Families of real projective algebraic hypersurfaces with large asymptotic Betti numbers

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Preliminaries

Definition

A real algebraic hypersurface X of degree d in the n-dimensional projective space $\mathbb{P}^n$ is a non-trivial homogeneous element $P \in \mathbb{R}[X_0, \ldots, X_n]/\mathbb{R}^*$ of degree d, where $\mathbb{R}^*$ acts by multiplication.

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One can consider the real part of X,

$$\mathbb{R}X := \{[x_0, \ldots, x_n] \in \mathbb{RP}^n | \; P([x_0, \ldots, x_n]) = 0\},$$

as well as its complex part,

$$\mathbb{C}X := \{[x_0, \ldots, x_n] \in \mathbb{CP}^n | \; P([x_0, \ldots, x_n]) = 0\}.$$

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We always assume our hypersurfaces to be smooth.

Topology of real algebraic hypersurfaces

We focus on the Betti numbers of the real part

$$b_i(\mathbb{R}X) := \dim_{\mathbb{Z}_2} H_i(\mathbb{R}X; \mathbb{Z}_2),$$

with the notation $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

Topology of real algebraic hypersurfaces

For given degree and ambient dimension, what Betti numbers can be achieved?

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$\rightarrow$ Constructions

Some constraints

Smith-Thom inequality

Let X be a real projective algebraic hypersurface. Then

$$\sum_q b_q(\mathbb{R}X) \leq \sum_q b_q(\mathbb{C}X).$$

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Let X be a real projective algebraic hypersurface. Then

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$\rightarrow$ maximal

Some constraints

Finer inequalities can be obtained under stricter hypotheses.

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Renaudineau, Shaw (2018)

Let X be a real projective algebraic hypersurface obtained by patchworking using a primitive triangulation. Then

$$b_q(\mathbb{R}X) \leq \sum_p h^{p,q}(\mathbb{C}X)$$

where $h^{p,q}(\mathbb{C}X)$ is the (p,q)-th Hodge number of $\mathbb{C}X$.

Some constraints

Related to the following conjecture by Viro, itself related to the famous Ragsdale conjecture.

Viro's conjecture

Let X be a simply-connected real projective algebraic surface. Then

$$b_1(\mathbb{R}X) \leq h^{1,1}(\mathbb{C}X).$$

Some constraints

Guiding principle

$$b_q(\mathbb{R}X) \leq \sum_p h^{p,q}(\mathbb{C}X)$$

Asymptotic Betti numbers

We consider families $\{X_d\}_{d\in\mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$, where d is the degree of X_d, and the asymptotic behaviour of $b_q(\mathbb{R}X_d)$ as $d \to \infty$.

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Given $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ two functions, we use the notation $f(d) \overset{n}{\leq} g(d)$ (respectively, $f(d) \overset{n}{=} g(d)$) to signify that $f(d) \leq g(d) + \mathcal{O}(d^{n-1})$ (respectively, that $f(d) = g(d) + \mathcal{O}(d^{n-1})$) as $d \longrightarrow \infty$.

Asymptotic Betti numbers

Smith-Thom inequality gives the asymptotic upper bound

$$\sum_q b_q(\mathbb{R}X_d) \overset{n}{\leq} d^n.$$

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$$\sum_q b_q(\mathbb{R}X_d) \overset{n}{\leq} d^n.$$

$\rightarrow$ asymptotically maximal

We try to make the asymptotic value of $\{b_q(\mathbb{R}X_d)\}_{d\in\mathbb{N}}$ as large as possible.

Asymptotic Betti numbers

For any $n \in \mathbb{N}$ and $q = 0, \ldots, n-1$, there exists $a_q^n \in \mathbb{R}_{>0}$ such that

$$\sum_p h^{p,q}(\mathbb{C}X_d) \overset{n}{=} a_q^n \cdot d^n$$

for any (smooth) real projective algebraic hypersurface X_d in $\mathbb{P}^n$.

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for any (smooth) real projective algebraic hypersurface X_d in $\mathbb{P}^n$.

For all $x \in \mathbb{R}$,

$$a^n_{\lfloor \frac{n-1}{2} + x\sqrt{n} \rfloor} = \sqrt{\frac{6}{\pi(n+1)}} \exp\left(-6x^2\right) + \mathcal{O}\left(n^{-\frac{3}{2}}\right).$$

Asymptotic Betti numbers

Various combinatorial interpretations of a_q^n :

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In terms of the volume of a certain "thick" slice of the cube $[0,1]^n$.

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In terms of the volume of a certain "thick" slice of the cube $[0,1]^n$.

$a_q^n = \frac{1}{n!} E(n,q)$, where $E(n,q)$ is the number of permutations of $\{1,\dots,n\}$ in which exactly q elements are greater than the previous element - the (n,q)-th Euler number.

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Also related to B-splines and lattice paths.

Pre-existing results

Itenberg, Viro (2007)

For any $n \in \mathbb{N}$, there exists a family $\{I_d^n\}_{d \in \mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$ such that

$$b_q(\mathbb{R}I_d^n) \overset{n}{=} a_q^n \cdot d^n$$

for $q = 0, \ldots, n-1$.

Goal

We try to find extreme asymptotic counterexamples to our guiding principle

$$b_q(\mathbb{R}X) \leq \sum_p h^{p,q}(\mathbb{C}X)$$

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$$b_q(\mathbb{R}X) \leq \sum_p h^{p,q}(\mathbb{C}X)$$

$\longrightarrow$ Asymptotic families $\{X_d\}_{d\in\mathbb{N}}$ of hypersurfaces in $\mathbb{P}^n$ such that $b_q(\mathbb{R}X_d)$ is asymptotically much larger than $a_q^n \cdot d^n$.

New asymptotic results

First asymptotic theorem (A.)

For any $n \geq 3$ and any $q = 0, \ldots, n-1$, there exists $b_q^n > a_q^n$ and an asymptotically maximal family $\{Y_d^n\}_{d \in \mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$ such that

$$b_q(\mathbb{R}Y_d^n) \overset{n}{\geq} b_q^n \cdot d^n.$$

New asymptotic results

Second asymptotic theorem, first part (A.)

For any $n \geq 3$, there exists an asymptotically maximal family $\{Y_d^{+,n}\}_{d \in \mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$, as well as $c_q^n \in \mathbb{R}$ (for every $q \in \mathbb{Z}$), such that for $q = 0, \ldots, n-1$, we have

$$b_q(\mathbb{R}Y_d^{+,n}) \overset{n}{=} c_q^n \cdot d^n$$

and such that we have, for all $x \in \mathbb{R}$,

$$c^n_{\left\lfloor \frac{n-1}{2} + x\sqrt{n} \right\rfloor} = \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{n}} \exp\left(-4x^2\right) + o\left(n^{-\frac{1}{2}}\right),$$

where the error term $o\left(n^{-\frac{1}{2}}\right)$ is uniform in x.

New asymptotic results

Second asymptotic theorem, second part (A.)

For any $n \geq 3$, there exists an asymptotically maximal family $\{Y_d^{-,n}\}_{d\in\mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$, as well as $d_q^n \in \mathbb{R}$ (for every $q \in \mathbb{Z}$), such that for $q = 0, \dots, n-1$, we have

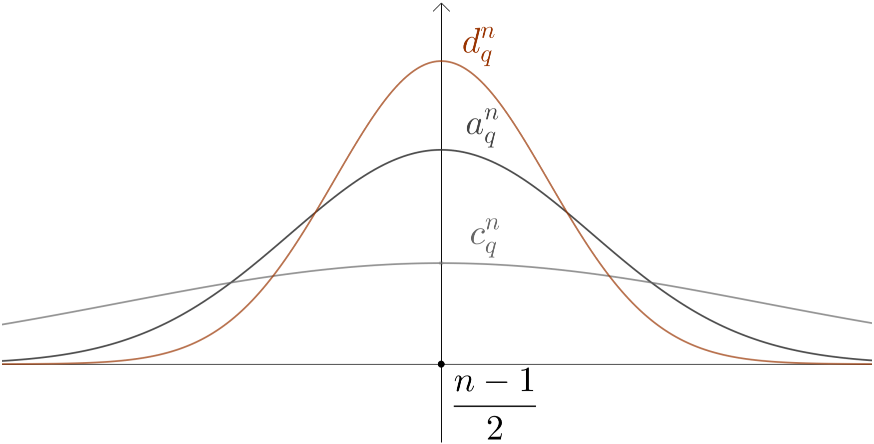
$$b_q(\mathbb{R}Y_d^{-,n}) \overset{n}{=} d_q^n \cdot d^n$$

and such that we have, for all $x \in \mathbb{R}$,

$$d^n_{\left\lfloor \frac{n-1}{2} + x\sqrt{n} \right\rfloor} = \frac{\sqrt{20}}{\sqrt{3\pi}} \frac{1}{\sqrt{n}} \exp\left(\frac{-20x^2}{3}\right) + o\left(n^{-\frac{1}{2}}\right),$$

where the error term $o\left(n^{-\frac{1}{2}}\right)$ is uniform in x.

New asymptotic results

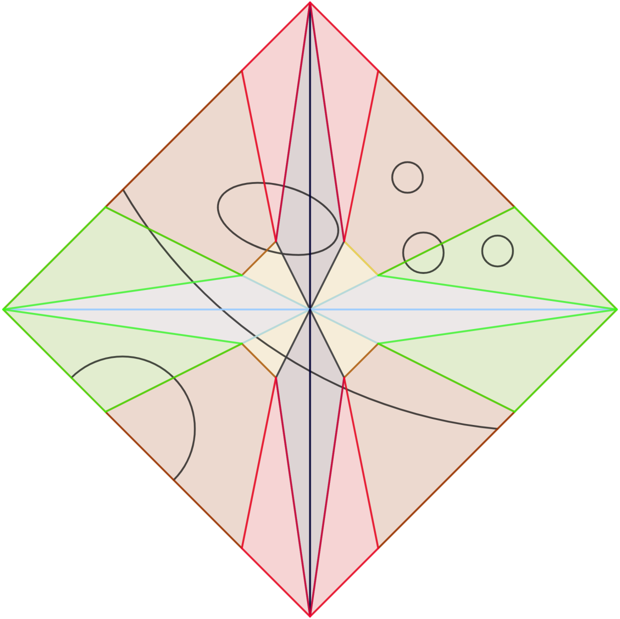


The construction method

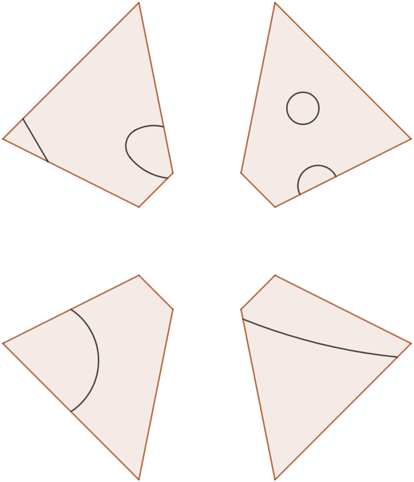
Viro's patchworking method

A method for "gluing" together real algebraic hypersurfaces to get more complicated hypersurfaces.

Viro's patchworking method



Viro's patchworking method



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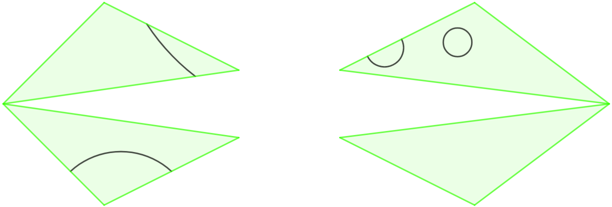
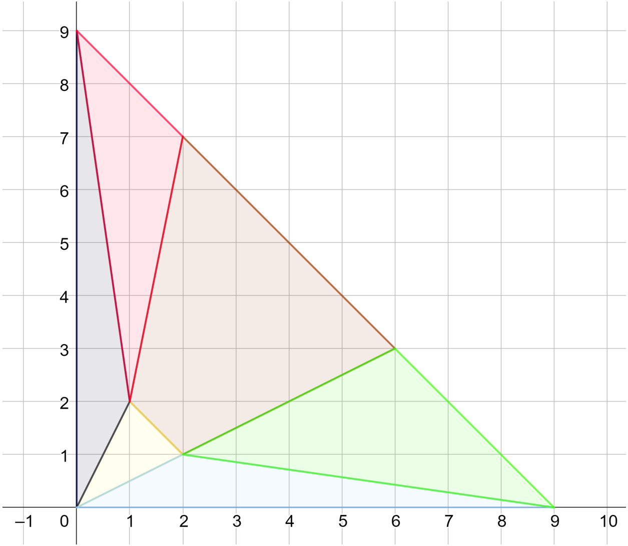


Viro's patchworking method

The Newton polytope $\Delta(P)$ of a polynomial $P(X) = \sum_{\lambda=(\lambda_1,\dots,\lambda_n)\in\Lambda} c_\lambda X_1^{\lambda_1}\dots X_n^{\lambda_n}$, where Λ is a finite subset of $\mathbb{Z}^n$ and $c_\lambda \in \mathbb{R}^*$ for all $\lambda \in \Lambda$, is the convex hull in $\mathbb{R}^n$ of Λ.

Viro's patchworking method



The ingredients

An improvement on an idea by Itenberg and Viro (see [IV]).

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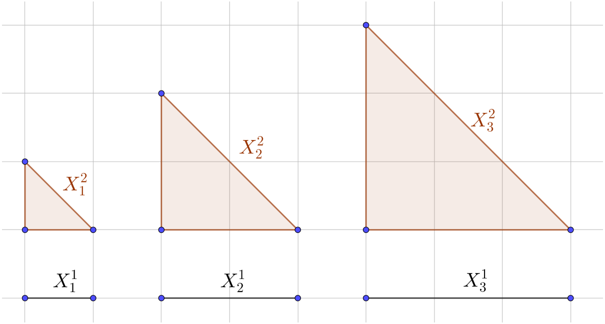
We start with families $\{X_d^k\}_{d\in\mathbb{N}}$ of real projective algebraic hypersurfaces in ambient dimension $k = 1, \dots, n-1$, which we call ingredients.

The ingredients

Pictured below are the Newton polytopes of polynomials representing the hypersurfaces X_d^k.



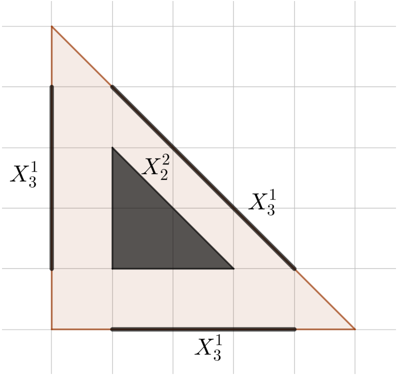
The main idea

We use these ingredients to define real projective algebraic hypersurfaces in ambient dimension $n-1$, which we call $\tilde{Y}_d^{n-1}$, then in dimension n, which we call Y_d^n.

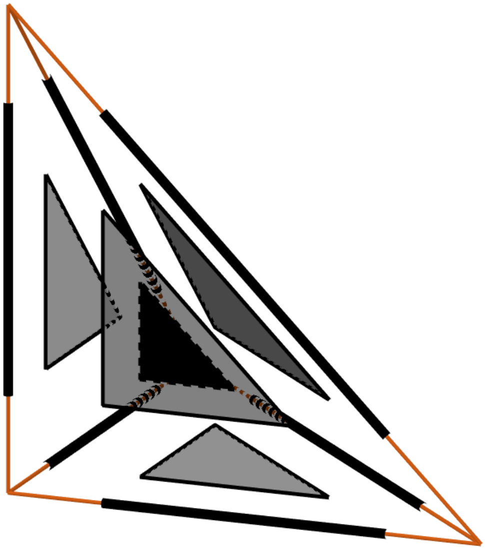
Intermediate constructions $\tilde{Y}_d^{n-1}$

Pictured below is the Newton polytope of a polynomial associated to $\tilde{Y}_5^2$ ($n = 3$ here).

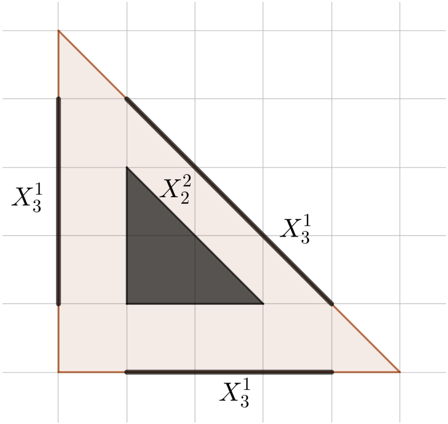
The polynomials corresponding to the faces of dimension $k = 1, 2$ are the same as those defining the ingredients X_d^k (up to a change of variables).



Intermediate constructions $\tilde{Y}_d^{n-1}$

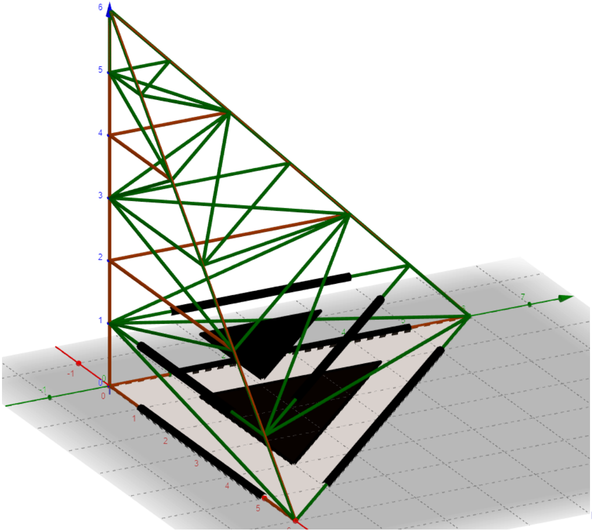


Final constructions Y_d^n



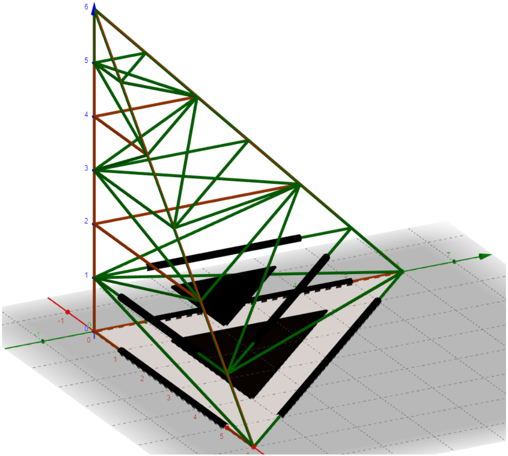
Final constructions Y_d^n

We suspend the previous construction to obtain something in dimension n ($n = 3$ here).



Final constructions Y_d^n

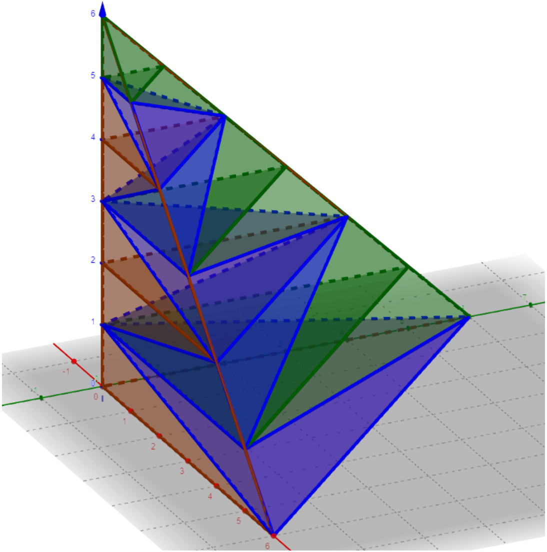
Pictured below is the Newton polytope of a polynomial associated to Y_d^3 ($n = 3$ here).



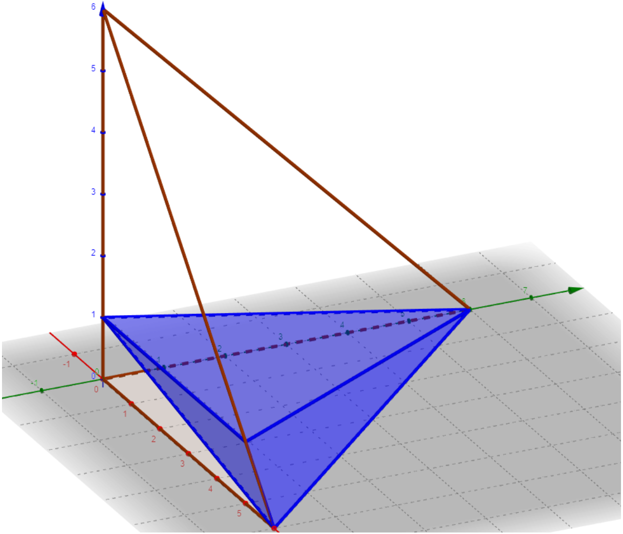
Applying the patchworking method

We choose coefficients such that the Patchworking method applies, and such that the pieces glued together correspond to suspensions and joins of the ingredients.

Applying the patchworking method

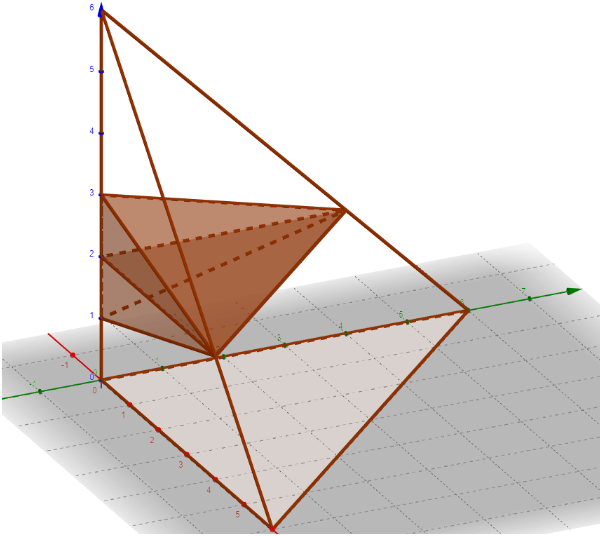


Applying the patchworking method



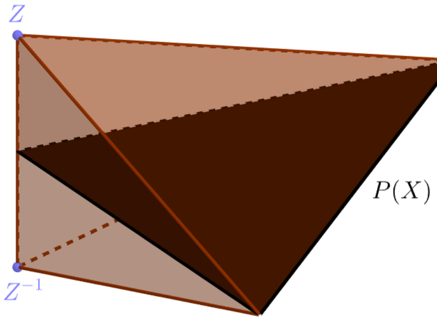
Join

Applying the patchworking method



Suspension

Suspensions

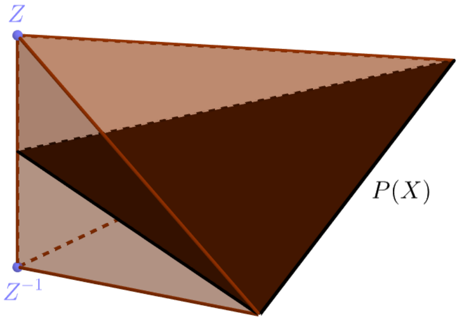


$$P(X) \in \mathbb{R}[X_1^{\pm}, \ldots, X_{n-1}^{\pm}]$$

$$\Downarrow$$

$$P(X) + Z + Z^{-1} \in \mathbb{R}[X_1^{\pm}, \ldots, X_{n-1}^{\pm}, Z^{\pm}]$$

Suspensions



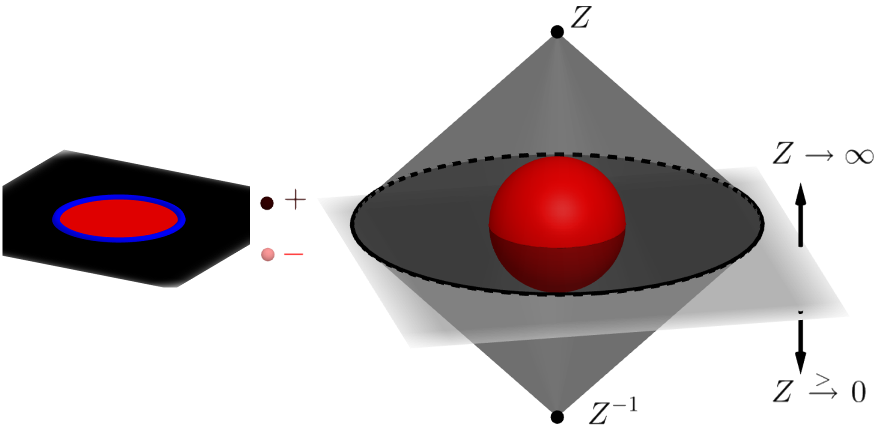
$$P(X) \in \mathbb{R}[X_1^{\pm}, \ldots, X_{n-1}^{\pm}]$$

$$\Downarrow$$

$$P(X) + Z + Z^{-1} \in \mathbb{R}[X_1^{\pm}, \ldots, X_{n-1}^{\pm}, Z^{\pm}]$$

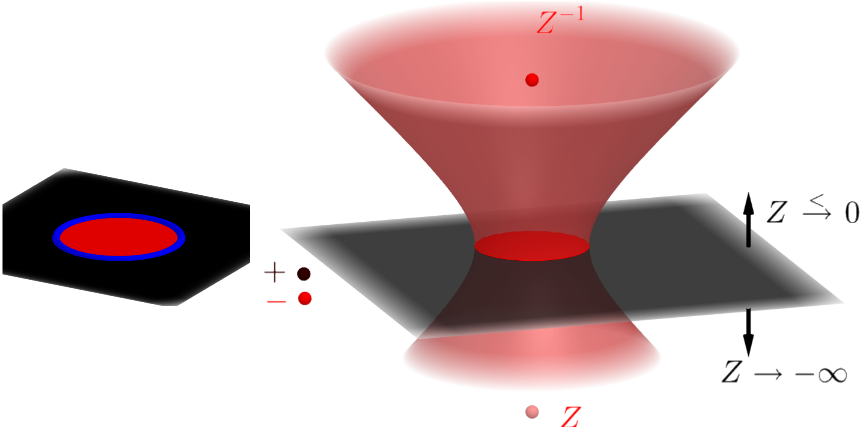
Each k-cycle in $\{P(X) = 0\}$ yields a k-cycle and a $(k+1)$-cycle in $\{P(X) + Z + Z^{-1} = 0\}$.

Suspensions



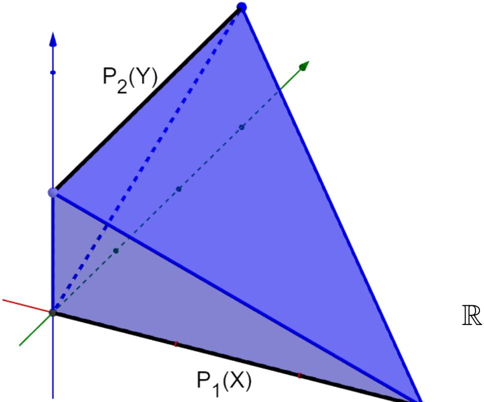
For positive Z.

Suspensions



For negative Z.

Joins

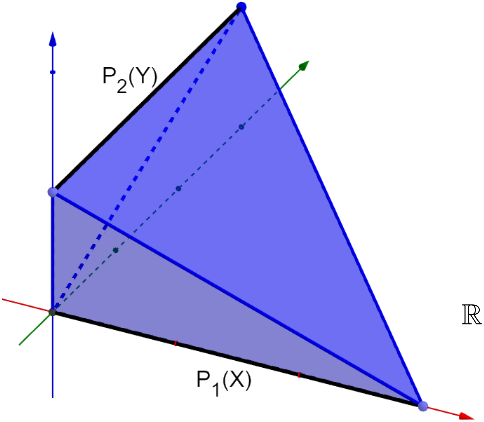


$$P_1(X) \in \mathbb{R}[X_1^{\pm}, \ldots, X_{m_1}^{\pm}],$$
$$P_2(Y) \in \mathbb{R}[Y_1^{\pm}, \ldots, Y_{m_2}^{\pm}]$$

$$\Downarrow$$

$$P_1(X) + Z \cdot P_2(Y) \in$$
$$\mathbb{R}[X_1^{\pm}, \ldots, X_{m_1}^{\pm}, Y_1^{\pm}, \ldots, Y_{m_2}^{\pm}, Z^{\pm}]$$

Joins



$$P_1(X) \in \mathbb{R}[X_1^{\pm}, \dots, X_{m_1}^{\pm}],$$
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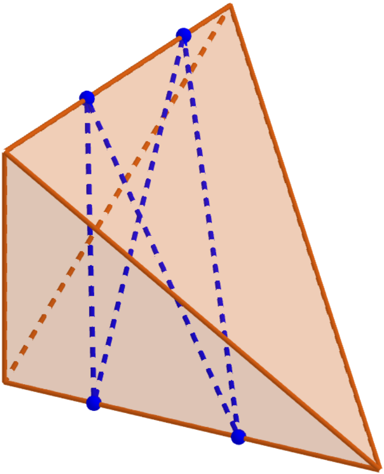
$$P_1(X) + Z \cdot P_2(Y) \in$$
$$\mathbb{R}[X_1^{\pm}, \dots, X_{m_1}^{\pm}, Y_1^{\pm}, \dots, Y_{m_2}^{\pm}, Z^{\pm}]$$

Each k_1-cycle in $\{P_1(X) = 0\}$ and k_2-cycle in $\{P_2(Y) = 0\}$ yields a $(k_1 + k_2 + 1)$-cycle in $\{P_1(X) + Z \cdot P_2(Y) = 0\}$.

Joins



Join of cycles

Counting cycles

Given a family of k-cycles $\{\alpha_t\}_t$ in the real part $\mathbb{R}X$ of some real algebraic projective hypersurface X of dimension $n-1$, we say that the cycles of a family $\{\beta_t\}_t$ of $(n-1-k)$-cycles in the complement of $\mathbb{R}X$ in the ambient space ($(\mathbb{R}^*)^n$, $\mathbb{R}^n$ or $\mathbb{RP}^n$) are axes for the cycles α_t if their linking numbers in the ambient space are well-defined and verify

$$l(\alpha_t, \beta_s) = \delta_{s,t}.$$

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$$I(\alpha_t, \beta_s) = \delta_{s,t}.$$

In particular, this implies that the classes $[\alpha_t]$ are linearly independent in $H_k(\mathbb{R}X)$.

Counting cycles

•We choose cycles $\{\alpha_t\}_t$ in the hypersurfaces $\mathbb{R}X_d^k$.

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•Using a variant of Alexander duality, we find axes $\{\beta_t\}_t$ for the cycles $\{\alpha_t\}_t$ in the complement of $\mathbb{R}X_d^k$.

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•Using a variant of Alexander duality, we find axes $\{\beta_t\}_t$ for the cycles $\{\alpha_t\}_t$ in the complement of $\mathbb{R}X_d^k$.

•Cycles yield new cycles $\{\tilde{\alpha}_t\}$ in the suspensions and joins, hence in $\mathbb{R}Y_d^n$. Similarly, axes yield new axes $\{\tilde{\beta}_t\}$ for those new cycles in the complement of $\mathbb{R}Y_d^n$.

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•Cycles yield new cycles $\{\tilde{\alpha}_t\}$ in the suspensions and joins, hence in $\mathbb{R}Y_d^n$. Similarly, axes yield new axes $\{\tilde{\beta}_t\}$ for those new cycles in the complement of $\mathbb{R}Y_d^n$.

•This shows that the classes of the new cycles are linearly independent in the homology of $\mathbb{R}Y_d^n$.

Cooking Theorem

Cooking Theorem (A.)

Let $n \geq 2$. For $k = 1, \ldots, n-1$, let $\{X_d^k\}_{d \in \mathbb{N}}$ be a family of real projective algebraic hypersurfaces in $\mathbb{P}^k$ such that X_d^k is of degree d. Suppose additionally that for $k = 1, \ldots, n-1$ and $i = 0, \ldots, k-1$,

$$b_i(\mathbb{R}X_d^k) \overset{k}{\geq} x_i^k \cdot d^k$$

for some $x_i^k \in \mathbb{R}_{\geq 0}$. Then there exists a family $\{Y_d^n\}_{d \in \mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$ such that Y_d^n is of degree d and such that for $i = 0, \ldots, n-1$

$$b_i(\mathbb{R}Y_d^n) \overset{n}{\geq} \frac{1}{n}\left(x_i^{n-1} + x_{i-1}^{n-1} + \sum_{k=1}^{n-2}\sum_{j=0}^{i-1} x_j^k \cdot x_{i-1-j}^{n-1-k}\right) \cdot d^n,$$

where x_j^k is set to be 0 for $j \notin \{0, \ldots, k-1\}$.

Cooking Theorem

$$b_i(\mathbb{R}Y_d^n) \overset{n}{\geq} \frac{1}{n}\left(x_i^{n-1} + x_{i-1}^{n-1} + \sum_{k=1}^{n-2}\sum_{j=0}^{i-1} x_j^k \cdot x_{i-1-j}^{n-1-k} \right) \cdot d^n,$$

Suspension Join

Asymptotically large Betti numbers

First asymptotic result

First asymptotic theorem (A.)

For any $n \geq 3$ and any $q = 0, \ldots, n-1$, there exists $b_q^n > a_q^n$ and an asymptotically maximal family $\{Y_d^n\}_{d \in \mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$ such that

$$b_q(\mathbb{R}Y_d^n) \overset{n}{\geq} b_q^n \cdot d^n.$$

Second asymptotic result

Second asymptotic theorem, first part (A.)

For any $n \geq 3$, there exists an asymptotically maximal family $\{Y_d^{+,n}\}_{d\in\mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$, as well as $c_q^n \in \mathbb{R}$ (for every $q \in \mathbb{Z}$), such that for $q = 0, \ldots, n-1$, we have

$$b_q(\mathbb{R}Y_d^{+,n}) \overset{n}{=} c_q^n \cdot d^n$$

and such that we have, for all $x \in \mathbb{R}$,

$$c^n_{\left\lfloor \frac{n-1}{2} + x\sqrt{n} \right\rfloor} = \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{n}} \exp\left(-4x^2\right) + o\left(n^{-\frac{1}{2}}\right),$$

where the error term $o\left(n^{-\frac{1}{2}}\right)$ is uniform in x.

Second asymptotic result

Second asymptotic theorem, second part (A.)

For any $n \geq 3$, there exists an asymptotically maximal family $\{Y_d^{-,n}\}_{d \in \mathbb{N}}$ of real projective algebraic hypersurfaces in $\mathbb{P}^n$, as well as $d_q^n \in \mathbb{R}$ (for every $q \in \mathbb{Z}$), such that for $q = 0, \ldots, n-1$, we have

$$b_q(\mathbb{R}Y_d^{-,n}) \overset{n}{=} d_q^n \cdot d^n$$

and such that we have, for all $x \in \mathbb{R}$,

$$d^n_{\left\lfloor \frac{n-1}{2} + x\sqrt{n} \right\rfloor} = \frac{\sqrt{20}}{\sqrt{3\pi}} \frac{1}{\sqrt{n}} \exp\left(\frac{-20x^2}{3}\right) + o\left(n^{-\frac{1}{2}}\right),$$

where the error term $o\left(n^{-\frac{1}{2}}\right)$ is uniform in x.

A crucial ingredient

In the proofs of both asymptotic theorems, the main ingredient used in the Cooking Theorem is a result by E. Brugallé (from [Bru]), which yields the most extreme known asymptotic values of b_0 and b_1 in ambient dimension 3.

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Brugallé himself made use of a method developped by F. Bihan.

A crucial ingredient

Brugallé (2006)

There exist asymptotically maximal families $\{B_d^+\}_{d\in\mathbb{N}}$ and $\{B_d^-\}_{d\in\mathbb{N}}$ of real projective algebraic surfaces in $\mathbb{P}^3$ such that

$$b_0(\mathbb{R}B_d^+) \overset{3}{=} \frac{3}{8}\cdot d^3 = \left(\frac{1}{6}+\frac{5}{24}\right)\cdot d^3 = \left(a_0^3+\frac{5}{24}\right)\cdot d^3,$$

$$b_1(\mathbb{R}B_d^+) \overset{3}{=} \frac{1}{4}\cdot d^3 = \left(\frac{2}{3}-\frac{5}{12}\right)\cdot d^3 = \left(a_1^3-\frac{5}{12}\right)\cdot d^3,$$

and

$$b_0(\mathbb{R}B_d^-) \overset{3}{=} \frac{1}{8}\cdot d^3 = \left(\frac{1}{6}-\frac{1}{24}\right)\cdot d^3 = \left(a_0^3-\frac{1}{24}\right)\cdot d^3,$$

$$b_1(\mathbb{R}B_d^-) \overset{3}{=} \frac{3}{4}\cdot d^3 = \left(\frac{2}{3}+\frac{1}{12}\right)\cdot d^3 = \left(a_1^3+\frac{1}{12}\right)\cdot d^3.$$

The construction yielding the second asymptotic theorem

We start with the asymptotic constructions $\{I_d^n\}_{d\in\mathbb{N}}$ by Itenberg and Viro (from [IV]) in ambient dimension $n = 1$ and $n = 2$, and the families $\{B_d^{\pm}\}_{d\in\mathbb{N}}$ by Brugallé in ambient dimension $n = 3$.

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We proceed by induction, and use the families of real projective algebraic hypersurfaces available in dimension less than or equal to $n - 1$ as ingredients to obtain a family in dimension n using the Cooking Theorem.

The construction yielding the second asymptotic theorem

$n = 4$:

$\{I_d^1\}_d$, $\{I_d^2\}_d$ and $\{B_d^{\pm}\}_d \xrightarrow{\text{Cooking Thm}} \{Y_d^{\pm,4}\}_d$

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$n = 5$:

$$\{I_d^1\}_d,\ \{I_d^2\}_d,\ \{B_d^{\pm}\}_d,\ \{Y_d^{\pm,4}\}_d \xrightarrow{\text{Cooking Thm}} \{Y_d^{\pm,5}\}_d$$

...

$n + 1$:

$$\{I_d^1\}_d,\ \{I_d^2\}_d,\ \{B_d^{\pm}\}_d,\ \{\{Y_d^{\pm,k}\}_d\}_{k=4}^{n} \xrightarrow{\text{Cooking Thm}} \{Y_d^{\pm,n+1}\}_d$$

Computing the asymptotic Betti numbers

Yields the recursive formula

$$x_i^n = \frac{1}{n}\left(x_i^{n-1} + x_{i-1}^{n-1} + \sum_{k=1}^{n-2}\sum_{j=0}^{i-1} x_j^k \cdot x_{i-1-j}^{n-1-k}\right),$$

where

$$b_i(\mathbb{R}Y_d^{\pm,n}) \overset{n}{=} x_i^n \cdot d^n.$$

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where

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How to compute the asymptotic behavior of the x_i^n?

Computing the asymptotic Betti numbers

After a change of variables, the formula becomes

$$\tilde{x}_i^{n+1} = \frac{1}{n} \sum_{k=1}^{n} \sum_{j \in \mathbb{Z}} \tilde{x}_j^k \cdot \tilde{x}_{i-j}^{n+1-k}.$$

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This can be interpreted in terms of discrete random variables, whose probability density functions are the $\tilde{x}_i^k$.

Computing the asymptotic Betti numbers

$\{\tilde{x}_j^k\}_{j\in\mathbb{Z}} \rightsquigarrow$ random variable $\tilde{X}^k$.

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Allows us to conclude using an analog of the Local Limit Theorem.

What now?

- Generalize to more general ambient toric varieties and to complete intersections.

- Find new low-dimensional asymptotic families (in particular, families obtained using the combinatorial patchworking) to which we could recursively apply the Cooking Theorem.

- Cleverer ways of applying the Cooking Theorem.

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