# Punctured logarithmic invariants and gluing


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Work with Qile Chen, Mark Gross and Bernd Siebert


Other work by Parker, Tehrani, Dhruv, Fan-Tseng-Wu-You

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# The gluing result

Here is a minimalist statement:

## Theorem (ACGS 2020)

*The evaluation maps $\widetilde{\mathcal{M}}(X,\tau) \to X^n$ of the moduli stack $\widetilde{\mathcal{M}}(X,\tau)$ of stable marked punctured curves of type $\tau$ in a log scheme $X$ are virtually idealized log smooth.*

*Given an edge of $\tau$ with splitting $\tau'$ we have a cartesian splitting diagram*

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(X/B,\tau) \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}(X,\tau') \arrow[d] \\
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There is a lot I need to explain and motivate.

# Rational plane curves

## Definition

$$N_d = \#\left\{\begin{array}{l} C \subset \mathbb{P}^2 \text{ a rational curve,} \\ \deg C = d, \text{ and} \\ p_1, \dots p_{3d-1} \in C \end{array}\right\}.$$

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## Theorem (Kontsevich)

*For $d > 1$ we have*

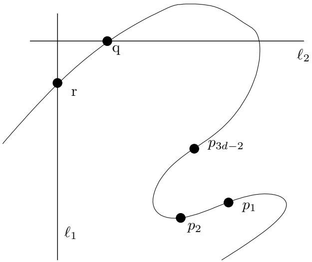
$$N_d = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} N_{d_1} N_{d_2} \left( d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

The first few numbers are

$$N_1 = 1,\ N_2 = 1,\ N_3 = 12,\ N_4 = 620,\ N_5 = 87304.$$

# Kontsevich's theorem: setup

1-parameter family $C \to B$: fix only $p_1 \ldots, p_{3d-2}$,
and two lines $\ell_1, \ell_2$ meeting at a point called $p_{3d-3}$:




We choose $q$ among $C \cap \ell_1$, and $r$ among $C \cap \ell_2$.

# Kontsevich's theorem: preview

The equation

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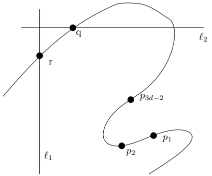


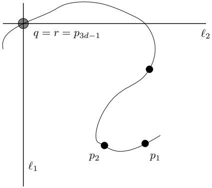

translates to

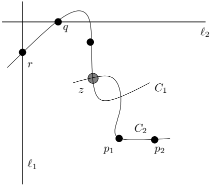
$$\deg_B(p_1, p_2 | q, r) = \deg_B(p_1, q | p_2, r)$$

coming from the cross ratio map $\lambda : B \to \mathbb{P}^1$.

# Kontsevich's theorem: computation

$$\deg_B(p_1, p_2 | q, r) \qquad = \qquad \deg_B(p_1, q | p_2, r)$$



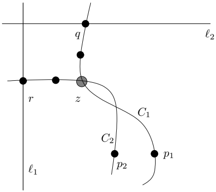




Figure: $N_d \quad + \quad \sum d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2} \quad = \quad \sum d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2},$

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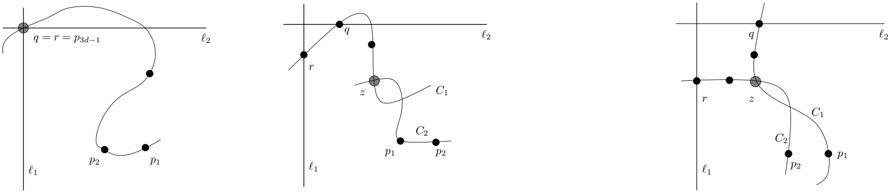


Figure: $N_d \quad + \quad \sum d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2} \quad = \quad \sum d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2},$

as needed! ♠

# Moduli spaces

- Let $\overline{\mathcal{M}}(X, \tau)$ be the Kontsevich moduli stack of stable maps in $X$
- with type specified by decorated graph $\tau = (G, h, \beta)$.
- to each vertex $v$ of $G$ we assign a genus $h(v)$ and a curve class $\beta(v)$.
- The legs are marked:

# The gluing principle on moduli spaces

**Proposition**

*The evaluation maps $\overline{\mathcal{M}}(X,\tau) \to X^m$ are virtually smooth.*
*Given an edge of $\tau$ with splitting $\tau'$ we have a cartesian splitting diagram*

$$\begin{tikzcd}
\overline{\mathcal{M}}(X/B,\tau) \arrow[r] \arrow[d] & \overline{\mathcal{M}}(X,\tau') \arrow[d] \\
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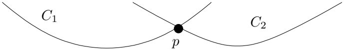
*of stacks with compatible virtual fundamental classes.*

# The gluing principle on curves




$$C = C_1 \overset{p}{\cup} C_2.$$

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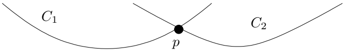


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implying

$$\begin{aligned} Hom(C, X) &= Hom(C_1, X) \underset{Hom(p,X)}{\times} Hom(C_2, X) \\ &= Hom(C_1, X) \underset{X}{\times} Hom(C_2, X) \end{aligned}$$

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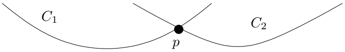


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spreading out to

$$\begin{tikzcd} \overline{\mathcal{M}}(X/B,\tau) \arrow[r] \arrow[d] & \overline{\mathcal{M}}(X,\tau') \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

# From gluing to quantum cohomology

Note that we relied on

$$X = Hom(p, X).$$

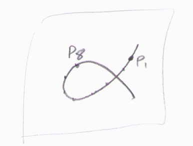
One defines quantum cohomology based on the operation

$$\langle \gamma_1 \dots \gamma_n, * \rangle_\beta \quad = \quad e_{n+1\,*}\left([M]^{\text{virt}} \cap e_1^*\gamma_1 \cdots e_n^*\gamma_n\right).$$

Associativity is a result of gluing.

# Example: plane sections of a cubic

$12 =$ number of rational cubics through $p_1, \ldots, p_8$



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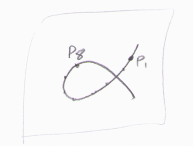


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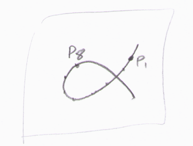

More generally

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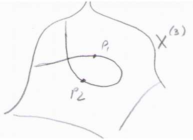
specifically

$12 =$ number of rational plane sections of $X^{(3)} \subset \mathbb{P}^3$ through $p_1, p_2$



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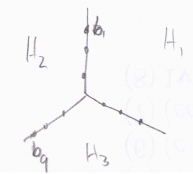
$$H_1 H_2 H_3 + tX^{(3)} = 0$$



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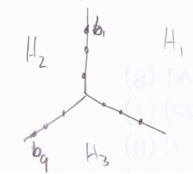
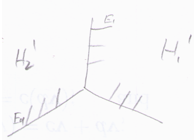



- To make it a normal crossings degeneration, blow up $H_1$ and then $H_2$.



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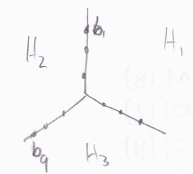


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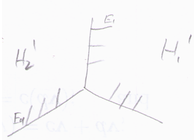


- *log smooth degeneration*

# Log geometry (K. Kato, Fontaine, Illusie; Ogus)

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- log schemes are étale glued from closed subsets of affine toric varieties - the standard-issue log smooth spaces.
- idealized log schemes are étale glued from closed subsets of monomial subschemes of affine toric varieties - the standard-issue idealized log smooth spaces.

# Log structures (K. Kato, Fontaine–Illusie)

- a log structure is a monoid homomorphism $\alpha : M \to \mathcal{O}_X$
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- Morphisms are given by natural commutative diagrams...
- A key example is the log structure associated to an open $U \subset X$,
- where $M = \mathcal{O}_X \cap \mathcal{O}_U^\times$.

# Idealized log structures (Ogus)

- a idealized log structure is a log structure $\alpha : M \to \mathcal{O}_X$
- along with a monoid ideal $K \subset M$,
- such that $\alpha(K) = 0 \in \mathcal{O}$.

# Toric and log smooth Log structures (K. Kato)

- When $X$ is a toric variety and $U$ the torus this is a prototypical example of a log smooth structure.
- In this case the monoid is associated to the regular monomials, with $\mathcal{O}^\times$ thrown in.

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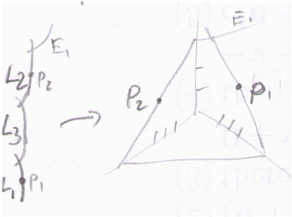
- When $X$ is a toric variety and $U$ the torus this is a prototypical example of a log smooth structure.
- In this case the monoid is associated to the regular monomials, with $\mathcal{O}^\times$ thrown in.
- In general $X$ is log smooth if it is étale locally toric.
- A morphism $X \to Y$ is log smooth if it is étale locally a base change of a dominant morphism of toric varieties.

# Back to example:

For each singular point $b_i$ there is a plane $H_i$ through $p_1, p_2$ and $b_i$



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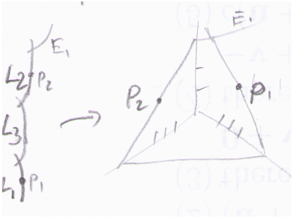


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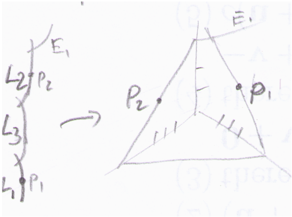


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$$12 = 9$$

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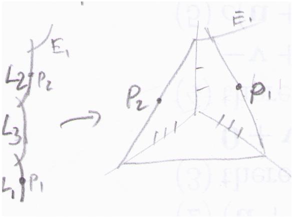
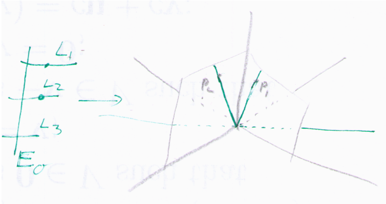



get

$$\begin{aligned} 12 &= 9 \quad \text{Anomaly?!?} \\ 12 &\stackrel{?}{=} 9 + \hbar(D \dots) \end{aligned}$$

# The balance

There is a unique plane through $p_1, p_2, O$.



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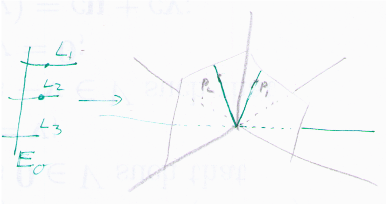


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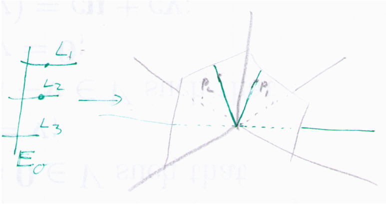


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$$12 \quad = \quad 9+1$$

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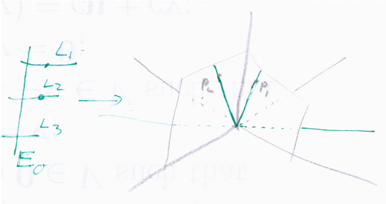


get

$$12 \quad = \quad 9 + 1 \times 3$$

What's with this multiplicity 3? Another talk!

# Log curves

- A log curve is a reduced 1-dimensional fiber of a flat log smooth morphism.
- F. Kato showed that these are the same as nodal marked curves, with "the natural" log structure.
- A punctured curve is the idealized version of the above.

# Log curves under the microscope

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- A marked point looks like $\mathrm{Spec}(M \to k[x])$ where

$$M = M_S \oplus \mathbb{N} \log x.$$

# Punctured curves under the microscope

- A puncturing of a marked curve is a log structure $M$ at a marked point with
$$M_S + \mathbb{N}\log x \quad \subseteq \quad M \quad \subsetneq \quad M_S + \mathbb{Z}\log x.$$
- It is an instance of an idealized log smooth scheme.
- In particular the splitting of a node is a punctrured curve.
- In what follow, I insist that every marking is given with a section.

# Splitting

- Consider $X \to \mathbb{A}^1$ the total space of $xy = t$, and
- $C \to S$ given by $\{y = 0\} \to \{t = 0\}$.
- At the origin $M_S + \mathbb{N}\log x \quad \subsetneq \quad M \quad \subsetneq \quad M_S + \mathbb{Z}\log x$.
- It is not a log curve, but rather a punctured curve.

# Stable punctured log maps

- Fix $X$ a nice log smooth scheme. It has a cone complex $\Sigma(X)$ with integer lattice.
- A stable punctured log map $C \to X$ is a log morphism with stable underlying morphism of schemes.

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- Stable punctured log maps have "standard issue" log structure, called minimal.

**Theorem ([ACGS])**

*$\mathcal{M}(X,\tau)$, the stack of minimal stable punctured log maps of type $\tau$, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(\underline{X},\underline{\tau})$.*

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- There is a range of choices for the punctured structure.
- For muduli of maps purposes, we use prestable structures:
- It is the minimal puncturing accommodating the map.
- For the purpose of gluing along sections, one can use cut curve structures.
- There is the maximal puncturing accommodating the section.
- The resulting categories are equivalent.

# Tropical picture

- $X$ has a cone complex $\Sigma(X)$ with integer lattice.
- $C \to S$ has cone complex $\Sigma(C) \to \Sigma(S)$. The fiber over $u \in \Sigma(S)$ is a tropical curve:
- Components give vertices, nodes give edges, and punctured points give legs.
- Usual marked points give infinite legs.
- Truly punctured points (and cut curves) give finite legs.
- A stable punctured log map gives $\Sigma(C) \to \Sigma(X)$, a family of tropical curves in $\Sigma(X)$.
- The sections mark the legs.
- Minimality is beautifully encoded in this picture...

# Gluing punctured curves

**Lemma**

*Let $C_1^\circ, C_2^\circ$ be two* cut curves *with underlying curves $\underline{C}_i$, over a log scheme $W$ with sections $W \to C_i^\circ$ along the puncture.*
*There is a unique log structure $C$, log smooth over $W$ on the nodal curve $\underline{C} = \underline{C}_1 \cup^p \underline{C}_2$, with a section at the node, restricting to $C_i^\circ$. Moreover, $C$ has the coproduct property:*

$$Hom(C, X) = Hom(C_1^\circ, X) \times_{Hom(W,X)} Hom(C_2^\circ, X).$$

This has a slightly more involved implication in terms of pre-stable punctured maps.

# Gluing punctured curves: moduli

This gives part of the first claim:

**Theorem (ACGS 2020)**

*... The following is cartesian:*

$$\begin{tikzcd} \widetilde{\mathcal{M}}(X/B,\tau) \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}(X,\tau') \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

# The end

Thank you for your attention