Global charts for moduli spaces of stable maps

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Outline

1. Gromov-Witten invariants in symplectic topology
2. Global charts
3. Applications

Conventions

We fix

- a closed symplectic manifold (X, ω)
- a homology class $\beta \in H_2(X; \mathbb{Z})$
- an ω-tame almost complex structure J

Remarks

- an almost complex structure is an auxiliary datum
- the space of ω-tame almost complex structures is contractible

J-holomorphic curves

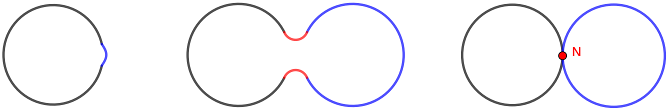
We are interested in maps $u\colon C \to X$ satisfying the elliptic PDE

$$\overline{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j).$$

The moduli space of these solutions enjoys certain analytical advantages. In particular, we have compactness modulo bubbling of J-holomorphic maps with uniformly bounded energy

$$E(u) = \frac{1}{2}\int_C u^*\omega.$$

Figure: Bubbling off



GW invariants

Question: How many J-holomorphic curves exist?

$\rightsquigarrow$ *Gromov Nonsqueezing:* If $B_r^{2n} \hookrightarrow B_R^2 \times \mathbb{C}^{n-1}$ symplectically, then $r \leqslant R$.

- Kill isotropy
- Add perturbations to obtain a manifold.
- Add constraints to reduce the dimension.

In algebraic geometry, the theory of GW invariants is much more developped; in particular, with respect to computational methods.

Stable maps

A *prestable map* is a J-holomorphic map $u\colon C \to X$ defined on a marked nodal curve $(C, \boldsymbol{x})$. It is *stable* if $\mathrm{Aut}(u, C, \boldsymbol{x})$ is finite.

The *moduli space of stable J-holomorphic maps of type* (g, n) is

$$\overline{\mathcal{M}}^J_{g,n}(X, \beta) := \left\{ (u, C, \boldsymbol{x}) : {\scriptstyle u_*[C]=\beta,\ \bar{\partial}_J u = 0, \atop \text{stable with } n \text{ marked points}} \right\} / \sim$$

$\overline{\mathcal{M}}^J_{g,n}(X, \beta)$ is compact and metric, but

- multiply covered curves
- singular domains

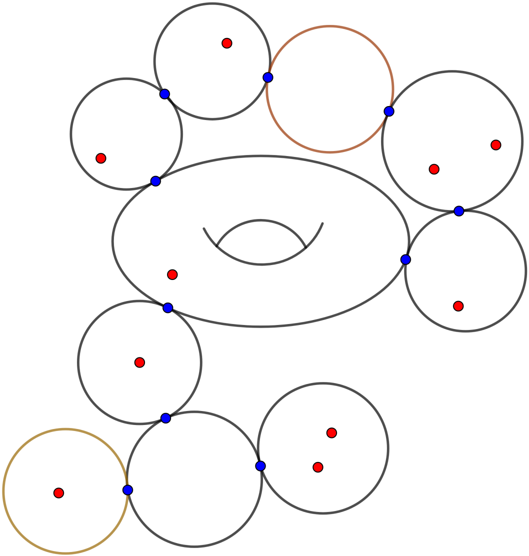


Figure: A nodal surface of genus $g = 2$ with $n = 10$ marked points

Virtual fundamental class

A suitable fundamental class over which to integrate is usually not available.

Solution: Construct a virtual fundamental class instead.

- lives in degree of the 'expected dimension';
- reduces to the ordinary fundamental class if the moduli space is cut out transversely.
- the vfc is often locally modelled on the Euler class

In algebraic geometry there are two definitions (Behrend-Fantechi, Li-Tian) which are known to agree.

Frameworks in symplectic geometry

- *Pseudocycles:* Ruan-Tian,...
- *Kuranishi approach:* Fukaya-Oh-Ohta-Ono, Li-Tian, Joyce, McDuff-Wehrheim, Pardon,...
- *Obstruction bundle approach:* Ruan, Liu-Tian, Siebert,...
- *Polyfolds:* Hofer-Wysocki-Zehnder, Wehrheim,...

Global charts

A *global chart* for a moduli space $\mathcal{M}$ is a tuple $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ where

- G compact Lie group,
- $\mathcal{T}$ a topological manifold, the *thickening*, with an almost free G-action,
- $\mathcal{E} \to \mathcal{T}$ a vector bundle, the *obstruction bundle*, with a fibrewise linear lift of the G-action,
- $\mathfrak{s}\colon \mathcal{T} \to \mathcal{E}$ an equivariant section such that

$$\mathfrak{s}^{-1}(0)/G \cong \mathcal{M}.$$

We call the chart *smooth* if $\mathcal{T}$ and $\mathcal{E}$ are smooth.

The *virtual fundamental class* is the composition

$$\check{H}^d(\mathcal{M};\mathbb{Q}) \xrightarrow{\simeq} H^G_{\mathrm{rank}(\mathcal{E})}(\mathcal{T},\mathcal{T}\backslash\mathfrak{s}^{-1}(0);\mathbb{Q}) \xrightarrow{s^*\tau\cap -} H^G_0(\mathcal{T};\mathbb{Q}) \to H^G_0(*;\mathbb{Q})$$

where

$$d = \dim(\mathcal{T}) - \dim(G) - \mathrm{rank}(\mathcal{E})$$

is the *virtual dimension* of $\mathcal{M}$.

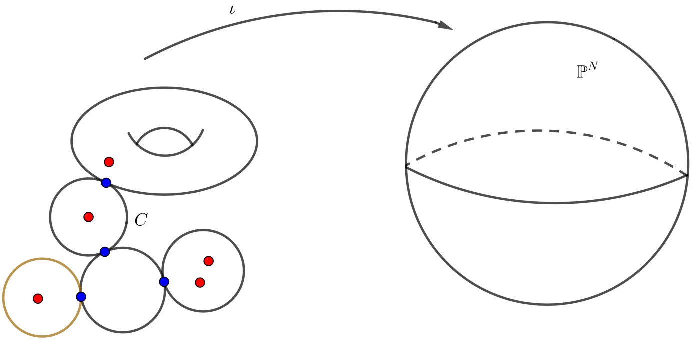
Main result

Theorem (H.-Swaminathan)

- *Given any* $g, n \geqslant 0$ *the moduli space* $\overline{\mathcal{M}}^{J}_{g,n}(X, \beta)$ *admits a global chart.*
- *(in progress) Its equivalence class is independent of the choices made during the construction and of the choice of* J*.*

Framings

Idea (McLean): Stabilise via framings instead of adding marked points.



Framed stable maps

A **framed smooth stable map** is a tuple $(C, \boldsymbol{x}, u, \iota)$ where $(C, \boldsymbol{x}, u)$ is a smooth stable map to X and $\iota \colon C \to \mathbb{P}^N$ is a nondegenerate holomorphic embedding.

We denote by $\overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N, m)$ the space of stable nondegenerate regular embeddings of degree m.

N and m depend on certain choices and will be determined later.

How to obtain a framing

Given Hermitian line bundle $O_X(1) \to X$ with Hermitian connection ∇ such that

$$F^{\nabla} = -2\pi i \Omega$$

with Ω a symplectic form taming J. Define

$$\mathfrak{L}_u := \omega_C(x_1 + \cdots + x_n) \otimes u^* \mathcal{O}_X(1)^{\otimes 3}$$

for $u \colon (C, \boldsymbol{x}) \to X$ a stable smooth map of type (g, n). Set

$$m := \deg(\mathfrak{L}_u^{\otimes p}) \qquad\qquad N := m - g$$

for $p \gg 0$ depending on g, n, and $\langle [\Omega], \beta \rangle$.

The **thickening** $\mathcal{T}$ consists of tuples $((C, \boldsymbol{x}, u, \iota), \alpha, \eta)$ where

- $(C, \boldsymbol{x}, u, \iota)$ is a framed smooth stable map with $u_*[C] = \beta$ and ι of degree m,
- $\alpha \in H^1(C, \mathcal{O}_C)$ is such that

$$[\iota^* \mathcal{O}_{\mathbb{P}^N}(1)] = \alpha + p \cdot [\mathfrak{L}_u] \text{ in } \mathrm{Pic}(C),$$

- η is an element of

$$E_{(u,\iota)} := H^0(C, \iota^* T^{*,0,1}_{\mathbb{P}^N} \otimes u^* TX \otimes \iota^* \mathcal{O}_{\mathbb{P}^N}(k)) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))}.$$

such that

$$\bar{\partial}_J u + \langle \eta \rangle \circ d\iota = 0$$

$k \gg 1$ is an integer that is yet to be determined.

Obstruction bundle and section

We let $\mathcal{E} \to \mathcal{T}$ be the vector bundle whose fibre over $((C, \boldsymbol{x}, u, \iota), \alpha, \eta)$ is

$$\mathfrak{su}(N+1) \oplus H^1(C, \mathcal{O}_C) \oplus E_{(u,\iota)}$$

and we let $\mathfrak{s} \colon \mathcal{T} \to \mathcal{E}$ be given by

$$\mathfrak{s}((C, \boldsymbol{x}, u, \iota), \alpha, \eta) = (i \log(\lambda(C, \boldsymbol{x}, u, \iota)), \alpha, \eta).$$

There exists a canonical action by $G := PU(N+1)$ on $\mathcal{T}$ and $\mathcal{E}$ with respect to which $\mathfrak{s}$ is equivariant.

Notes

- $\eta \in E_{(u,\iota)}$ is the perturbation term which will ensure transversality;
- α chooses the 'correct' line bundle for the framing;
- λ is a G-equivariant map to PGL/G.
- $i\log\colon PGL/G \to \mathfrak{su}(N+1)$ is given by the polar decomposition.

$$\mathfrak{s}((C, \boldsymbol{x}, u, \iota), \alpha, \eta) = (i\log(\lambda(C, \boldsymbol{x}, u, \iota)), \alpha, \eta).$$

If $g = 0$, then $H^1(C, \mathcal{O}_C)$ vanishes, so α is not necessary.

Transversality - choice of k

Theorem (H.-Swaminathan)

There exists a positive integer k_0 such that for any $(C, \boldsymbol{x}, u)$ in $\overline{\mathcal{M}}^J_{g,n}(X, A)$, any basis $\mathcal{F}$ of $H^0(C, \mathfrak{L}_u^{\otimes p})$ with

$$\lambda(C, \boldsymbol{x}, u, \iota_{\mathcal{F}}) = [\mathrm{Id}],$$

and any $k \geqslant k_0$ we have

$$H^1(C, \iota_{\mathcal{F}}^*(T^{*0,1}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k)) \otimes u^* T_X) = 0$$

and $D(\bar{\partial}_J)_u + \langle \cdot \rangle \circ d\iota_{\mathcal{F}}$ surjects onto $\Omega^{0,1}(C, u^* T_X)$.

Proof

- For $k \gg 1$ (depending only on N and m) the restriction map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(C, \mathcal{O}_C(k))$$

  is surjective for any embedded nondegenerate nodal curve of degree m.

- For $k \gg 1$ we have

$$H^1(C, \iota^*_{C,\mathcal{F}}(T^{*0,1}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k)) \otimes u^* T_X) = 0$$

  for a (specific) J-holomorphic stable framed map (u, ι) by Serre vanishing.

- Use Hörmander peak sections coming from the tensor powers of $\mathcal{O}_{\mathbb{P}^N}(1)$.

Gluing

Let $E \to \mathbb{P}^N \times X$ be a suitable complex vector bundle. We can choose an a.c.s. $\tilde{J}$ on E such that for a smooth map

$$v = (\iota, u, s) \colon C \to E$$

we have

$$\overline{\partial}_{\tilde{J}} v = 0$$

if and only if

- $\iota \colon C \to \mathbb{P}^N$ is holomorphic,
- $\nabla^{0,1} s = 0$
- $\overline{\partial}_J u + \langle s \rangle = 0$

By forgetting α, we see that $\mathcal{T}$ determines a subset

$$\overline{\mathcal{M}}^* \subseteq \overline{\mathcal{M}}_{g,n}^{\tilde{J}}(E, \tilde{\beta}).$$

To account for the choice of α we need the following additional structure.

Letting

$$\pi \colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}^*(\mathbb{P}^N, m)$$

be the universal curve, set

$$\mathcal{L} := \Pi^* R^1 \pi_* \mathcal{O}_{\mathcal{C}_{g,n}} \to \overline{\mathcal{M}}^*.$$

where

$$\Pi \colon \overline{\mathcal{M}}_{g,n}^{\tilde{J}}(E, \tilde{\beta}) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)$$

is the natural forgetful map.

Lemma

There exists an open subset $\overline{\mathcal{M}}^{*,\mathrm{reg}} \subseteq \overline{\mathcal{M}}^{*}$ *such that*

1. $\overline{\mathcal{M}}^{*,\mathrm{reg}}$ *is a topological manifold;*
2. $\Pi\colon \overline{\mathcal{M}}^{*,\mathrm{reg}} \to \overline{\mathcal{M}}^{*}_{g,n}(\mathbb{P}^N, m)$ *is a topological submersion;*
3. $\overline{\mathcal{M}}^{J}_{g,n}(X,\beta) \hookrightarrow \overline{\mathcal{M}}^{*,\mathrm{reg}}$.

This determines $\mathcal{T}^{\mathrm{reg}} \subseteq \mathcal{T}$ admitting an embedding $\mathcal{T}^{\mathrm{reg}} \hookrightarrow \mathcal{L}$. The bundle map $\mathcal{L} \to \overline{\mathcal{M}}^{*}$ restricts to an étale map

$$\mathcal{T}^{\mathrm{reg}} \to \overline{\mathcal{M}}^{*,\mathrm{reg}}.$$

$(G, \mathcal{T}^{\mathrm{reg}}, \mathcal{E}|_{\mathcal{T}^{\mathrm{reg}}}, \mathfrak{s}|_{\mathcal{T}^{\mathrm{reg}}})$ *is our desired global chart.*

Independence of choices made in the construction

We made the following choices: $\mathcal{O}_X(1), p$, and k.

Different choices give rise to different charts. Via a double sum construction we can relate them using the following operations.

(Germ) Replace with $(G, U, \mathcal{E}|_U, \mathfrak{s}|_U)$ for $U \subseteq \mathcal{T}$ open and G-invariant.

(Stabilisation) Replace with $(G, W, p^*\mathcal{E} \oplus p^*W, \mathfrak{s} \oplus \delta)$ where $W \xrightarrow{p} \mathcal{T}$ is a G-vector bundle.

(Group extension) Replace with $(G \times G', P, q^*\mathcal{E}, q^*\mathfrak{s})$ where $P \xrightarrow{q} \mathcal{T}$ is a principal G'-bundle with a compatible G-action.

Independence of almost complex structure

(Cobordism) Replace $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ with $(G, \mathcal{T}', \mathcal{E}', \mathfrak{s}')$ if there exists a G-cobordism

$$\begin{tikzcd} \tilde{\mathcal{E}} \arrow[r] & \tilde{\mathcal{T}} \arrow[l, bend left, "\tilde{\mathfrak{s}}"] \end{tikzcd}$$

between the two charts.

The vfc does not change when switching to an equivalent chart.

Applications

Already seen:

- Additive splitting

$$H^\bullet(P_\phi;\mathbb{Z}) \cong H^\bullet(S^2;\mathbb{Z}) \otimes H^\bullet(X;\mathbb{Z})$$

  (Abouzaid-McLean-Smith)

- $\mathbb{Z}$-valued GW invariants (Bai-Xu)

Possible applications:

- product formula
- localisation
- comparison with algebro-geometric GW invariants

Localisation

Lemma (H.-Swaminathan)

The construction of the global chart can be done equivariantly with respect to a nice group action on the target X.

- Construction of vfc requires equivariant Poincaré duality.
- Symplectic toric manifolds have the equivalent of a fan but are less rigid than toric varieties.

Comparison with algebraic GW invariants

- Given a chart with thickening $\mathcal{T}$ construct a cone C in $\mathcal{E}|_{\mathfrak{s}^{-1}(0)}$ using a deformation to the normal cone:

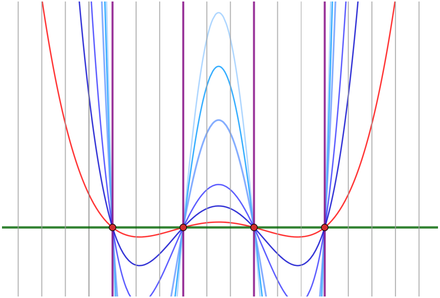


Figure: graph construction

  This cone is possibly badly behaved as the section is not algebraic.

- Construct *local* holomorphic Kuranishi models.
- Compare with algebraic construction (intrinsic normal cone).

Thank you for your attention.