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Tropicalizing Psi Classes

Based on joint work with Renzo Cavalieri and Hannah Markwig
and joint work in progress with Renzo Cavalieri

Motivation: correspondence theorems

Mikhalkin '05: Curves on toric surfaces can be counted using tropical methods.

More precisely: one computes degrees of Severi varieties. For toric Fano surfaces these numbers are Gromov-Witten invariants.

Shustin '05: This can be done with algebro-geometric methods.

Tropical moduli spaces play an important role in controlling the combinatorics.

Question: Can one do intersection theory on them to define tropical Gromov-Witten invariants?

Mikhalkin '06: Outlines the first steps in tropical intersection theory, including intersections with tropical Cartier divisors.

The development of tropical intersection theory

Gathmann-Kerber-Markwig '09: Degree of morphisms of tropical fans.

Allermann-Rau '10: Tropical intersection product on $\mathbb{R}^n$.

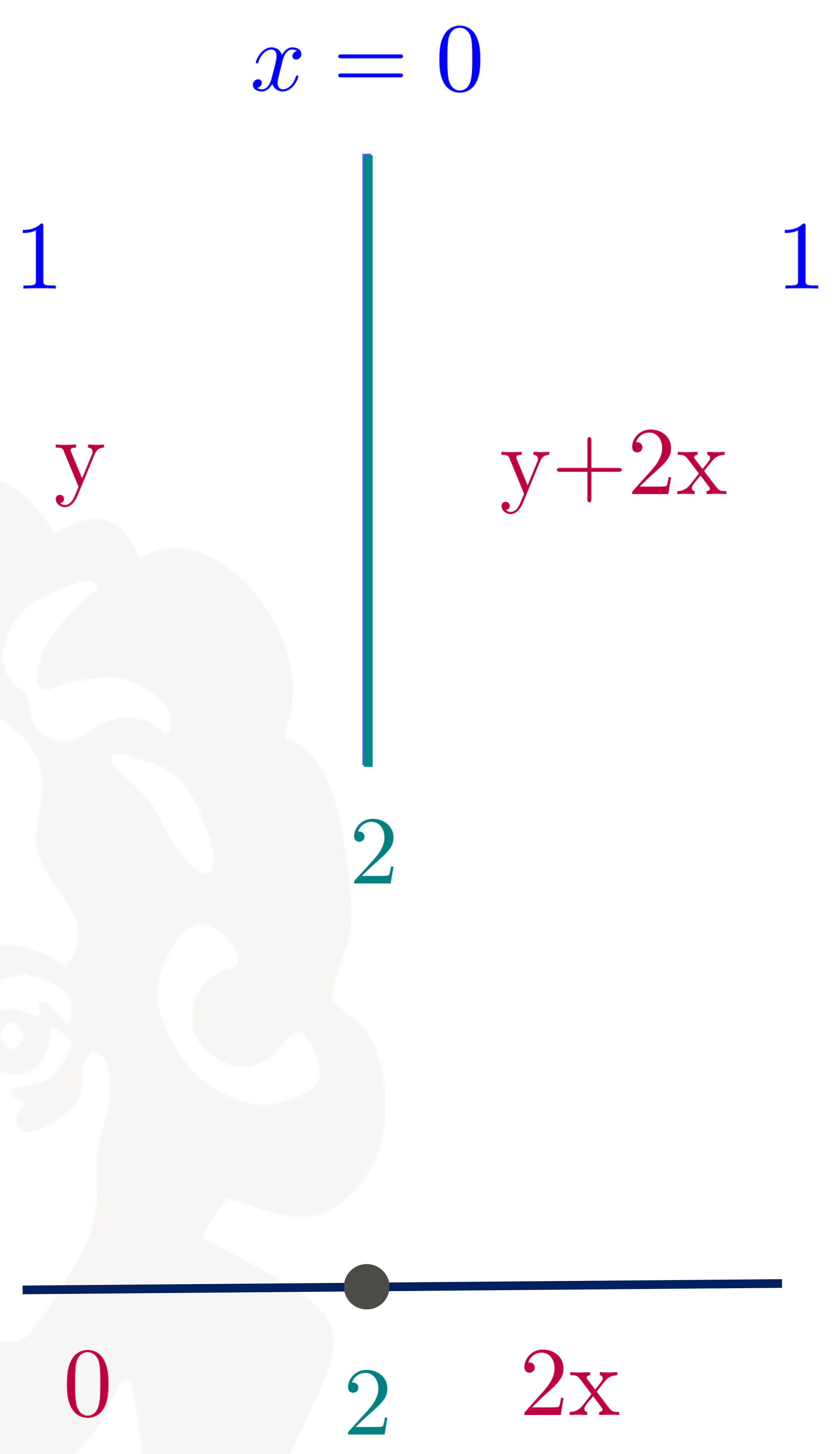
Francois-Rau, Shaw '13: Intersection product on locally matroidal varieties.

Note: $M_{0,n}^{\text{trop}}$ is matroidal.

Francois '13: Tropical cocycles

G-Shokrieh '21 Tropical cycles and cocycles are Poincaré dual on locally matroidal tropical varieties.

Intersecting with a divisor



Given:

1) Tropical k-cycle (balanced weighted fan)

2) Piecewise linear function with integer slopes

Output:

Tropical $(k-1)$-cycle

Psi classes in genus 0

Mikhalkin '07: class ψ_i^{trop} on $M_{0,n}^{\text{trop}}$ represented by locus of tropical curves with i-th mark at 4-valent vertex.

Kerber-Markwig '09: These loci can be intersected. We have

$$\int_{M_{0,n}^{\text{trop}}} \prod_{i=1}^{n} (\psi_i^{\text{trop}})^{k_i} = \int_{\overline{M}_{0,n}} \prod_{i=1}^{n} \psi_i^{k_i} \; .$$

Katz '12: ψ_i^{trop} on $M_{0,n}^{\text{trop}}$ is the tropicalization of ψ_i on $\overline{M}_{0,n}$.

What is special in genus 0?

Speyer-Sturmfels '04: $M_{0,n}^{\mathrm{trop}}$ has an embedding into $\mathbb{R}^N$ induced by an embedding of $M_{0,n}$ into $(\mathbb{C}^*)^N$.

G '18: Given a complete toroidal embedding $X^\circ \subseteq X$ (e.g. $M_{0,n} \subseteq \overline{M}_{0,n}$), the elements of $\Gamma(X^\circ, \mathcal{O}_X^*)/\mathbb{C}^*$ define affine functions on the cone complex Σ_X.

This framework allows to do tropical intersection theory on Σ_X without embedding it.

It also allows to tropicalize. And again one gets $\mathrm{Trop}(\psi_i) = \psi_i^{\mathrm{trop}}$ on $\Sigma_{\overline{M}_{0,n}}$

Applying this machinery, one obtains a correspondence theorem for genus-0 logarithmic descendant Gromov-Witten invariants.

But: $\Gamma(M_{g,n}, \mathcal{O}_{\overline{M}_{g,n}}^*)/\mathbb{C}^* = 0$.

Moduli of tropical curves (w/ Hannah Markwig and Renzo Cavalieri)

Goal: define linear functions on $M_{g,n}^{\text{trop}}$ to do intersection theory with.

Idea: Consider $M_{g,n}^{\text{trop}}$ as a stack on the category of pairs (X, Aff_X) consisting of

- an integral piecewise-linear space X
- a subsheaf Aff_X of the sheaf PL_X of piecewise linear functions with integer slopes.

Francois-Hampe '13: defined a functor for $M_{0,n}^{\text{trop}}$ on category of locally matroidal spaces.

The tropical moduli stack

$\mathcal{M}_{g,n}^{\text{trop}}(B) = \{\text{families of stable tropical curves over B}\}$

$\mathcal{C} \to B$ is a family of tropical curves if:

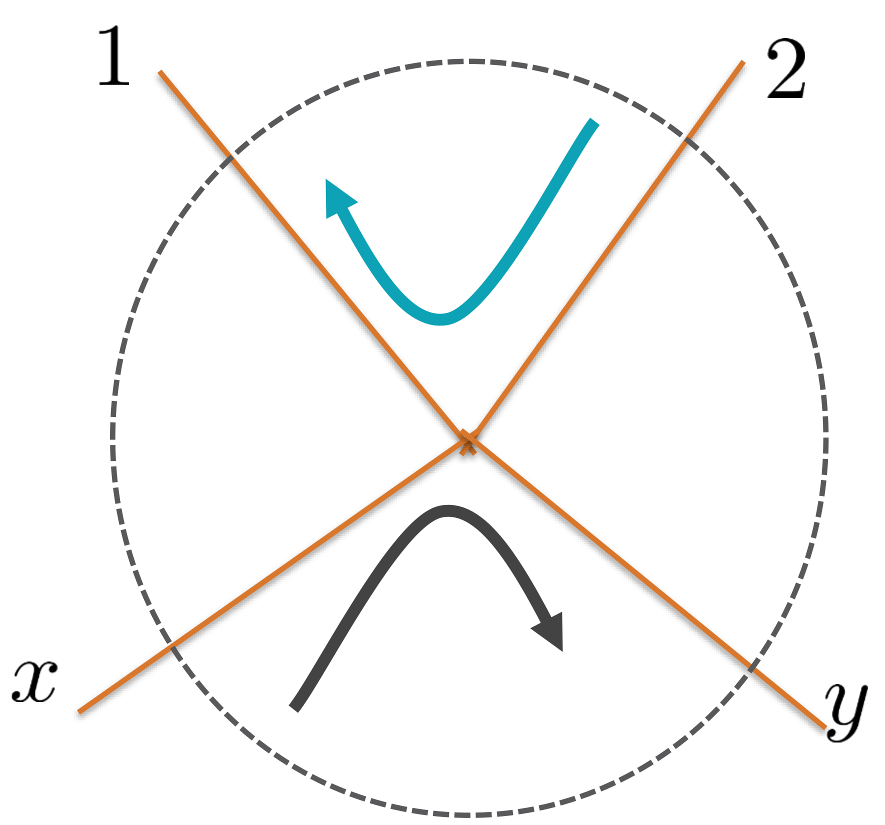
- fibers are smooth tropical curves
- affine functions on $\mathcal{C}$ that are constant on fibers are pull-backs of affine functions from B.

Theorem (Cavalieri-G-Markwig '22): The functor $\mathcal{M}_{0,n}^{\text{trop}}$ is represented by $M_{0,n}^{\text{trop}}$.
For $g > 0$, the stack $\mathcal{M}_{g,n}^{\text{trop}}$ has a dense open substack $\mathcal{M}_{g,n}^{\text{Mf}}$ that has an étale cover that is representable.

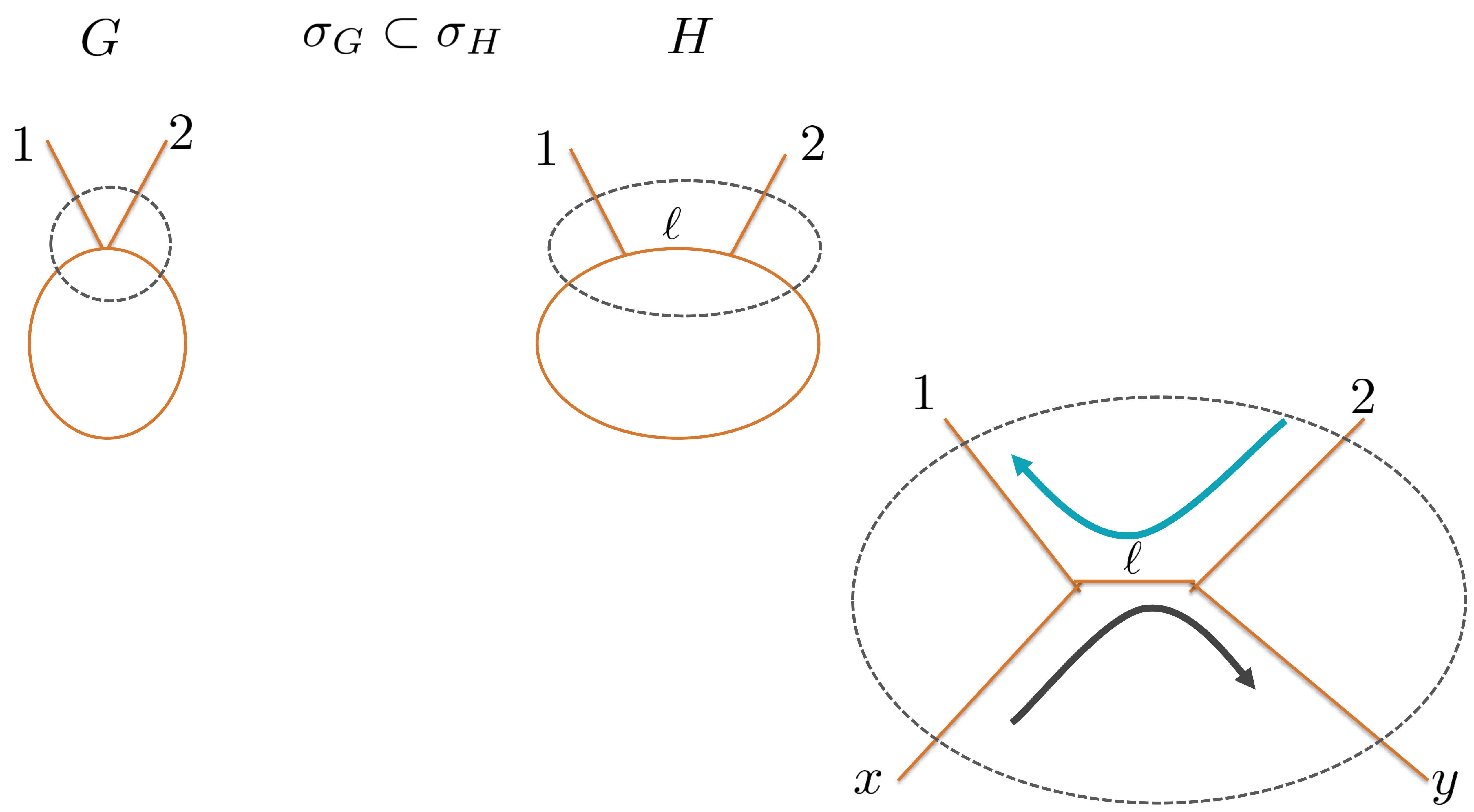
Affine functions on tropical moduli space

Functions are given by tropical cross ratios:

$M_{1,2}^{\text{trop}}$:



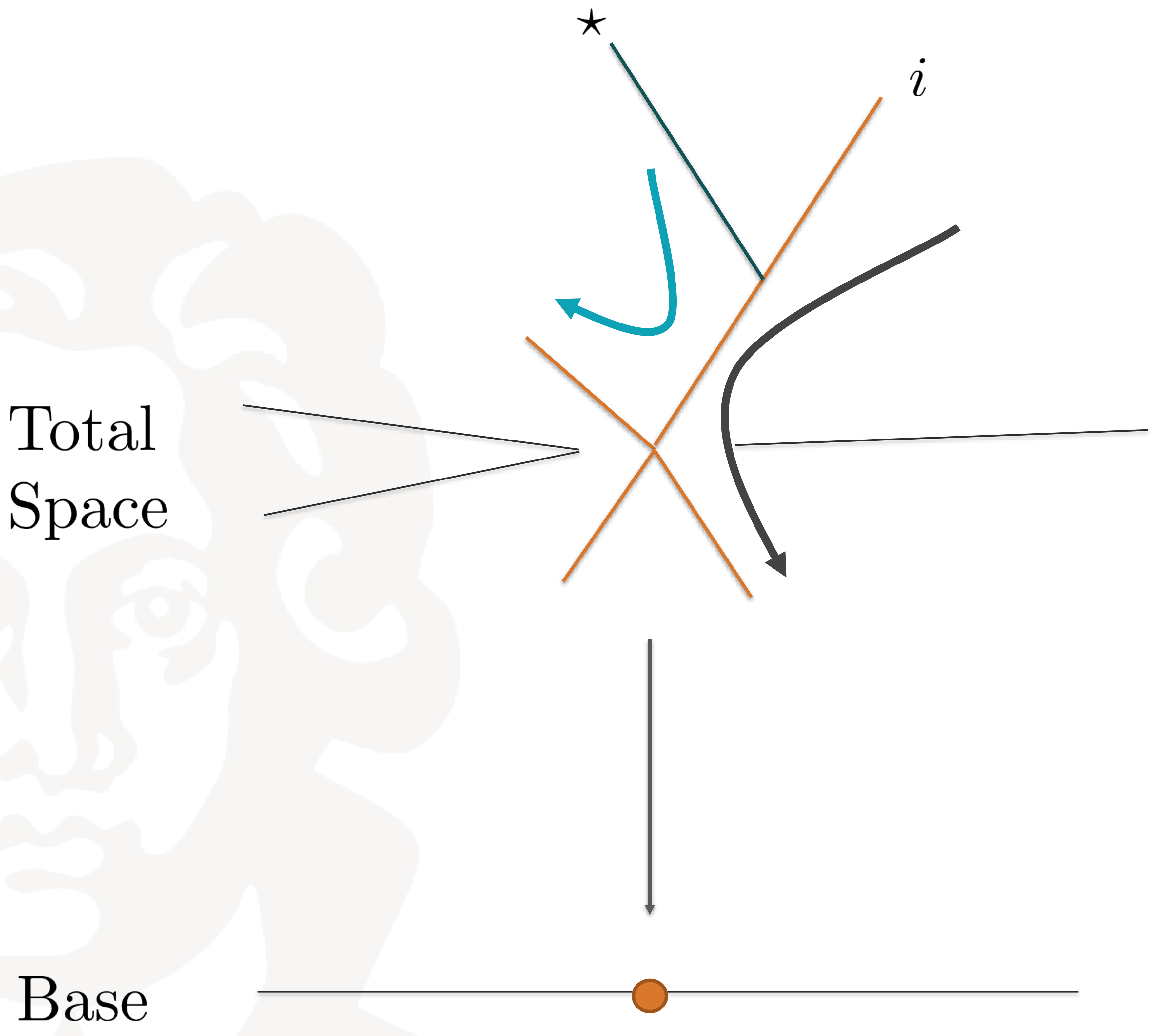
Cross ratio datum



value of cross ratio: $-\ell$

Psi classes in higher genus

The i-th ψ class is the negative self intersection of the i-th section of a family of curves.



If the i-marked leg is adjacent to a genus-0 vertex, there is a cross ratio having slope 1 on that leg.

In favorable situation we can compute the ψ class from the image of the base in $\mathcal{M}_{g,n}^{\mathrm{trop}}$

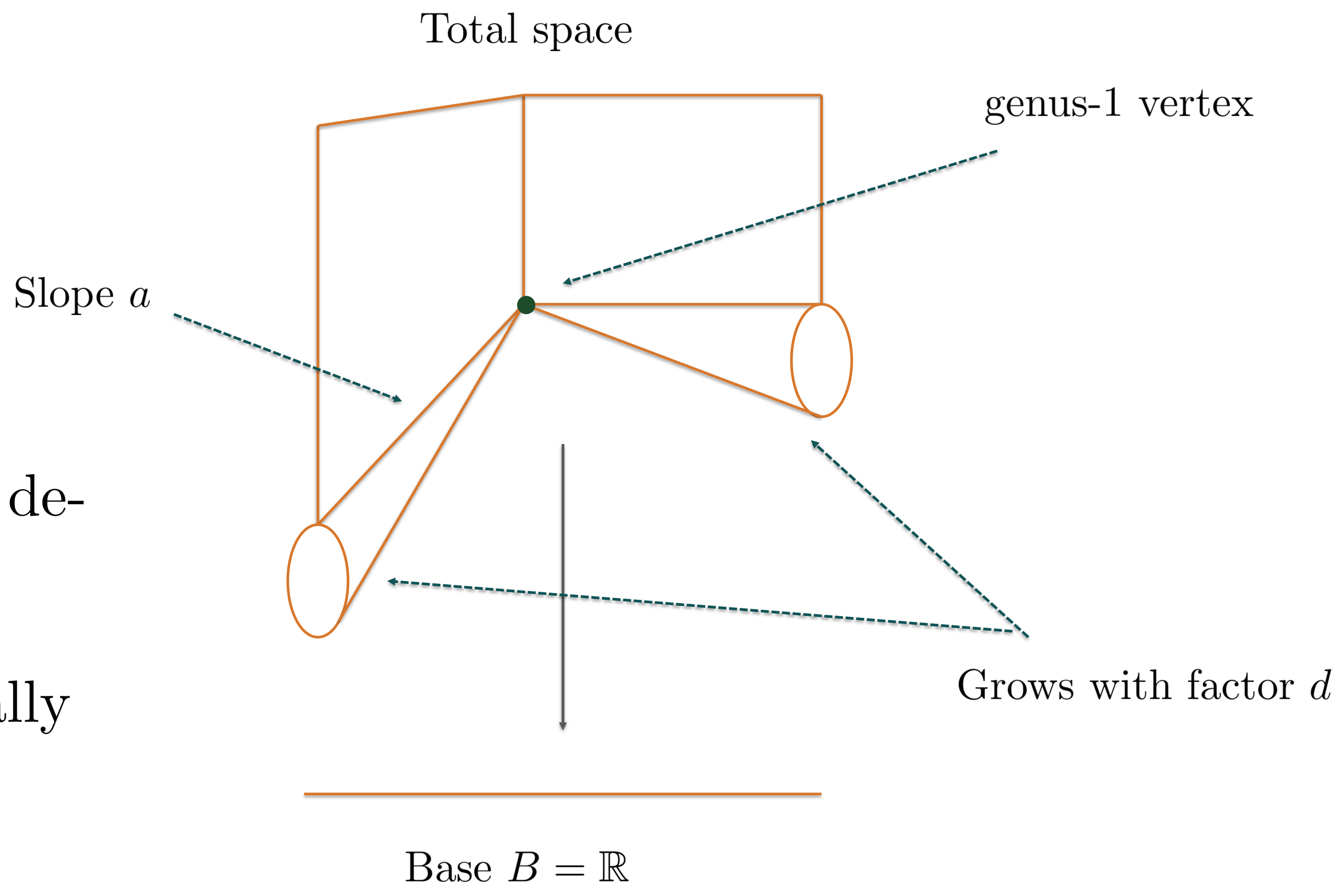
A new realizability problem

$B \to \mathcal{M}_{1,1}^{\text{trop}}$ has degree $2d$

$\int_B \psi_1 = -a$

Degree of ψ_1 is independent of the degree of $B \to \mathcal{M}_{1,1}^{trop}$!

So $\psi_1 = \frac{1}{24}[pt]$ does not hold tropically



However: Get expected answer for families that come from algebro-geometric input

Tropicalizing familes of curves (with Renzo Cavalieri)

Problem: Given a family $\mathcal{C} \xrightarrow{\pi} B$ of stable curves, what is its tropicalization?

Step 1: Assume $\mathcal{C}$ and B are toroidal and π a logarithmic stable curve.
Now take the map of associated cone complexes $\Sigma_\pi : \Sigma_{\mathcal{C}} \to \Sigma_B$

Step 2: Assume Σ_π is a family of tropical curves in the PL-category.

Affine functions on cone complexes of toroidal embeddings

Step 3: Define affine functions on $\Sigma_{\mathcal{C}}$ and Σ_B as in following definition.

Definition: For toroidal X, a piecewise linear function defined in neighborhood of $\sigma \in \Sigma_X$ is affine at σ iff

$$\mathcal{O}_X(\phi)|_{V(\sigma)} \cong \mathcal{O}_{V(\sigma)}$$

This generalizes previous definition:

Families almost tropicalize to families

Theorem (Cavalieri-G): Let $\Sigma_\pi : \Sigma_{\mathcal{C}} \to \Sigma_B$ be obtained as before. Let $x \in \Sigma_{\mathcal{C}}$ be a point of genus 0. Then we have

1) All affine functions at x are harmonic on fibers.

2) All affine functions at x that are constant on fibers are pull-backs of affine functions on Σ_B

3) All cross ratios defined at x and $\Sigma_\pi(x)$ are linear.

In particular, $(\Sigma_{\mathcal{C}})_{\Sigma_\pi(x)}$ is smooth at x if x is not on an edge between higher-genus vertices.

Tropicalizing psi classes

We can tropicalize line bundles by tropicalizing their total spaces

But: The result might not be a tropical line bundle

If it is, we can also tropicalize the first Chern class.

Theorem (Cavalieri-G): Let $\Sigma_\pi : \Sigma_{\mathcal{C}} \to \Sigma_B$ be the tropicalization of a family of n-marked stable curves.

For $1 \leq i \leq n$, if ψ_i^{trop} is defined, then ψ_i is tropicalizable and

$$\mathrm{Trop}(\psi_i) = \psi_i^{\mathrm{trop}} \ .$$

Thank you!