Global charts for moduli spaces of stable maps

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Outline

1. Gromov-Witten invariants in symplectic topology
2. Global charts
3. Applications
Conventions

We fix

- a closed symplectic manifold \((X, \omega)\)
- a homology class \(\beta \in H_2(X; \mathbb{Z})\)
- an \(\omega\)-tame almost complex structure \(J\)

Remarks

- an almost complex structure is an auxiliary datum
- the space of \(\omega\)-tame almost complex structures is contractible
**$J$-holomorphic curves**

We are interested in maps $u: C \to X$ satisfying the elliptic PDE

$$
\bar{\partial}_J u := \frac{1}{2} (du + J \circ du \circ j).
$$

The moduli space of these solutions enjoys certain analytical advantages. In particular, we have compactness modulo bubbling of $J$-holomorphic maps with uniformly bounded energy

$$
E(u) = \frac{1}{2} \int_C u^* \omega.
$$

**Figure:** Bubbling off
GW invariants

**Question:** How many $J$-holomorphic curves exist?

$\rightsquigarrow$ **Gromov Nonsqueezing:** If $B_r^{2n} \hookrightarrow B_R^2 \times \mathbb{C}^{n-1}$ symplectically, then $r \leq R$.

- Kill isotropy
- Add perturbations to obtain a manifold.
- Add constraints to reduce the dimension.

In algebraic geometry, the theory of GW invariants is much more developed; in particular, with respect to computational methods.
Stable maps

A *prestable map* is a $J$-holomorphic map $u: C \to X$ defined on a marked nodal curve $(C, x)$. It is *stable* if $\text{Aut}(u, C, x)$ is finite.

The *moduli space of stable $J$-holomorphic maps of type* $(g, n)$ is

$$\overline{M}^J_{g, n}(X, \beta) := \left\{ (u, C, x) : u_*[C]=\beta, \bar{\partial}_J u=0, \text{stable with } n \text{ marked points} \right\} / \sim$$

$\overline{M}^J_{g, n}(X, \beta)$ is compact and metric, but

- multiply covered curves
- singular domains
Figure: A nodal surface of genus $g = 2$ with $n = 10$ marked points
Virtual fundamental class

A suitable fundamental class over which to integrate is usually not available.

*Solution:* Construct a virtual fundamental class instead.

- lives in degree of the ‘expected dimension’;
- reduces to the ordinary fundamental class if the moduli space is cut out transversely.
- the vfc is often locally modelled on the Euler class

In algebraic geometry there are two definitions (Behrend-Fantechi, Li-Tian) which are known to agree.
Frameworks in symplectic geometry

- *Pseudocycles*: Ruan-Tian,…

- *Kuranishi approach*: Fukaya-Oh-Ohta-Ono, Li-Tian, Joyce, McDuff-Wehrheim, Pardon,…

- *Obstruction bundle approach*: Ruan, Liu-Tian, Siebert,…

- *Polyfolds*: Hofer-Wysocki-Zehnder, Wehrheim,…
Global charts

A global chart for a moduli space $\mathcal{M}$ is a tuple $(G, \mathcal{T}, \mathcal{E}, s)$ where

- $G$ compact Lie group,
- $\mathcal{T}$ a topological manifold, the thickening, with an almost free $G$-action,
- $\mathcal{E} \to \mathcal{T}$ a vector bundle, the obstruction bundle, with a fibrewise linear lift of the $G$-action,
- $s: \mathcal{T} \to \mathcal{E}$ an equivariant section such that

$$s^{-1}(0)/G \cong \mathcal{M}.$$ 

We call the chart smooth if $\mathcal{T}$ and $\mathcal{E}$ are smooth.
The \textit{virtual fundamental class} is the composition

\[ \tilde{H}^d(\mathcal{M}; \mathbb{Q}) \xrightarrow{\sim} H^G_{\text{rank}(\mathcal{E})}(\mathcal{T}, \mathcal{T}\backslash s^{-1}(0); \mathbb{Q}) \xrightarrow{s^* \tau \cap -} H^G_0(\mathcal{T}; \mathbb{Q}) \rightarrow H^G_0(*; \mathbb{Q}) \]

where

\[ d = \dim(\mathcal{T}) - \dim(G) - \text{rank}(\mathcal{E}) \]

is the \textit{virtual dimension} of \( \mathcal{M} \).
Main result

Theorem (H.-Swaminathan)

- Given any $g, n \geq 0$ the moduli space $\overline{M}_{g,n}^{J}(X, \beta)$ admits a global chart.

- (in progress) Its equivalence class is independent of the choices made during the construction and of the choice of $J$. 
Framings

Idea (McLean): Stabilise via framings instead of adding marked points.
A **framed smooth stable map** is a tuple \((C, x, u, \iota)\) where \((C, x, u)\) is a smooth stable map to \(X\) and \(\iota: C \to \mathbb{P}^N\) is a nondegenerate holomorphic embedding.

We denote by \(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)\) the space of stable nondegenerate regular embeddings of degree \(m\).

\(N\) and \(m\) depend on certain choices and will be determined later.
How to obtain a framing

Given Hermitian line bundle $\mathcal{O}_X(1) \to X$ with Hermitian connection $\nabla$ such that

$$F^\nabla = -2\pi i \Omega$$

with $\Omega$ a symplectic form taming $J$. Define

$$\mathcal{L}_u := \omega_C(x_1 + \cdots + x_n) \otimes u^* \mathcal{O}_X(1)^\otimes 3$$

for $u: (C, x) \to X$ a stable smooth map of type $(g, n)$. Set

$$m := \deg(\mathcal{L}_u^\otimes p) \quad N := m - g$$

for $p \gg 0$ depending on $g$, $n$, and $\langle [\Omega], \beta \rangle$. 
The **thickening** \( \mathcal{T} \) consists of tuples \(((C, x, u, \iota), \alpha, \eta)\) where

- \((C, x, u, \iota)\) is a framed smooth stable map with \(u_*[C] = \beta\) and \(\iota\) of degree \(m\),
- \(\alpha \in H^1(C, \mathcal{O}_C)\) is such that
  \[
  [\iota^*\mathcal{O}_{\mathbb{P}^N}(1)] = \alpha + p \cdot [\mathcal{L}_u] \text{ in } \text{Pic}(C),
  \]
- \(\eta\) is an element of
  \[
  E_{(u, \iota)} := H^0(C, \iota^*T_{\mathbb{P}^N}^{0,1} \otimes u^*TX \otimes \iota^*\mathcal{O}_{\mathbb{P}^N}(k)) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))}.
  \]

such that

\[
\bar{\partial}_J u + \langle \eta \rangle \circ d\iota = 0
\]

\(k \gg 1\) is an integer that is yet to be determined.
We let $\mathcal{E} \rightarrow \mathcal{T}$ be the vector bundle whose fibre over $((C, x, u, \nu), \alpha, \eta)$ is

$$su(N + 1) \oplus H^1(C, \mathcal{O}_C) \oplus E_{(u, \nu)}$$

and we let $s: \mathcal{T} \rightarrow \mathcal{E}$ be given by

$$s((C, x, u, \nu), \alpha, \eta) = (i \log(\lambda(C, x, u, \nu)), \alpha, \eta).$$

There exists a canonical action by $G := PU(N + 1)$ on $\mathcal{T}$ and $\mathcal{E}$ with respect to which $s$ is equivariant.
Notes

- $\eta \in E_{(u, \nu)}$ is the perturbation term which will ensure transversality.
- $\alpha$ chooses the ‘correct’ line bundle for the framing;
- $\lambda$ is a $G$-equivariant map to $PGL/G$.
- $i \log : PGL/G \rightarrow \mathfrak{su}(N + 1)$ is given by the polar decomposition.

$$s((C, x, u, \nu), \alpha, \eta) = (i \log(\lambda(C, x, u, \nu)), \alpha, \eta).$$

If $g = 0$, then $H^1(C, \mathcal{O}_C)$ vanishes, so $\alpha$ is not necessary.
Transversality - choice of $k$

**Theorem (H.-Swaminathan)**

There exists a positive integer $k_0$ such that for any $(C, x, u)$ in $\overline{M}_{g,n}(X, A)$, any basis $\mathcal{F}$ of $H^0(C, \mathcal{L}_u^\otimes \mathcal{P})$ with

$$\lambda(C, x, u, \iota_{\mathcal{F}}) = [\text{Id}],$$

and any $k \geq k_0$ we have

$$H^1(C, \iota_{\mathcal{F}}^*(T^*_{P^N} \otimes \mathcal{O}_{P^N}(k)) \otimes u^* T_X) = 0$$

and $D(\bar{\partial}_J)_u + \langle \cdot \rangle \circ d\iota_{\mathcal{F}}$ surjects onto $\Omega^{0,1}(C, u^* T_X)$. 
Proof

• For $k \gg 1$ (depending only on $N$ and $m$) the restriction map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(C, \mathcal{O}_C(k))$$

is surjective for any embedded nondegenerate nodal curve of degree $m$.

• For $k \gg 1$ we have

$$H^1(C, \nu^*_C, \mathcal{F}(T^{0,1}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k)) \otimes u^* T_X) = 0$$

for a (specific) $J$-holomorphic stable framed map $(u, \nu)$ by Serre vanishing.

• Use Hörmander peak sections coming from the tensor powers of $\mathcal{O}_{\mathbb{P}^N}(1)$. 


Let $E \to \mathbb{P}^N \times X$ be a suitable complex vector bundle. We can choose an a.c.s. $\tilde{J}$ on $E$ such that for a smooth map

$$v = (\iota, u, s): C \to E$$

we have

$$\bar{\partial} \tilde{j} v = 0$$

if and only if

- $\iota: C \to \mathbb{P}^N$ is holomorphic,
- $\nabla^{0,1}s = 0$
- $\bar{\partial}Ju + \langle s \rangle = 0$
By forgetting \( \alpha \), we see that \( \mathcal{T} \) determines a subset

\[
\overline{\mathcal{M}}^* \subseteq \overline{\mathcal{M}}_{g,n}(E, \tilde{\beta}).
\]

To account for the choice of \( \alpha \) we need the following additional structure.

Letting

\[
\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\tilde{\beta}}(\mathbb{P}^N, m)
\]

be the universal curve, set

\[
\mathcal{L} := \pi^* R^1 \pi_* \mathcal{O}_{\mathcal{C}_{g,n}} \to \overline{\mathcal{M}}^*.
\]

where

\[
\Pi : \overline{\mathcal{M}}_{g,n}(E, \tilde{\beta}) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)
\]

is the natural forgetful map.
Lemma
There exists an open subset $\overline{M}^{*,\text{reg}} \subseteq \overline{M}^*$ such that
1. $\overline{M}^{*,\text{reg}}$ is a topological manifold;
2. $\Pi: \overline{M}^{*,\text{reg}} \to \overline{M}_{g,n}(\mathbb{P}^N, m)$ is a topological submersion;
3. $\overline{M}^J_{g,n}(X, \beta) \hookrightarrow \overline{M}^{*,\text{reg}}$.

This determines $\mathcal{T}^{\text{reg}} \subseteq \mathcal{T}$ admitting an embedding $\mathcal{T}^{\text{reg}} \hookrightarrow \mathcal{L}$.
The bundle map $\mathcal{L} \to \overline{M}^*$ restricts to an étale map

$$\mathcal{T}^{\text{reg}} \to \overline{M}^{*,\text{reg}}.$$

$(G, \mathcal{T}^{\text{reg}}, \mathcal{E}|_{\mathcal{T}^{\text{reg}}}, \mathfrak{s}|_{\mathcal{T}^{\text{reg}}})$ is our desired global chart.
Independence of choices made in the construction

We made the following choices: $\mathcal{O}_X(1)$, $p$, and $k$.

Different choices give rise to different charts. Via a double sum construction we can relate them using the following operations.

\textbf{(Germ)} Replace with $(G, U, \mathcal{E}|_U, s|_U)$ for $U \subseteq \mathcal{T}$ open and $G$-invariant.

\textbf{(Stabilisation)} Replace with $(G, \mathcal{W}, p^*\mathcal{E} \oplus p^*\mathcal{W}, s \oplus \delta)$ where $\mathcal{W} \xrightarrow{p} \mathcal{T}$ is a $G$-vector bundle.

\textbf{(Group extension)} Replace with $(G \times G', P, q^*\mathcal{E}, q^*s)$ where $P \xrightarrow{q} \mathcal{T}$ is a principal $G'$-bundle with a compatible $G$-action.
Independence of almost complex structure

(Cobordism) Replace \((G, \mathcal{T}, \mathcal{E}, \mathfrak{s})\) with \((G, \mathcal{T}', \mathcal{E}', \mathfrak{s}')\) if there exists a \(G\)-cobordism

\[
\tilde{\mathcal{E}} \xrightarrow{\tilde{s}} \tilde{\mathcal{T}}
\]

between the two charts.

The vfc does not change when switching to an equivalent chart.
Applications

Already seen:

- Additive splitting

\[ H^\bullet(P_\phi; \mathbb{Z}) \cong H^\bullet(S^2; \mathbb{Z}) \otimes H^\bullet(X; \mathbb{Z}) \]

(Abouzaid-McLean-Smith)

- \(\mathbb{Z}\)-valued GW invariants (Bai-Xu)

Possible applications:

- product formula

- localisation

- comparison with algebro-geometric GW invariants
Localisation

Lemma (H.-Swaminathan)

*The construction of the global chart can be done equivariantly with respect to a nice group action on the target \(X\).*

- Construction of vfc requires equivariant Poincaré duality.
- Symplectic toric manifolds have the equivalent of a fan but are less rigid than toric varieties.
Comparison with algebraic GW invariants

- Given a chart with thickening $\mathcal{T}$ construct a cone $C$ in $E|_{s^{-1}(0)}$ using a deformation to the normal cone:

![Graph construction](image)

**Figure**: graph construction

This cone is possibly badly behaved as the section is not algebraic.

- Construct *local* holomorphic Kuranishi models.
- Compare with algebraic construction (intrinsic normal cone).
Thank you for your attention.