derived category $D^b(X, \alpha)$

- $Br(X) \simeq H^3(X, \mathbb{Z})_{\text{tors}}$ suggests $\mathcal{K}_\alpha$ parametrizes Bridgeland stability conditions on $D^b(X, \alpha)$.
- $H^2(C, \mathcal{O}_C^*) = 0$ for $C$ a curve.
- Twisting can be ignored in DT computations.
- $Br(X) \simeq H^3(X, \mathbb{Z})_{\text{tors}}$ suggests $\mathcal{K}_\alpha$ parametrizes Bridgeland stability conditions on $D^b(X, \alpha)$. 

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Conclusions

- Torsion provides additional structure which can be exploited in many ways
  - A feature, not a bug
- GV invariants of non-Kähler resolutions and their flops can be computed by B-model techniques
- A general description in terms of noncommutative resolutions is anticipated
THANK YOU!