Tropical Enumeration of Real Curves in Toric Varieties

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- Correspondence theorems between algebraic curves in toric varieties and tropical curves in $\mathbb{R}^n$.
- Toric degenerations of toric varieties.
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- Lifting maximally degenerate curves to log curves.
- Global tropical multiplicities of Nishinou–Siebert.
- Real log curves and their enumeration.
- Tropical Welschinger invariants.
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Tropical Enumerative Geometry In $\mathbb{R}^2$: Mikhalkin

- **An enumerative problem in $\mathbb{CP}^2$:** Fix $d \in \mathbb{Z}_{>0}$, and a generic set of points $\mathbf{Pt}_{3d-1}(\mathbb{CP}^2) := \{p_1, \ldots, p_{3d-1}\}$.
- $N_d = \#\text{Alg. curves, deg.} = d, g = 0, \text{ matching } \mathbf{Pt}_{3d-1}(\mathbb{CP}^2)$.
-

d	1	2	3	4	5	6	...
N_d	1	1	12	640	84000	$26 \cdot 10^6$	...

Theorem (Mikhalkin)

$\mathcal{M}^{\text{trop}} = \text{Trop. curves in } \mathbb{R}^2, \text{ deg.} = d, g = 0, \text{ matching } \mathbf{Pt}_{3d-1}(\mathbb{R}^2)$.

Then,

$$N_d = \sum_{\Gamma \in \mathcal{M}^{\text{trop}}} \text{Mult}(\Gamma)$$

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Let X_Σ be a toric variety of $\dim_{\mathbb{C}} = n$. Fix the following data:

- Genus: $g = 0$.
- Degree: Δ, a map $\Delta : M \setminus \{0\} \to \mathbb{N}$ with support contained in the union of rays of Σ. This defines the degree d of a curve in X_Σ.
- $\mathbf{A} = (A_1, \ldots, A_\ell)$: a tuple of affine linear subspaces A_j of $\mathbb{R}^n$ with codimension $d_j + 1$ such that $\sum_{j=1}^{\ell} d_j = (n-3)(1-g) + |\Delta|$.
- $\mathbf{P} = (P_1, \ldots, P_\ell)$: a tuple of real points in the big torus of X.
- Incidences $Z_{\mathbf{A},\mathbf{P}} \in X_\Sigma$: defined by $\mathbf{A}$ and $\mathbf{P}$.

Theorem (Nishinou–Siebert)

Let $\mathcal{M}^{\mathrm{trop}} = \mathrm{Trop.\ curves\ in\ } \mathbb{R}^n,\ \mathrm{deg.} = d, g = 0,\ \mathrm{matching\ } \mathbf{A}$, *and* $\mathcal{M}^{alg} = \{\mathrm{Algebraic\ curves\ in\ } X_\Sigma \mathrm{\ of\ degree\ } d \mathrm{\ matching\ } Z_{\mathbf{A},\mathbf{P}}\}$. *Then,*

$$\#\{\mathcal{M}^{alg}\} = \sum_{\Gamma \in \mathcal{M}^{\mathrm{trop}}} \mathrm{Mult}^{NS}(\Gamma)$$

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Algebraic curves in toric degenerations

Theorem (Nishinou–Siebert)

Every curve $(\phi : C \to X_\Sigma) \in \mathcal{M}^{alg}$ is obtained as a deformation of a ***maximally-degenerate log curve*** *$\phi_0 : C_0 \to X_0$ into the central fiber of a* ***toric degeneration*** *of X_Σ.*

- A *toric degeneration* of X_Σ is a degeneration

$$\pi : \mathcal{X} \longrightarrow \mathbb{A}^1$$

where the central fiber X_0 is a **broken toric variety**, that is, a union of toric varieties glued along toric strata.

- We construct $\pi\colon \mathcal{X} \longrightarrow \mathbb{A}^1$ from the data of $(\Delta, \mathbf{A})$:
- Consider all tropical curves matching $(\Delta, \mathbf{A})$, to obtain a polyhedral decomposition $\mathcal{P}$ of $\mathbb{R}^n$.
- We construct the toric fan $\Sigma_{\mathcal{X}}$ by taking the cones over cells of $\mathcal{P}$ in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$.
- There is a map of fans $\Sigma_{\mathcal{X}} \to \Sigma_{\mathbb{A}^1}$ given by projection onto height.
- At height zero the asymptotic fan prescribes the general fiber.

Example:

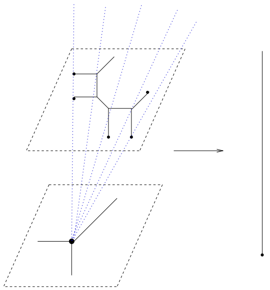

Figure : A degeneration of $\mathbb{P}^2$ into $\mathcal{X}_0 = \coprod_4 \mathbb{P}^2$

The toric degeneration constructed from $\mathcal{P}$

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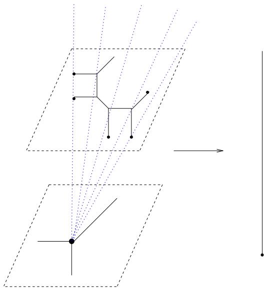

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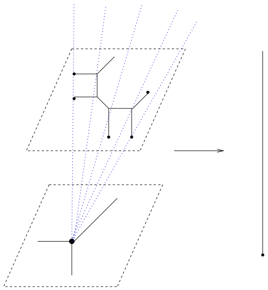

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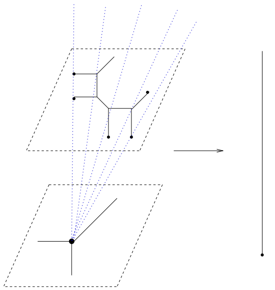

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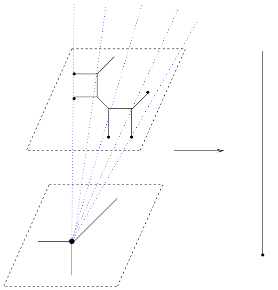

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Maximally degenerate curves in X_0

Definition

Let $\underline{\varphi}_0 \colon C_0 \to X_0 = \prod_v X_v$ be a stable map. We call $\underline{\varphi}_0$ *maximally degenerate* if it is

- **Torically transverse**; that is, the image is disjoint from all toric strata of codimension greater than 1.
- The **kissing condition** is satisfied: If $P \in C_0$ maps to the singular locus of X_0, then C_0 has a node at P, and $\underline{\varphi}_0$ has the same intersection index with the two irreducible components of X_0.
- For every $v \in \mathscr{P}^{[0]}$ the projection $C_0 \times_{X_0} X_v \to X_v$ is a **line**.

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The tropicalization of a maximally degenerate curve $\underline{\varphi}_0 \colon C_0 \to X_0$ is the **dual intersection graph** of $\underline{\varphi}_0$.

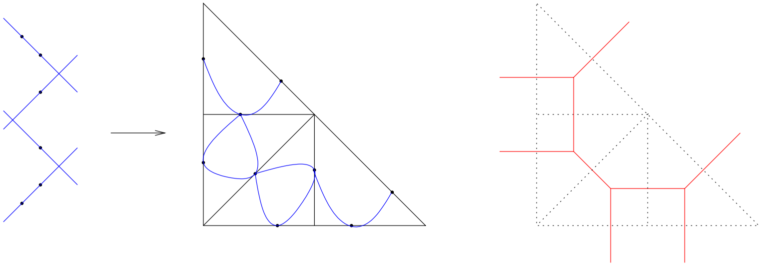

Figure : A maximally degenerate stable map $\underline{\varphi}_0 : C_0 \to X_0 \cong \coprod_4 \mathbb{P}^2$, and the associated tropical curve.

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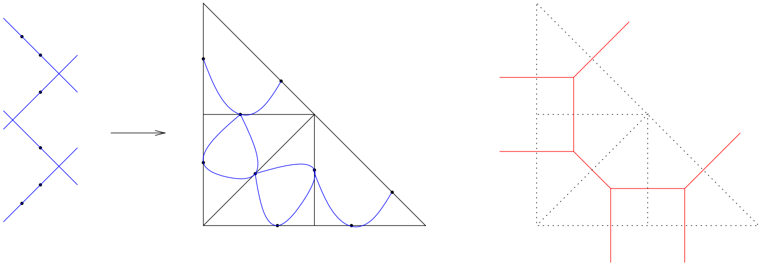

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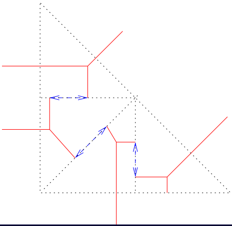
Counts of maximally degenerate curves in X_0

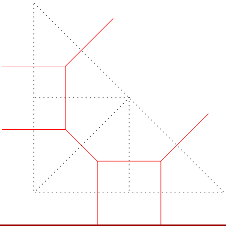
Theorem (Nishinou–Siebert)

The number of maximally degenerate stable maps to X_0 with associated tropical curve $(\Gamma, \mathbf{E}, h)$ equals the lattice index $\mathcal{D}$ of

$$\begin{aligned} \mathrm{Hom}(\Gamma^{[0]}, M) &\to \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} M/\mathbb{Z}u_{(\partial^- E, E)} \times \prod_{j=1}^{\ell} M/(\mathbb{Z}u_{(\partial^- E, E)} + L(A_j)) \\ \phi &\mapsto ((\phi(\partial^+ E) - \phi(\partial^- E))_E, (\phi(\partial^- E))_j). \end{aligned}$$

where $\{\partial^+ E, \partial^- E\}$ is the set of vertices adjacent to E.





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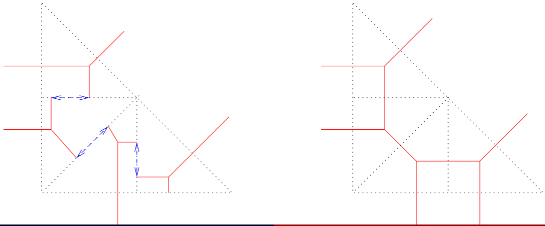

Log Structures

Definition

A *log structure* on a scheme X is a sheaf of monoids $\mathcal{M}$ on X together with a homomorphism $\alpha : \mathcal{M} \longrightarrow (\mathcal{O}_X, \cdot)$ which induces an isomorphism

$$\alpha|_{\alpha^{-1}(\mathcal{O}_X^\times)} : \alpha^{-1}(\mathcal{O}_X^\times) \longrightarrow \mathcal{O}_X^\times$$

Example (The divisorial log structure)

Let $D \subset X$ be a divisor, and $j : X \setminus D \to X$. Define $\mathcal{M}_{(X,D)} := j_*(\mathcal{O}_{X\setminus D}^\times) \cap \mathcal{O}_X$, and $\alpha_X : \mathcal{M}_{(X,D)} \hookrightarrow \mathcal{O}_X$ to be the inclusion.

Example (The standard log point)

Let $X := \operatorname{Spec}\mathbb{C}$, $\mathcal{M}_X := \mathbb{C}^\times \oplus \mathbb{N}$, and define $\alpha_X : \mathcal{M}_X \to \mathbb{C}$ as follows:

$$\alpha_X(x,n) := \begin{cases} x & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

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Log maps in X_0

- On the total space $\mathcal{X}$ we consider the divisorial log structure $\mathcal{M}_{(\mathcal{X},X_0)}$.
- Pulling back $\mathcal{M}_{(\mathcal{X},X_0)}$ to X_0 we obtain a log structure $\mathcal{M}_{X_0}$.

Definition

A morphism of log schemes $f : (C_0, \mathcal{M}_{C_0}) \to (X_0, \mathcal{M}_{X_0})$ is called a **log map** if it fits into the following commutative diagram

$$\begin{array}{ccc} (C_0, \mathcal{M}_{C_0}) & \xrightarrow{f} & (X_0, \mathcal{M}_{X_0}) \\ & \searrow^{\pi} & \downarrow \\ & & (\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^{\star}) \end{array}$$

where π is *log smooth.*

- The log structure of log smooth curves $(C_0, \mathcal{M}_{C_0}) \to (\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^{\star})$ is classified by Kato.

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Lifting maximally degenerate curves to log maps in X_0

Theorem (Nishinou–Siebert)

The number of log maps $\varphi : C_0 \to X_0$ with underlying maximally degenerate map $\underline{\varphi} : C_0 \to X_0$ with associated tropicalization $(\Gamma, \mathbf{E}, h)$ is

$$\prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_{\infty}} w(E) \cdot \prod_{i=1}^{\ell} w(E_i)$$

Hence, the tropical multiplicity of Nishinou–Siebert is given by

$$\boxed{\mathrm{Mult}^{NS}(\Gamma) = \mathcal{D} \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_{\infty}} w(E) \cdot \prod_{i=1}^{\ell} w(E_i)}$$

Real maximally degenerate curves in X_0

- A *real structure* on a scheme X over $\mathbb{C}$ is an anti-holomorphic involution $\iota\colon X \to X$ on the set of complex points of X.
- A *real morphism* between complex algebraic varieties with real structures (X, ι_X) and (Y, ι_Y) is a morphism $f : X \to Y$ such that $\iota_Y \circ f = f \circ \iota_X$ as morphisms of R-schemes.

Theorem

The number of ***real*** *maximally degenerate stable maps to X_0 with associated tropical curve $(\Gamma, \mathbf{E}, h)$ equals the* ***twisted real lattice index*** *$\mathcal{D}^{\mathbb{R}}$ of*

$$\begin{aligned} \mathrm{Hom}(\Gamma^{[0]}, M) &\to \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} M/\mathbb{Z}u_{(\partial^- E, E)} \times \prod_{j=1}^{\ell} M/(\mathbb{Z}u_{(\partial^- E, E)} + L(A_j)) \\ \phi &\mapsto ((\phi(\partial^+ E) - \phi(\partial^- E))_E, (\phi(\partial^- E))_j). \end{aligned}$$

where $\{\partial^+ E, \partial^- E\}$ is the set of vertices adjacent to E.

The twisted real lattice index

Definition

Let $\Psi : M_1 \to M_2$ be an inclusion of lattices of finite index. Let

$$\mathrm{Coker}(\Psi) = \mathbb{Z}/_{(p_1)^{e_1}\mathbb{Z}} \times \ldots \times \mathbb{Z}/_{(p_n)^{e_n}\mathbb{Z}}$$

be the primary decomposition of the free abelian group $\mathrm{Coker}(\Psi)$. We define the *real index* of Ψ as $\mathcal{D}^{\mathbb{R}}_{\Psi} := 2^{\#\{i \mid p_i = 2\}}$.

Lemma

Let $\Psi_{\mathbb{R}} : M_1 \otimes_{\mathbb{Z}} \mathbb{R}^{\times} \to M_2 \otimes_{\mathbb{Z}} \mathbb{R}^{\times}$. *Then,* $\#\{\mathrm{Ker}\Psi_{\mathbb{R}}\} = \mathcal{D}^{\mathbb{R}}_{\Psi}$.

- We introduce a **twisting** which deals with the issue that the map $\Psi_{\mathbb{R}}$ is not necessarily surjective.

From maximally degenerate real curves to real log maps

Definition

Let $(X, \mathcal{M}_X)$ be a log scheme over $\mathbb{C}$ with a real structure $\iota_X \colon X \to X$ on the underlying scheme. A *real structure* on $(X, \mathcal{M}_X)$ (*lifting* ι_X) is an involution

$$\tilde{\iota}_X = (\iota_X, \iota_X^\flat) : (X, \mathcal{M}_X) \longrightarrow (X, \mathcal{M}_X)$$

of log schemes over $\mathbb{R}$ with underlying scheme-theoretic morphism ι_X.

- We define **real log maps** and obtain the following result:

Theorem (A. Bousseau)

The number of real log maps lifting a maximally degenerate curve in X_0, with associated tropical curve $(\Gamma, \mathbf{E}, h)$ equals $\prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} w^{\mathbb{R}}(E) \cdot \prod_{j=1}^{\ell} w^{\mathbb{R}}(E_j)$ where $w^{\mathbb{R}}(E) = 2$ if $w(E)$ is even, and $w^{\mathbb{R}}(E) = 1$ if $w(E)$ is odd.

The higher dimensional real correspondence theorem

- For every $t \in \mathbb{A}^1(\mathbb{R}) \setminus \{0\} \simeq \mathbb{R}^\times$, let $M^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t}$ be the set of genus g **real** stable maps to X_t of degree Δ and matching incidences $\mathbf{Z}_{\mathbf{A},\mathbf{P},t}$. We denote

$$\boxed{N^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t} := \sharp M^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t}}.$$

- Every real log map in $M^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t}$ is obtained by a deformation of a real log map in $M^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),0}$.
- Let $\mathcal{T}_{g,\ell,\Delta}(\mathbf{A})$ be the set of ℓ-marked tropical curves $h \colon \Gamma \to M_{\mathbb{R}}$ of genus g and degree Δ and matching the tropical incidences $\mathbf{A}$.

Theorem (A. Bousseau)

$$N^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t} = \sum_{(\Gamma,\mathbf{E},h)\in\mathcal{T}_{g,\ell,\Delta}(\mathbf{A})} \prod_{E\in\Gamma^{[1]}\setminus\Gamma^{[1]}_{\infty}} w^{\mathbb{R}}(E) \cdot \prod_{j=1}^{\ell} w(E_j) \cdot \mathcal{D}^{\mathbb{R}}.$$

- Note that $N^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t}$ is not an invariant.

Welschinger Invariants

- **A real enumerative problem:** Fix $d \in \mathbb{Z}_{>0}$, and a generic set of points $\mathbf{Pt}_{3d-1}(\mathbb{RP}^2) := \{p_1, \dots, p_{3d-1}\}$.
- $N_d^{\mathbb{R}} = \#\text{Realrational curves , deg.} = d, \text{ matching } \mathbf{Pt}_{3d-1}(\mathbb{RP}^2)$.
- **The Welschinger sign** is $(-1)^s$, where s is the number of solitary (elliptic) nodes.
- Let $N_d^{\mathbb{R}^+}$ denote the number of real rational curves with an even number of solitary node, and $N_d^{\mathbb{R}^-}$ the ones with an odd number of solitary nodes.

Theorem

The number $\mathcal{W}_d^{\mathbb{R}} := N_d^{\mathbb{R}^+} - N_d^{\mathbb{R}^-}$ does not depend on the choice of the point configuration $\mathbf{Pt}_{3d-1}(\mathbb{RP}^2)$.

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Welschinger signs in degenerations

- There are two possible types of nodes for curves in $\mathcal{X}_t$:
  . Nodes of curves in $\mathcal{X}_0$ that are preserved in the deformation.
  . Nodes that are generated during the smoothing.
- Nodes that are preserved in the deformation are captured by considering the associated tropical curve and counting integral points in the interiors of dual triangulation.
- We analyse nodes that are generated during the smoothing using log structures.

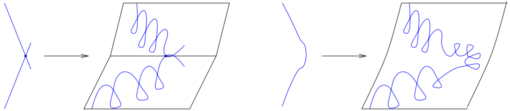

Figure : A maximally degenerate curve and its smoothing

Analysing the new nodes via log structures

- After a blow-up and base change the local equation around a point in double locus of X_0 in the total space is given by $\mathrm{Spec}\mathbb{C}[x, y, \gamma, t]/(xy - t^e)$.
- On a nghd of a nodal point $P \in C_0$ the local equation is of the form $\{z, w, t \mid z \cdot w = t^{e/\mu}\}$

- We analyse the following morphism for $c, c', d \in \mathbb{R}$:

$$\begin{array}{rcl} \mathcal{M}_{\mathcal{X},\varphi_0(P)} & \longrightarrow & \mathcal{M}_{C_0,P} \\ x & \longmapsto & cz^{\mu} \\ y & \longmapsto & c'w^{\mu} \\ \gamma & \longmapsto & d(z+w-1) \\ t & \longmapsto & t \end{array}$$

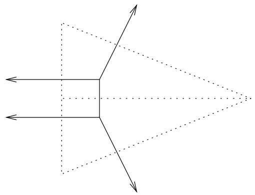

Figure : Tropical image after a shift

New nodes generated in the smoothing

Theorem (Shustin, A. Bousseau)

Let $C_0 \to X_0$ be a maximally degenerate real log curve, and let $P \in C_0$ be a nodal point corresponding to an edge E of the associated tropical curve with $w(E) = \mu$. Then, in the smoothing $\varphi : C \to X$ of φ_0 we have the following types of nodes:

- *If μ is odd, then all the $\mu - 1$ nodes of $\varphi(C)$ are elliptic.*
- *If μ is even, then there are two possibilities:*

. *Either all the $\mu - 1$ created nodes of $\varphi(C)$ are elliptic.*

. *Or one of the nodes of $\varphi(C)$ is hyperbolic and the $\mu - 2$ others form pairs of complex conjugated nodes.*

- Therefore, there is no contribution to the Welschinger sign of the new nodes generated during the smoothing

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Tropical Welschinger signs

- Defined by Mikhalkin, Itenberg–Kharlamov–Shustin.
- For a tropical curve $(h\colon \Gamma \to M_{\mathbb{R}}) \in \mathcal{T}_{g,\ell,\Delta}(\mathbf{A})$, with $V \in \Gamma^{[0]}$ set

$$\mathrm{Mult}_{\mathbb{R}}(V) := (-1)^{I_{\Delta_V}} ,$$

  where I_{Δ_V} is the number of integral points in the interior of Δ_V.
- Define the real tropical multiplicity of a tropical curve as

$$\mathrm{Mult}_{\mathbb{R}}(h) := \begin{cases} 0 & \text{if } \exists\ E \in \Gamma^{[1]} \text{ with } w(E) \text{ even} \\ \prod_V \mathrm{Mult}_{\mathbb{R}}(V) & \text{else.} \end{cases}$$

- Set

$$\mathcal{W}^{\mathbb{R}-trop}_{(g,\Delta,\mathbf{A},\mathbf{P})} := \sum_{(\Gamma,\mathbf{E},h)\in\mathcal{T}_{g,\ell,\Delta}(\mathbf{A})} \mathrm{Mult}_{\mathbb{R}}(h) .$$

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- The *log Welschinger sign* of $\varphi\colon C \to X$ is $\mathcal{W}^{\log}(\varphi) := (-1)^{m(\varphi)}$ where $m(\varphi)$ is the total number of elliptic nodes of $\varphi(C)$.
- The *log Welschinger numbers* are

$$\mathcal{W}^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t} := \sum_{\varphi \in M^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t}} \mathcal{W}^{log}(\varphi)\,.$$

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- *For all $t \in \mathbb{A}^1(\mathbb{R}) \setminus \{0\} \simeq \mathbb{R}^\times$ sufficiently close to 0, we have*

$$\mathcal{W}^{\mathbb{R}-log}_{(g,\Delta,\mathbf{A},\mathbf{P}),t} = \mathcal{W}^{\mathbb{R}-trop}_{(g,\Delta,\mathbf{A},\mathbf{P})}\,.$$

- *For a toric del Pezzo surface $\mathcal{W}^{\mathbb{R}-log}_{(0,\Delta_d,\mathbf{A},\mathbf{P}),t}$ is independent of t and agrees with the Welschinger invariant defined symplectically.*

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