A short proof of the multiple cover formula for point insertions in abelian surfaces

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- Ingredients :
  - explicit formula in primitive case
  - reduction to the primitive case
- Tools : tropical geometry and correspondence theorem.

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$$\begin{cases} \Omega Q^{-1}\Omega^{\intercal} = 0, \\ -i\Omega Q^{-1}\overline{\Omega}^{\intercal} > 0. \end{cases}$$

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It means $\mathbb{C}A$ is projective, and Q is the chern class of an ample line bundle:

$$Q = c_1(\mathcal{L}) \in H^2(\mathbb{C}A, \mathbb{Z}) \simeq \wedge^2 L^*.$$

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Up to a change of basis, we have for some $d_1 | d_2$,

$$Q = \begin{pmatrix} & & d_1 & 0 \\ & & 0 & d_2 \\ -d_1 & 0 & & \\ 0 & -d_2 & & \end{pmatrix}.$$

We can then assume that $\omega = (I \ Z)$ for some $Z \in \mathcal{M}_2(\mathbb{C})$.

Curves in abelian surfaces

- Consider stable maps

$$f : (C, p_1, \ldots, p_n) \longrightarrow \mathbb{C}A,$$

  where $(C, p_1, \ldots, p_n)$ is a genus g nodal curve with n marked points.
- Its degree is $\beta = f_*[C] \in H_2(\mathbb{C}A, \mathbb{Z})$.

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- If section of $\mathcal{L}$ with $c_1(\mathcal{L}) = Q$, then

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Problem

How many degree β genus g curves pass through g points inside $\mathbb{C}A$?

Example

If $g = 2$, look for curves whose jacobian is a covering of $\mathbb{C}A$. Map is determined up to translation.

Problem

Equivalently, for $n = g$, what is the degree of the evaluation map

$$\mathrm{ev} : \mathcal{M}_{g,n}(\mathbb{C}A, \beta) \longrightarrow \mathbb{C}A^n.$$

- The answer $N_{g,\beta}$ does not depend on the choice of $\mathbb{C}A$ (with the right polarization) nor the points.

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- The answer $N_{g,\beta}$ does not depend on the choice of $\mathbb{C}A$ (with the right polarization) nor the points.
- β is an element of $H_2(\mathbb{C}A, \mathbb{Z}) \simeq U^{\oplus 3} \simeq \mathbb{Z}^6$. Up to automorphism, it is determined by its divisibility $(= d_1)$ and square β^2 $(= 2d_1d_2)$.
- $N_{g,\beta}$ depends on β through divisibility and self-intersection.

Problem

What is the value of $N_{g,\beta}$?

Results

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- closed formula if β primitive (divisibility 1)

Theorem (Bryan-Leung)

If β is primitive with $\beta^2 = 2n$,

$$N_{g,\beta} = g \sum_{a_1+\cdots+a_{g-1}=n} \prod_{i=1}^{g-1} a_i\sigma_1(a_i), \text{ with } \sigma_1(a) = \sum_{d|a} d.$$

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- formula for non-primitive case

Theorem (B.)

For $k|\beta$ let $\widetilde{\beta/k}$ be a primitive class with $\widetilde{\beta/k}^2 = (\beta/k)^2$.

$$N_{g,\beta} = \sum_{k|\beta} k^{4g-3} N_{g,\widetilde{\beta/k}}.$$

Multiple cover formula

Example

Assume $\beta \in U \oplus 0 \oplus 0 \subset U^{\oplus 3} = H_2(\mathbb{C}A, \mathbb{Z})$, so that classes are $\beta = (a, b)$ with $\beta^2 = 2ab$.

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and

$$N_{g,(6,6)} = N_{g,(1,36)} + 2^{4g-3} N_{g,(1,9)} + 3^{4g-3} N_{g,(1,4)} + 6^{4g-3} N_{g,(1,1)}.$$

Recall we have a formula for $N_{g,(1,n)}$!

Proof of Bryan-Leung formula

Take $\mathbb{C}A = E \times F$ where E, F elliptic curves, and $\beta = [E] + n[F]$, which satisfies $\beta^2 = 2n$.

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$$N_{g,\beta} = g \sum_{a_1+\cdots+a_{g-1}=n} \prod_{i=1}^{g-1} a_i \sigma_1(a_i), \text{ with } \sigma_1(a) = \sum_{d|a} d.$$

It does not work if non-primitive, consider for instance $2[E] + 2n[F]$.

Strategy of proof of the multiple cover formula

- We have a tropical correspondence theorem from Nishinou: enumeration is transformed into piecewise affine graph combinatorics.
- Theoretically, if one is brave enough, we can compute everything and witness the formula.
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Contrarily to the toric case, provided one knows the primitive case, no tropical enumeration is required to prove the formula.

Tropical abelian surfaces

A real torus $\mathbb{T}A$ is a quotient $\mathbb{R}^2/S$ where $S : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ is a lattice in $\mathbb{R}^2$ with image $\Lambda \subset \mathbb{R}^2$.

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Definition

A real torus is a *tropical abelian surface* if there exists a *polarization* C satisfying Riemann condition $C^\intercal S \in \mathcal{S}_2^{++}(\mathbb{R})$.

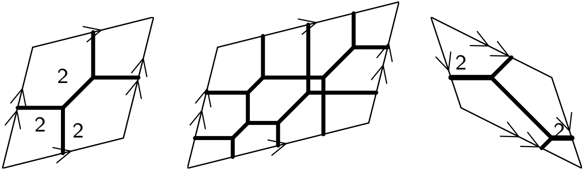
- Setting $B = \mathrm{com}(C)$, it equivalently means $BS^\intercal \in \mathcal{S}_2^{++}(\mathbb{R})$.
- C is the tropical chern class and plays the role of Q,
- B is the tropical degree and plays the role of β,
- C and B are linked through Poincaré duality: $C = \mathrm{com}(B)$.

Tropical curves

Definition

A parametrized tropical curve is a map $h : \Gamma \to \mathbb{T}A$ where

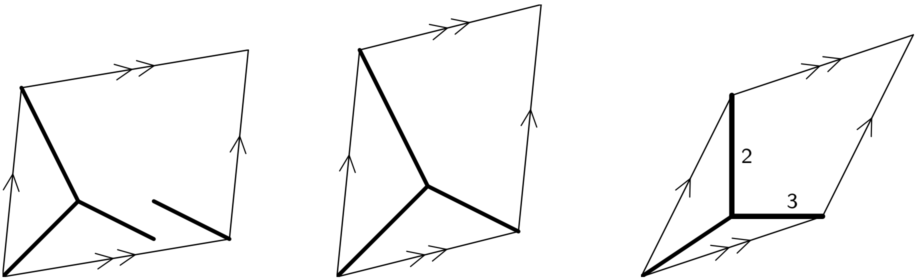
1. Γ is a metric graph (with $n = g$ marked points)
2. h is affine with integer slope on the edges
3. h is balanced: at each vertex v, $\sum_{e \ni v} \partial_e h = 0$.



$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\qquad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $\qquad \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

Curves have a degree B, a genus g and a gcd δ_Γ (the gcd of integer lengths of slopes)

The symmetry is due to the *gluing condition*.



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Proposition

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Problem

How many genus g curves in the class C pass through g points in general position ?

Tropicalization

Definition

A *Mumford family* of complex torus is of the form $\mathbb{C}^2/\Omega_t$ for $0 < |t| < 1$ with $\Omega_t = (I\ Z + \frac{\log t}{2i\pi}S)$ for some S with $\mathbb{Z}$-coefficients. Its tropicalization is $\mathbb{R}^2/S$.

In multiplicative coordinates, a Mumford family is of the form

$$(\mathbb{C}^*)^2/\langle e^{2i\pi z_{ij}} t^{s_{ij}} \rangle.$$

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It fits in the exact sequence

$$0 \to (S^1)^2 \to \mathbb{C}A_t \to \mathbb{R}^2/\log_t|\Omega_t| \to 0.$$

The second map is the amoeba map, and amoeba of complex curves converge to tropical curves (correspondence).

From complex to tropical

Consider a Mumford family $\Omega_t = (I\ Z + \frac{\log t}{2i\pi} S)$.

- If Q is a polarization for each value of t, it needs to be of the form

$$Q = \begin{pmatrix} 0 & C \\ -C^\intercal & T \end{pmatrix} \text{ or equivalently } \beta = \begin{pmatrix} -T & B \\ -B^\intercal & 0 \end{pmatrix},$$

  with $C = \mathrm{com}(B)$ a tropical polarization of $\mathbb{R}^2/S$, and $T = \left(\begin{smallmatrix} 0 & \tau \\ -\tau & 0 \end{smallmatrix}\right)$.
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  with $C = \mathrm{com}(B)$ a tropical polarization of $\mathbb{R}^2/S$, and $T = \left(\begin{smallmatrix} 0 & \tau \\ -\tau & 0 \end{smallmatrix}\right)$.
- *A complex polarization induces a tropical polarization.*
- Furthermore, we also have by Riemann condition for Ω_1

$$b_{11}z_{12} - b_{12}z_{11} + b_{21}z_{22} - b_{22}z_{21} = \tau.$$

From tropical to complex

- Starting with a tropical abelian variety $\mathbb{R}^2/S$, we have MANY Mumford families with this given tropicalization: pick the matrix $Z = (z_{ij})$. However, most of them are not polarized.

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  for some τ which completes B in Q.
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We have some freedom in the choice of τ.

Correspondence

Let $\mathbb{C}A_t$ be a Mumford family with tropicalization $\mathbb{T}A$. $\mathcal{P}_t$ a configuration of g points tropicalizing to $\mathcal{P} \subset \mathbb{T}A$.

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Theorem (Nishinou)

Let $h : \Gamma \to \mathbb{T}A$ be a trivalent tropical curve in $\mathbb{T}A$.

1. *The tropical curve can be deformed in the Mumford family if and only if it is phasable:* $\exp\left(2i\pi \frac{b_{11}z_{12} - b_{12}z_{11} + b_{21}z_{22} - b_{22}z_{21}}{\delta_\Gamma}\right) = (-1)^{\sum m_V/\delta_\Gamma}$.
2. *If it can be deformed and passes through $\mathcal{P}$, there are exactly $m_\Gamma = \prod_e w_e \cdot \mathrm{Ind}\Theta$ deformations passing through $\mathcal{P}_t$.*

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2. *If it can be deformed and passes through $\mathcal{P}$, there are exactly* $m_\Gamma = \prod_e w_e \cdot \mathrm{Ind}\Theta$ *deformations passing through $\mathcal{P}_t$.*

- The first point can be rewritten $e^{2i\pi\tau/\delta_\Gamma} = 1$, or equivalently $\delta_\Gamma | \tau$.
- Multiplicity is $(4g-3)$-homogeneous:

$$m_{k\Gamma} = k^{4g-3} m_\Gamma.$$

This is because there are $3g - 3 + g$ edges.

Proof of the multiple cover formula

$\mathbb{T}A$ tropical abelian surface with polarization B. Let $\mathcal{P} \subset \mathbb{T}A$ g points. $\mathscr{C}(B)$ set of tropical curves of degree B passing through $\mathcal{P}$. $\mathscr{C}_k(B)$ the subset of gcd k.

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- Let $\tau = 0$ and pick z_{ij} accordingly. The associated β has same divisibility and square as B. Condition $e^{2i\pi\tau/\delta_\Gamma} = 1$ is always satisfied.

$$\text{Correspondence ensures } N_{g,\beta} = \sum_{\mathscr{C}(B)} m_\Gamma.$$

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- Let $\tau = 1$ and pick z_{ij} accordingly. The associated $\widetilde{\beta}$ has now divisibility 1 and same square as B. Condition $e^{2i\pi\tau/\delta_\Gamma} = 1$ is satisfied when $\delta_\Gamma = 1$.

$$\text{Correspondence ensures } N_{g,\widetilde{\beta}} = \sum_{\mathscr{C}_1(B)} m_\Gamma.$$

The formula comes from the silly computation:

$$N_{g,\beta} = \sum_{k|B} \sum_{\mathscr{C}_k(B)} m_\Gamma = \sum_{k|B} k^{4g-3} \sum_{\mathscr{C}_1(B/k)} m_{\Gamma'} = \sum_{k|\beta} k^{4g-3} N_{g,\widetilde{\beta/k}}.$$

Example

For degree $(2,2)$, we have

$$N_{g,(2,2)} = \sum \text{curves with gcd } 1 + \sum \text{curves with gcd } 2.$$

The first term is actually $N_{g,(1,4)}$ by correspondence. By homogeneity, the second term is 2^{4g-3} times the sum over curves of degree $(1,1)$, which is $N_{g,(1,1)}$ by correspondence.

$$N_{g,(2,2)} = N_{g,(1,4)} + 2^{4g-3} N_{g,(1,1)}.$$

Thanks !