Lipschitz Geometry of Singularities of Real Surfaces

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Let $(X,0)$ be a germ of a semialgebraic **surface** (a two-dimensional semialgebraic set) in $\mathbb{R}^n$.

Two metrics on X induced from $\mathbb{R}^n$:

Inner metric: $\text{dist}_i(x,y)$ is the length of a shortest path in X from x to y.

Outer metric: $\text{dist}_o(x,y) = |y-x|$.

Obviously, $\text{dist}_o(x,y) \le \text{dist}_i(x,y)$ for all x,y in X.

Definition. $(X,0)$ is **normally embedded** if the two metrics are equivalent:

$\text{dist}_i(x,y) < c\,\text{dist}_o(x,y)$ for some $c>0$ and all x,y in X.

Lipschitz classification problems:

1) **Inner** Lipschitz equivalence: $(X,0) \sim_i (Y,0)$ if there is a homeomorphism $h : (X,0) \to (Y,0)$ bi-Lipschitz with respect to the inner metric.

2) **Outer** Lipschitz equivalence: $(X,0) \sim_o (Y,0)$ if there is a homeomorphism $h : (X,0) \to (Y,0)$ bi-Lipschitz with respect to the outer metric.

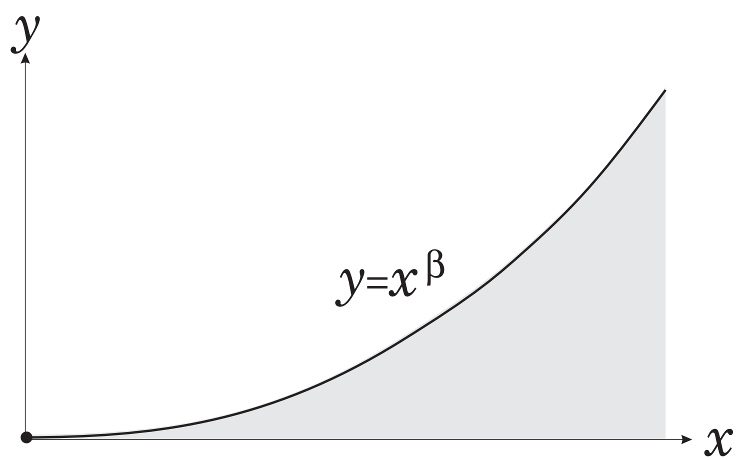
3) **Ambient** Lipschitz equivalence: $(X,0) \sim_a (Y,0)$ if there is an orientation preserving bi-Lipschitz homeomorphism $H : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ such that $H(X) = Y$.

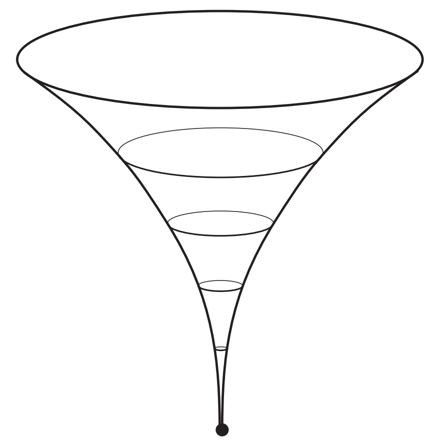
For $\beta \in \mathbb{F}$, $\beta \geq 1$, the **standard β-Hölder triangle** is the set

$$T_\beta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}.$$

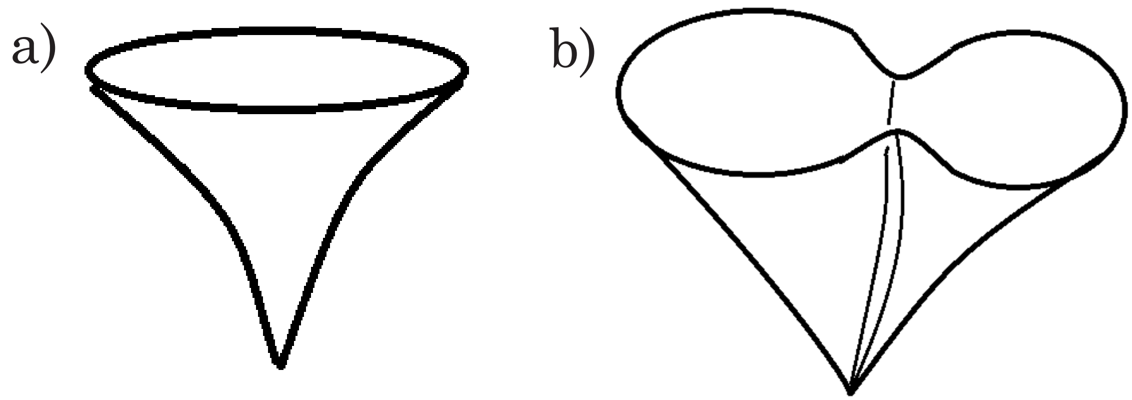
The **standard β-horn** is

$$C_\beta = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2 + y^2 = z^{2\beta}\}.$$





Example. A β-**horn** is a surface of revolution of a β-cusp $x = y^\beta$, $\beta \geq 1$.



a) A β-horn, normally embedded; b) Two β-horns glued along arcs with the tangency exponent $q > \beta$.

These two surfaces are inner Lipschitz equivalent but not outer Lipschitz equivalent.

Horn Theorem. A semialgebraic surface germ $(X, 0)$ with an isolated singular point 0 and connected link $X \cap \{|x| = \epsilon\}$ is **inner** Lipschitz equivalent to a β-horn, for some $\beta \geq 1$.

Corollary. A germ $(X, 0)$ of an irreducible complex curve (considered as a real surface) is **inner** Lipschitz equivalent to $(\mathbb{C}, 0)$.

Normal Embedding Theorem (LB, Mostowski)

Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set. Then there exists another semialgebraic set $\tilde{X} \subset \mathbb{R}^m$, such that

1. $\tilde{X}$ is bi-Lipschitz equivalent to X with respect to the inner metric.

2. $\tilde{X}$ is Normally embedded in $\mathbb{R}^m$.

Theorem (Pham-Tessier, Fernandes, Neumann-Pichon). Two germs of irreducible complex curves in $\mathbb{C}^2$ are **ambient** Lipschitz equivalent if and only if they are **outer** Lipschitz equivalent (have the same Puiseux exponents).

Unique Embedding Theorem

(LB, A.Fernades, Z.Jelonek)

If X_1 and X_2 are two semialgebraic sets of dimension k, embedded to $\mathbb{R}^m$, where $m > 2k + 1$, and if X_1 and X_2 are outer bi-Lipschitz equivalent, then they are ambient bi-Lipschitz equivalent.

Definition. The **tangent cone** of X at 0 is

$$C_0X = Cone\left(\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(X\cap\{|x|=\epsilon\}\Big)\right).$$

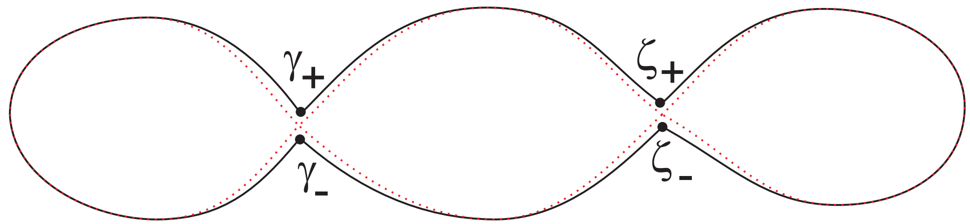
Theorem (Sampaio). Let $(X,0)$ and $(Y,0)$ be two outer (resp., ambient) Lipschitz equivalent germs of semialgebraic sets. Then their tangent cones are outer (resp., ambient) Lipschitz equivalent.

In other words, tangent cones are outer and ambient bi-Lipschitz invariants.

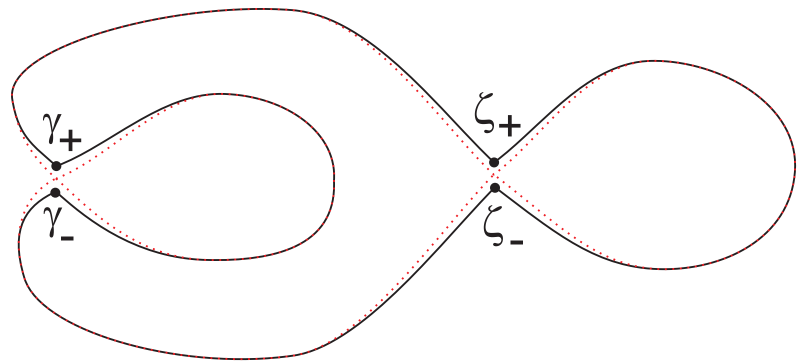
Sampaio's theorem is very useful for the following **Problem:** Does **outer** Lipschitz equivalence imply **ambient** Lipschitz equivalence? **Answer:** No!

There are examples of isotopic and outer Lipschitz equivalent, but **not** ambient Lipschitz equivalent, surface germs in $\mathbb{R}^3$ and $\mathbb{R}^4$ (-, Gabrielov, 2018).

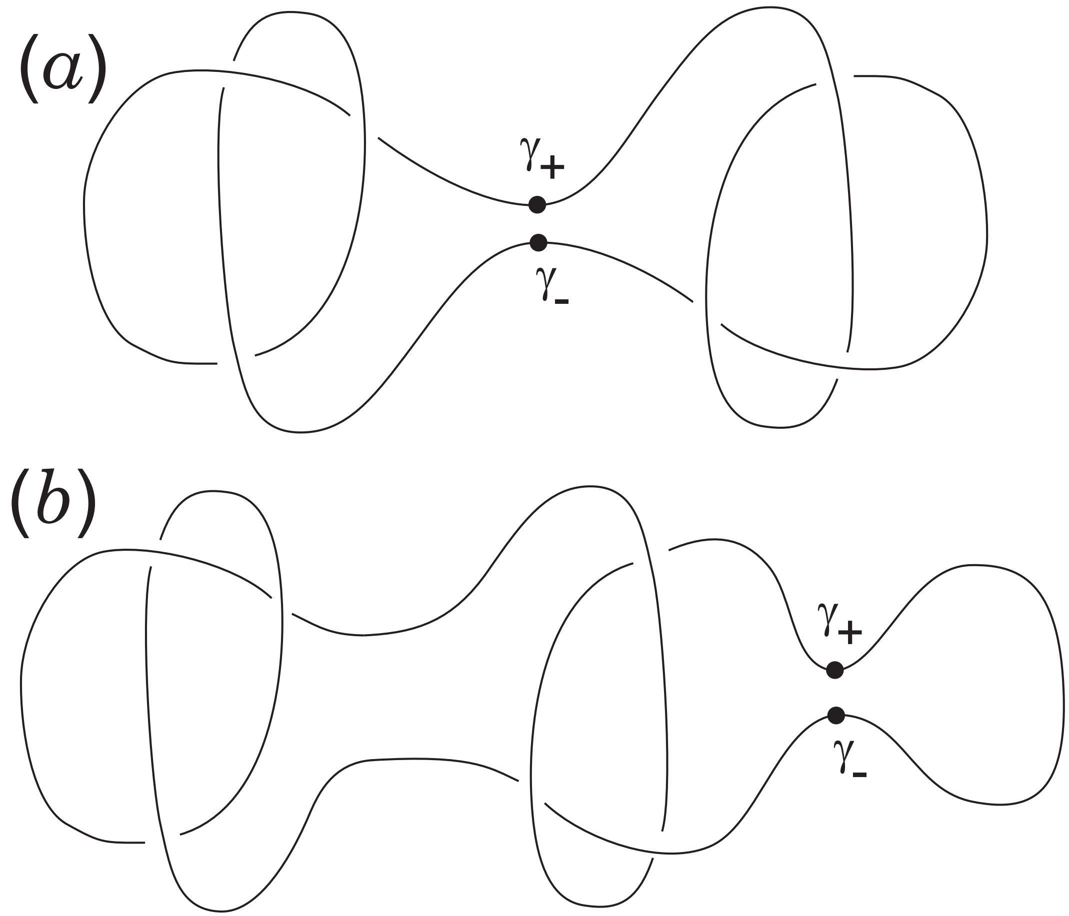
(a)



(b)



Links of two isotopic and outer Lipschitz equivalent, but not ambient Lipschitz equivalent, surfaces in $\mathbb{R}^3$.



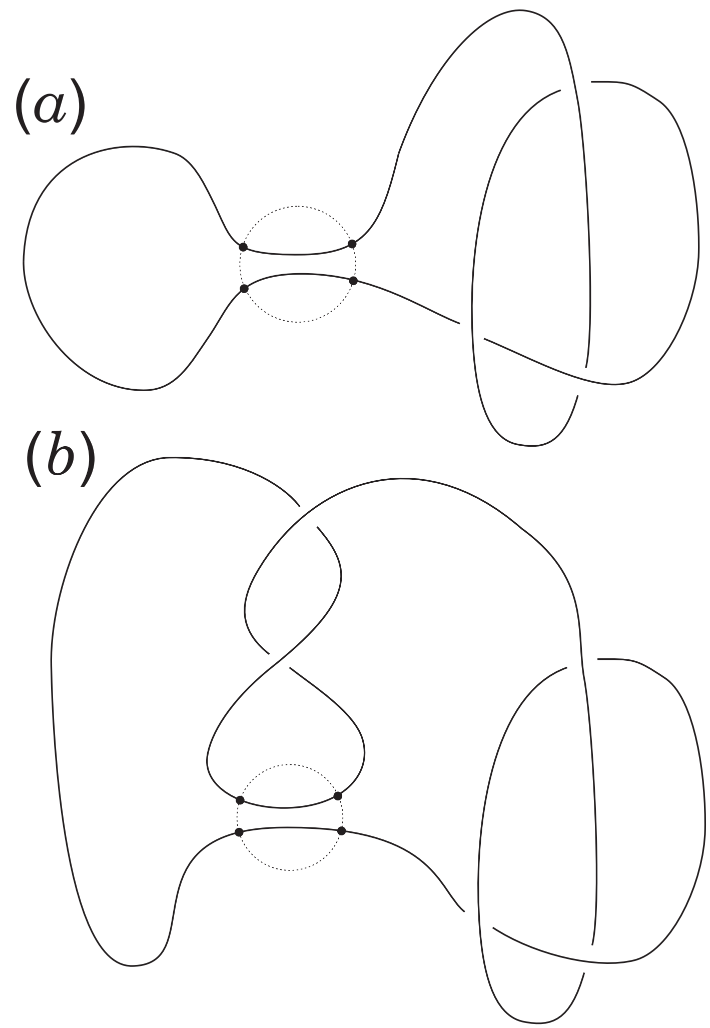
Links of two isotopic and outer Lipschitz equivalent, but not ambient Lipschitz equivalent, surfaces in $\mathbb{R}^4$.

Theorem (LB,Gabrielov). For any isotopy type of a real semialgebraic surface germ $(X, 0)$ in $\mathbb{R}^4$ there are infinitely many semialgebraic surface germs $(X_i, 0)$ such that

1) All X_i are isotopic to X;

2) All X_i are outer Lipschitz equivalent;

3) X_i and X_j are not ambient Lipschitz equivalent if $i \neq j$.

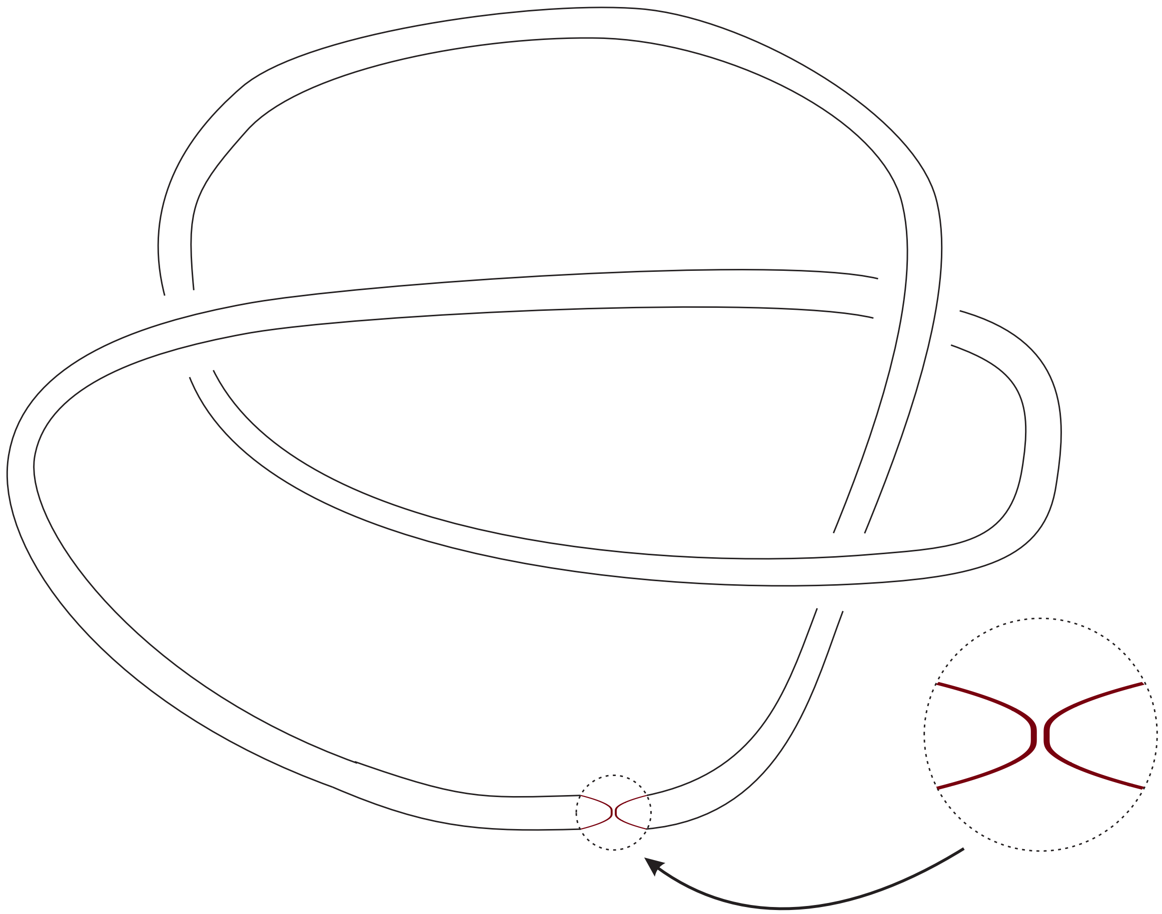

(a)
(b)

Universality Theorem (LB , Brandenbursky, Gabrielov). For each knot $K \subset S^3$ there exists a semialgebraic surface $X_K \subset \mathbb{R}^4$ such that

1) The link of X_K at 0 is a trivial knot in S^3.

2) All surfaces X_K are outer Lipschitz equivalent.

3) X_{K_1} is ambient bi-Lipschitz equivalent to X_{K_2}, only if the knots K_1 and K_2 are isotopic.



Link of the surface X_K for the trefoil knot K.

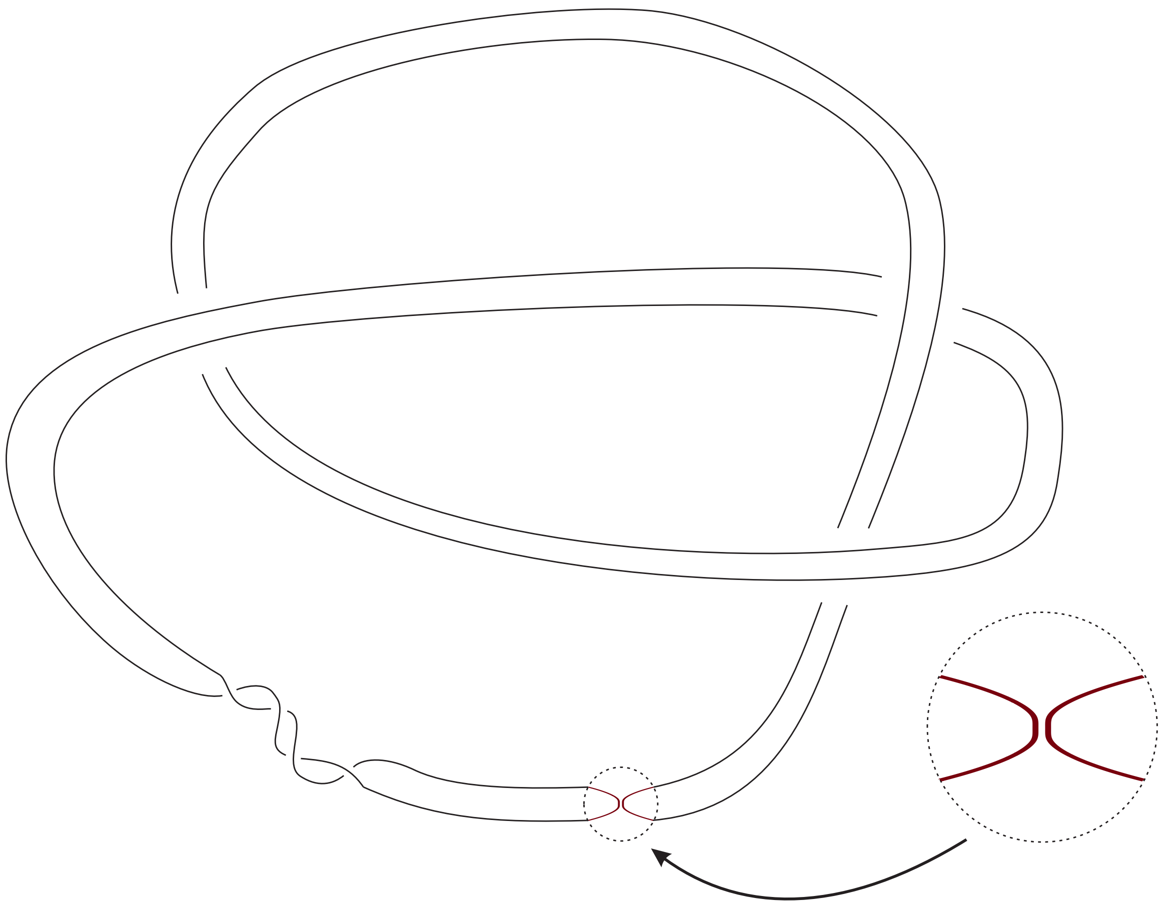
Combining the above constructions, we get the following

Theorem (LB, Brandenbursky, Gabrielov). For each knot $K \subset S^3$ there is an infinite sequence of semialgebraic surfaces $X_{K,i} \subset \mathbb{R}^4$ such that

1) The link of each $X_{K,i}$ at 0 is a trivial knot in S^3.

2) All surfaces $X_{K,i}$ are outer Lipschitz equivalent.

3) $X_{K,i}$ and $X_{L,j}$ are ambient Lipschitz equivalent only if the knots K and L are isotopic and $i = j$.



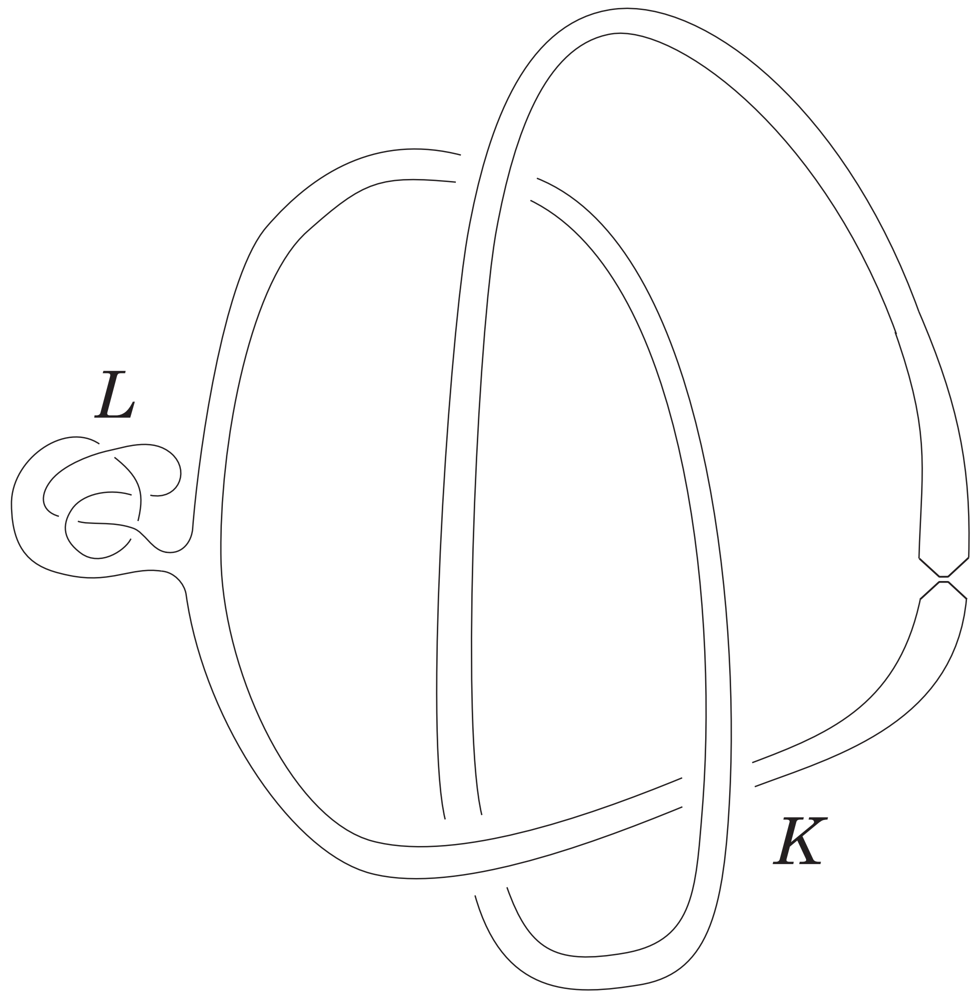
Surface $X_{K,1}$ (with one twist) for the trefoil knot K.

Universality Theorem 2 (LB, Brandenbursky, Gabrielov). For any two knots K and L, there exists a germ of a semialgebraic surface X_{KL} such that:

1. The link of X_{KL} at zero is isotopic to L.

2. For any knots K and L, all surface germs X_{KL} are outer bi-Lipschitz equivalent.

3. The tangent link of X_{KL} is isotopic to K.


L
K

Outer metric.

Kurdyka pancake decomposition.

Any semialgebraic surface can be decomposed into a union of LNE Hölder triangles.

The case of two LNE Hölder triangles.