On the Severi problem with point constraints

with K. Christ and I. Tyomkin

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Overview

1. Overview
2. Basic tropical geometry
3. Tropicalizing the Severi variety
4. Tropicalizing a family of curves
5. The proof

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- $\dim V_{g,d} = 3d + g - 1$ and it is regular at nodal curves of geometric genus g.
- The notion of Severi variety generalizes naturally: let S be a projective surface and $\mathcal{L}$ a line bundle on S. The *Severi variety* $V_{g,\mathcal{L}}$ is the closure of the following loci in $|\mathcal{L}|$:
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- Severi gave a proof of the irreducibility of $V_{g,d}$ which contains a gap. The correct proof is given by Harris in 1980s in characteristic zero. The result was generalized to other rational surfaces. In particular, Tyomkin proved the irreducibility for Hirzebruch surfaces (also in characteristic zero), and in a recent project of Christ, Tyomkin and the speaker, a characteristic free proof the plane case is given.

The proof mentioned above consist of two essential steps:

- To show that each irreducible component of $V_{g,d}$ contains the locus $V_{0,d}$ of rational curves of degree d (or the locus of certain more degenerated curves). This is basically by induction: showing that each component of $V_{g,d}$ contains a component of $V_{g-1,d}$ (or a codimension one sublocus of curves of genus less than g).

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- Show that there is a unique component of $V_{g,d}$ containing $V_{0,d}$. This is by a monodromy argument: the monodromy group of $V_{0,d}$ acts transitively on the nodes of a general curve in $V_{0,d}$.

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The first step is more involved and lead to substantial progress in various directions, including Caporaso and Harris' recursive formula of the number of plane curves of degree d and genus g passing through $3d + g - 1 = \dim V_{g,d}$ general points (the Severi degree).

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Theorem(Christ-He-Tyomkin)

The sublocus in $V_{g,d}$ consisting of curves passing through $p_1, ..., p_{n-1}$ is irreducible.

Basic tropical geometry

The tropicalization map

Let $T^n = (K^*)^n$ be the algebraic torus. We define $\mathrm{Trop}\colon T^n \to \mathbb{R}^n$ by

$$(x_1, ..., x_n) \to (-\mathrm{val}(x_1), ..., -\mathrm{val}(x_n)).$$

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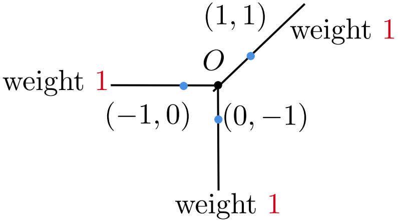
Example (The tropicalization of $X = \{x + y + 1 = 0\} \subset T^2$)



$$1 \cdot (0, -1) + 1 \cdot (-1, 0) + 1 \cdot (1, 1) = 0$$

Tropical intersection theory

Stable intersection

For two tropical varieties $V_1, V_2 \subset \mathbb{R}^n$, one can construct the *stable intersection* $V_1 \cdot V_2$ of V_1 and V_2 such that:

- $V_1 \cdot V_2$ is a tropical variety which is also a subcomplex of $V_1 \cap V_2$, its codimension in $\mathbb{R}^n$ is $\mathrm{codim} V_1 + \mathrm{codim} V_2$;

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- If τ is a face of $V_1 \cap V_2$ along which V_1 intersects V_2 *properly*, i.e. τ is maximal and $\mathrm{codim}\tau = \mathrm{codim} V_1 + \mathrm{codim} V_2$, then $\tau \subset V_1 \cdot V_2$.

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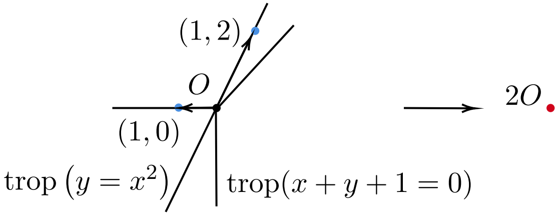

The tropical lifting theorem (Osserman-Payne)

Let $X, X' \subset T^n$. Assume that $\mathrm{trop}(X)$ and $\mathrm{trop}(X')$ intersect properly along a face τ of $\mathrm{trop}(X) \cap \mathrm{trop}(X')$. Then τ is contained in $\mathrm{trop}(X \cap X')$ and

$$i(\tau; \mathrm{trop}(X) \cdot \mathrm{trop}(X')) = \sum_{Z} i(Z; X \cdot X') m_{\mathrm{trop}(Z)}(\tau)$$

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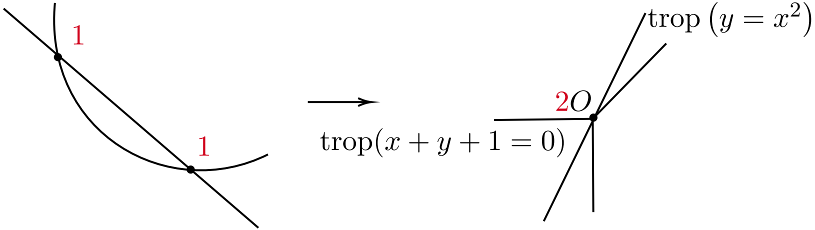
The theorem can be generalized to intersecting more than 2 subvarieties.

Example



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We have a natural inclusion $V_{g,d} \subset \mathbb{P}^N = |\mathscr{O}_{\mathbb{P}^2}(d)|$ where $N = \frac{d(d+3)}{2}$. Let T^N be the maximal torus in $\mathbb{P}^N$ consisting of points with non-zero coordinates. Pick general points $p_1, ..., p_n$ in $T^2 \subset \mathbb{P}^2$ tropicalizing to n general points $q_1, ..., q_n \subset \mathbb{R}^2$. Let H_i be the hyperplane in $\mathbb{P}^N$ cut out by passing through p_i. Then the Severi variety (with point constraints) is $V_{g,d} \cap H_1 \cap \cdots \cap H_{n-1}$.

Denote $V^\circ_{g,d} = V_{g,d} \cap T^N$ and $H^\circ_i = H_i \cap T^N$. We want to calculate $\mathrm{trop}(V^\circ_{g,d}) \cdot \mathrm{trop}(H^\circ_1) \cdot \cdots \cdot \mathrm{trop}(H^\circ_{n-1})$.

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- $\mathrm{trop}(H^\circ_i)$ consists of points in $\mathbb{R}^N$ defining tropical plane curves passing through q_i.

Calculate $\mathrm{trop}(V^{\circ}_{g,d}) \cdot \mathrm{trop}(H^{\circ}_1) \cdot \dots \cdot \mathrm{trop}(H^{\circ}_{n-1})$

Mikhalkin's Correspondence theorem (Mikhalkin; Nishinou, Shustin, Tyomkin, etc.)

For each degree-d and genus-g tropical plane curve Γ passing through all $\{q_i\}_{1\le i\le n}$ there are $\mathrm{mult}(\Gamma)$ distinct curves in $V_{g,d}$ passing through all p_i whose tropicalization is Γ. Here $\mathrm{mult}(\Gamma)$ is the product of the twice areas of all triangles in the dual subdivision of Γ.

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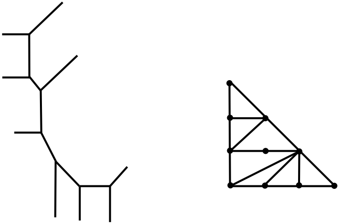
Example

A degree-3 tropical curve with genus 0 and multiplicity 4.



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Let Γ be a tropical curve passing through all q_i's such that $\mathrm{mult}(\Gamma) = 1$. Then there is a unique point $w \in \mathbb{R}^N$ whose corresponding tropical curve is Γ and $\mathrm{trop}(V^\circ_{g,d})$ intersects transversely with $\{\mathrm{trop}(H_i^\circ)\}_{i \leq n}$ along w.

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It follows from Mikhalkin's theorem and tropical lifting theorem that

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According to the tropical lifting theorem, it suffices to show that for any irreducible component Z of $V_{g,d} \cap H_1 \cap \cdots \cap H_{n-1}$, $\mathrm{trop}(Z)$ contains τ.

Tropicalizing a family of curves

Parametrized tropical curves

An *abstract tropical curve* is a metric graph $\Gamma = (\mathbb{G}, \ell)$ where $\mathbb{G}$ is a (connected) graph with legs, and $\ell\colon E(\mathbb{G}) \to \mathbb{R}_{>0}$ is the length function.

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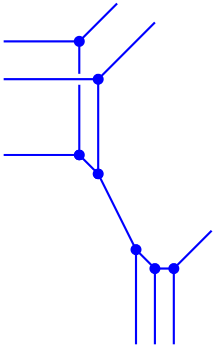
Parametrized tropical curves

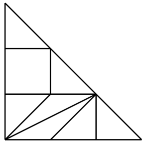
A *parametrized (plane) tropical curve* is a pair (Γ, h), where Γ is an abstract tropical curve, and $h\colon \Gamma \to \mathbb{R}^2$ is a map such that:

- for any edge $e \in E(\mathbb{G})$, the restriction $h|_e$ is an integral affine function;
- (Balancing condition) for any vertex $v \in V(\mathbb{G})$ we have $\sum_{\vec{e} \in \mathrm{Star}(v)} \frac{\partial h}{\partial \vec{e}} = 0$.

Example

An example of parametrized tropical curve with degree 3 and genus 0. The slopes are equal to the primitive vectors on the image of each edge.





Floor decomposed tropical curves

A parametrized tropical curve (Γ, h) is called *floor decomposed* if for any edge e of Γ, the slope $\frac{\partial h}{\partial \vec{e}}$ is either $(\pm 1, *)$ or $(0, *)$. In other words the edges in its dual subdivision has vertical length at most one.

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If the image of a non-contracted edge (resp. leg) is vertical, then the edge (resp. leg) is called an *elevator*. After removing the interior of all elevators in Γ, we are left with a disconnected graph. The non-contracted connected components of this graph are called the *floors* of Γ.

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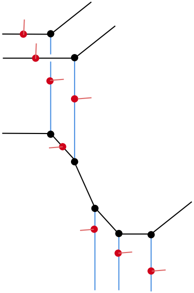

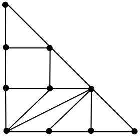

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Let $C \subset \mathbb{P}^2$ be an integral curve of degree d which do not contain any orbit of $\mathbb{P}^2$. Let $\{\sigma_i\}$ be a set of marked non-singular points of C that contains $C \backslash T^2$. Normalizing C gives a map $f \colon C^\nu \to \mathbb{P}^2$. Let C_0 be the stable reduction of C^ν and Γ_C the dual graph of C_0.

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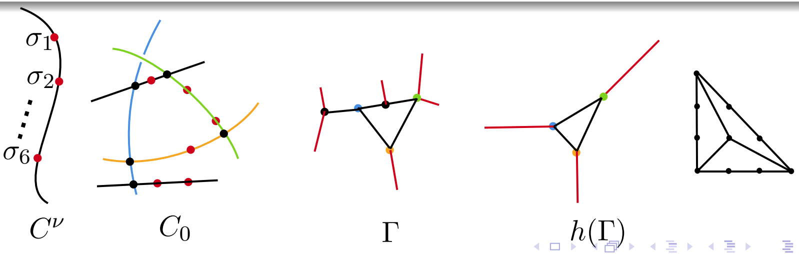
Γ_C is an abstract tropical curve with legs corresponding to σ_i, and f induces a map $h_C\colon \Gamma_C \to \mathbb{R}^2$ which realizes Γ_C as a parametrized tropical curve. The edges of Γ_C can be contacted by h_C. A leg l_i corresponding to σ_i is contracted by h_C if and only if $\sigma_i \in T^2$. Moreover, $h_C(\Gamma)$ is dual to a subdivision of the triangle $(0,0)-(0,d)-(d,0)$.



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There is a specialization map trop: $C^\nu \to \Gamma_C$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(T^2) & \xrightarrow{\text{trop}} & \Gamma_C \\ \downarrow f & & \downarrow h_C \\ T^2 & \xrightarrow{\text{trop}} & \mathbb{R}^2 \end{array}$$

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$$\begin{array}{ccc} f^{-1}(T^2) & \xrightarrow{\text{trop}} & \Gamma_C \\ \downarrow f & & \downarrow h_C \\ T^2 & \xrightarrow{\text{trop}} & \mathbb{R}^2 \end{array}$$

In particular, if $\sigma_i \in T^2$ then $h_C(l_i) = \text{trop}(\sigma_i)$.

The balancing theorem(Christ-He-Tyomkin)

Let Z be an irreducible component of $V_{g,d} \cap H_1 \cap \cdots \cap H_{n-1}$. For each $[C] \in Z$ mark the point $p_1, ..., p_{n-1}$ as well as the points in $C \backslash T^2$. Then $\Lambda := \{(\Gamma_C, h_C) | [C] \in Z \text{ general}\}$ is a family of paramatrized tropical curves such that:

(1) If $(\Gamma, h) \in \Lambda$ is a tri-valent submersed tropical curve then Λ contains certain degenerations of (Γ, h);

(2) If $(\Gamma, h) \in \Lambda$ is tri-valent and submersed except for a four-valent vertex v, then Λ contains all three deformation types of Γ such that v becomes two three-valent vertices.

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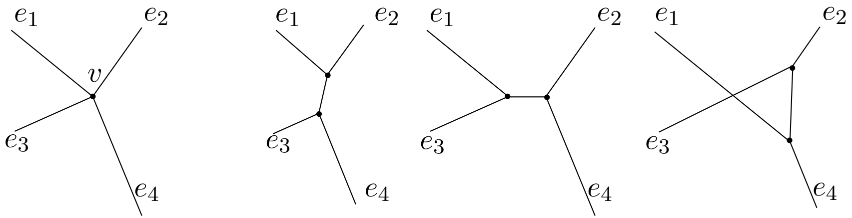
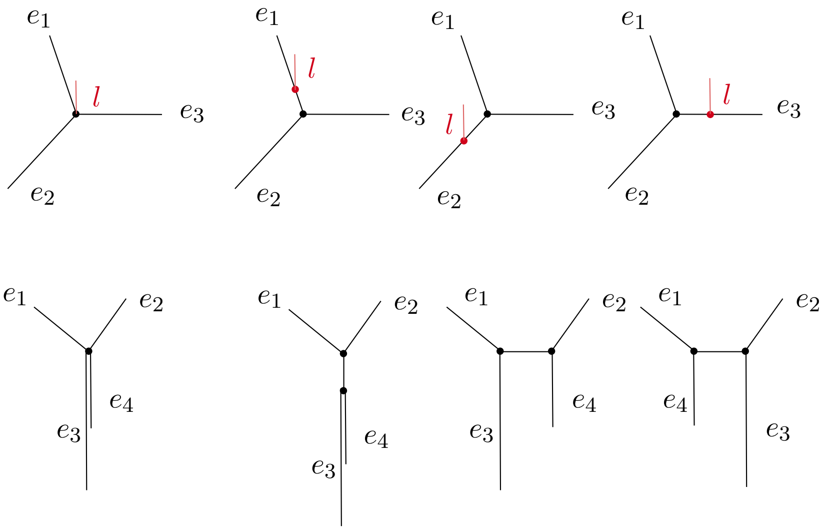
Example



Example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The proof

Recall

We take general points $p_1, ..., p_{n-1} \in T^2 \subset \mathbb{P}^2$ such that $q_i = \text{trop}(p_i) \in \mathbb{R}^2$ is in general position. Let $H_i \subset \mathbb{P}^N = |\mathcal{O}_{\mathbb{P}^2(d)}|$ be the locus of curves containing p_i. We want to show that for any irreducible component Z of $V_{g,d} \cap H_1 \cap \cdots \cap H_{n-1}$, we have $\text{trop}(Z) \subset \mathbb{R}^N$ contains τ, where τ is a face of $\text{trop}(V^\circ_{g,d}) \cap \text{trop}(H^\circ_1) \cap \cdots \cap \text{trop}(H^\circ_{n-1})$ containing a point w which induces a plane tropical curve of multiplicity one.

Strategy of the proof

Pick general $p_1, ..., p_n \in \mathbb{P}^2$ and $q_n = \text{trop}(p_n)$ such that for any $i > j$ we have: $y(q_i) - y(q_j) > \lambda \cdot |(x(q_i) - x(q_j))|$ for a fixed $\lambda >> 0$. Let $[C] \in Z \cap H_n$, then $\text{trop}(C)$ contains all q_i. Let (Γ_C, h_C) be the induced parametrized tropical curve from C by marking $C \backslash T^2$ and $p_1, ..., p_{n-1}$. It is tri-valent, submersed, and floor decomposed; each floor contains exaclty one contracted leg mapping to some q_i and all but one elevators contain a contracted leg that maps to some q_i, where $1 \leq i \leq n-1$; the image of the last elevator contains q_n.

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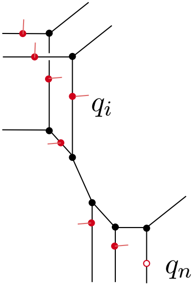

$d = 3, g = 0, n = 8$

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The idea

Start from any (Γ_C, h_C) as above (in particular, $h(\Gamma)$ can be of multiplicity one). We show that one can get a fixed parametrized tropical curve via (small) deformation and degeneration, which is contained in $\Lambda := \{(\Gamma_{C'}, h_{C'}) | [C'] \in Z\}$ by the balancing theorem. Since the process is revertable, Λ (which dominates $\text{trop}(Z) \subset \mathbb{R}^N$) contains a family of multiplicity-one tropical curves.

Step 1

To get a parametrized curve without self-intersections

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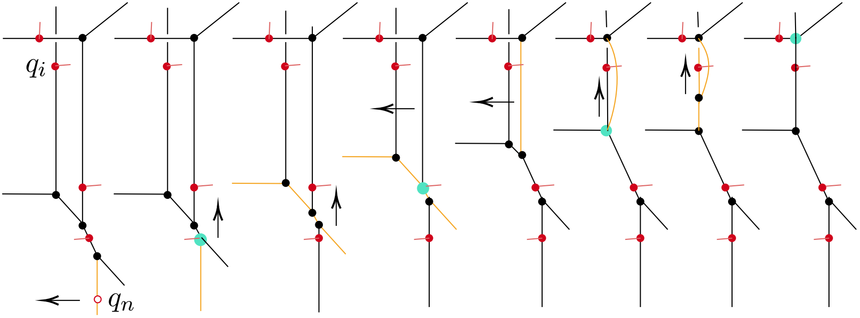

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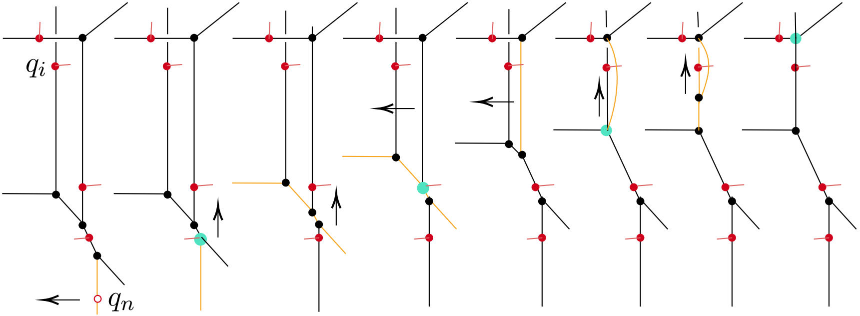
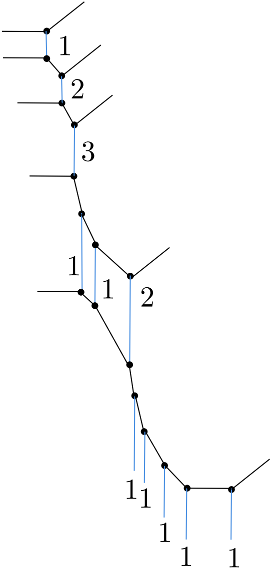


Step 2

To get a parametrized curve without self-intersections such that "the genus is concentrated on the bottom" and "the weights of elevators on the same floor are concentrated on the right side".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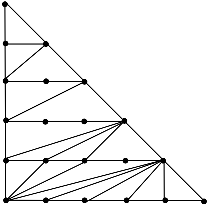

Further Directions

- Almost the same argument works for Hirzebruch surfaces.

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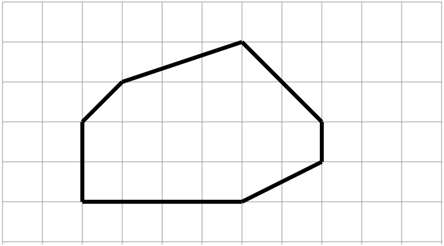
- Almost the same argument works for Hirzebruch surfaces.
- Also works for certain toric surfaces associated to an *h-transverse polygon* (the primitive vector on each edge is of the form $(*, 0)$ or $(*, \pm 1)$), provided the existence of multiplicity-one tropical curves with respect to given genus and complete linear system.



Thank You!