Enumeration of tropical curves in abelian surfaces

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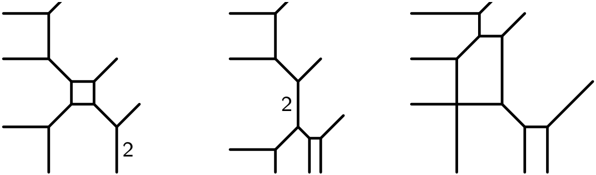
Online seminar, January 27th 2020

- Goal : solve enumerative problems and compute Gromov-Witten invariants.
- Tools: tropical geometry and correspondence theorem
- Upshot: Refined invariants.

The case of toric surfaces



Definition

Let Γ be a metric graph of genus g. A tropical curve in $\mathbb{R}^2$ is a map $h : \Gamma \to \mathbb{R}^2$ that is affine with integer slope on the edges and satisfies the balancing condition.

Problem

How many degree d genus g curves passing through $3d + g - 1$ points inside $\mathbb{C}P^2$

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Theorem (Mikhalkin, Nishinou-Siebert, Shustin, Tyomkin)

Let $h : \Gamma \to \mathbb{R}^2$ passing through $\mathcal{P}$. The number of complex curves passing through $\mathcal{P}_t$ and tropicalizing to Γ is equal to $m_\Gamma^{\mathbb{C}} = \prod m_V$. In particular, the number of curves $N_{d,g}^{\mathbb{T}}(\mathcal{P})$ passing through $\mathcal{P}$ does not depend on $\mathcal{P}$ and is equal to $N_{d,g}^{\mathbb{C}}$.

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Theorem (Itenberg-Mikhalkin)

Replacing $m_\Gamma^{\mathbb{C}}$ by $m_\Gamma^q = \prod \frac{q^{m_V/2} - q^{-m_V/2}}{q^{1/2} - q^{-1/2}}$ yields an invariant count, called refined.

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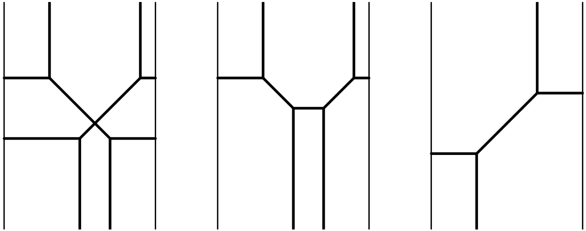

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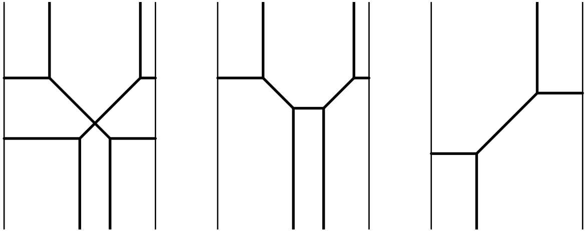
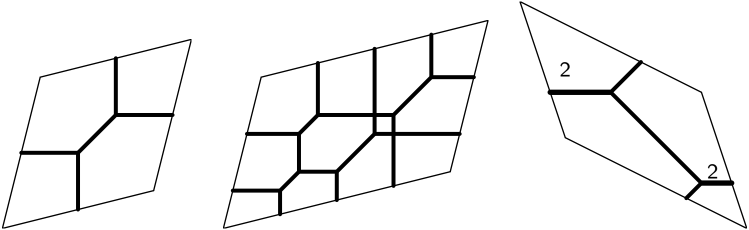


- Real tori



Plan

1. Tropical curves, degree and dimension of the moduli space.
2. Enumerative problem
3. Correspondence theorem and multiplicity
4. Refined invariance
5. (Computation ?)

1. Curves in line bundle over an elliptic curve

2. Curves in abelian surfaces

3. Curves in linear system in abelian surfaces

Tropical curves in cylinder

- A cylinder is obtained by quotient of $\mathbb{R}^2$ by a map of the form $(x, y) \mapsto (x + l, y(+\delta x) + a)$.
- Concretely, you identify both sides of the strip $[0; l] \times \mathbb{R}$.

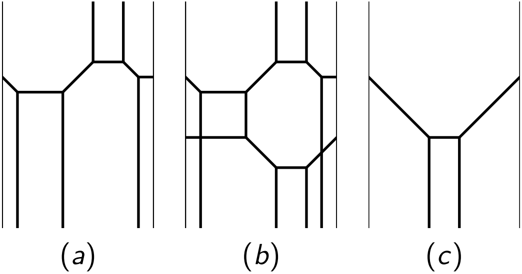


Figure: Examples of tropical curves inside $\mathbb{T}F_1$ ((a) and (b)) and in $\mathbb{T}F_2$ (c).

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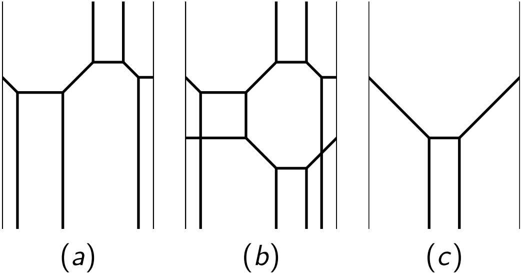

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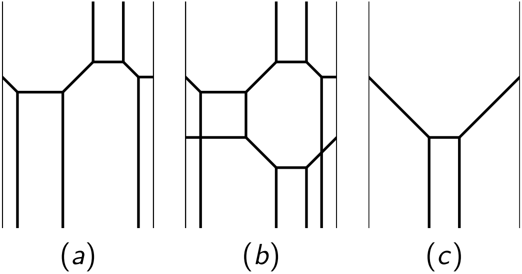


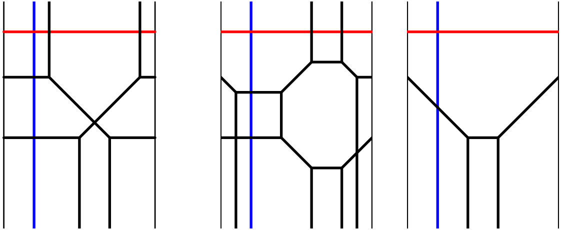
Figure: Examples of tropical curves inside $\mathbb{T}F_1$ ((a) and (b)) and in $\mathbb{T}F_2$ (c).

- You might change slope when crossing the boundary.
- It corresponds to *line bundle of degree δ over an elliptic curve*

Degree of a curve

Definition

A curve is of bidegree (d_1, d_2) if it has d_2 upper unbounded ends and intersects a fiber d_1 times. (counted with weights and multiplicities.)



$(2,2)$ in $E \times \mathbb{T}P^1$ $(2,2)$ in $\mathbb{T}F_1$ $(1,0)$ in $\mathbb{T}F_2$

By balancing condition, d_1 does not depend on the choice of the fiber.

Moduli

A genus g trivalent curve $h : \Gamma \to \mathbb{T}F_\delta$ has

- d_2 upper ends
- $\delta d_1 + d_2$ lower ends
- $3g - 3 + \delta d_1 + 2d_2$ bounded edges

Dimension of its deformation space should be

$$(3g - 3 + \delta d_1 + 2d_2) - 2g + 2.$$

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Proposition

The moduli space of genus g bidegree (d_1, d_2) curves is of dimension $\delta d_1 + 2d_2 + g - 1$.

Problem

How many genus g bidegree (d_1, d_2) curves pass through $\delta d_1 + 2d_2 + g - 1$ points in general position ?

Correspondence theorem

Definition

The multiplicity of a *simple* tropical curve is

$$m_\Gamma^{\mathbb{C}} = \prod m_V.$$

Let $\mathbb{C}F_t$ a family of line bundles over $\mathbb{C}E_t$ that tropicalizes to $\mathbb{T}F$ over $\mathbb{T}E$. Let $\mathcal{P}$ be a complex/tropical configuration of points and $N_{g,(d_1,d_2)}^{\mathbb{C}/\mathbb{T}}(\mathcal{P})$ the number of complex/tropical curves passing through $\mathcal{P}$.

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Theorem (B.)

Given a family of points $\mathcal{P}_t \subset \mathbb{C}F_t$ that tropicalizes to $\mathcal{P} \subset \mathbb{T}F$, and $h : \Gamma \to \mathbb{T}F_\delta$ there are $m_\Gamma^{\mathbb{C}}$ complex curves passing through $\mathcal{P}_t$ that tropicalize to Γ. In particular,

$$N_{g,(d_1,d_2)}^{\mathbb{C}} = N_{g,(d_1,d_2)}^{\mathbb{T}},$$

and $N_{g,(d_1,d_2)}^{\mathbb{T}}(\mathcal{P})$ does not depend on $\mathcal{P}$.

Refined invariants

Replace the complex multiplicity by the *refined multiplicity*:

$$m_\Gamma^q = \prod_V \frac{q^{m_V/2} - q^{-m_V/2}}{q^{1/2} - q^{-1/2}} \in \mathbb{Z}[q^{\pm 1/2}].$$

Let $BG_{g,(d_1,d_2)}(\mathcal{P})$ be the refined count of tropical curves passing through $\mathcal{P}$.

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A few remarks

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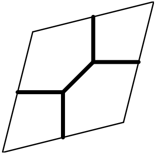
- It is possible to define relative invariants by prescribing intersection profile with zero and infinite-section (*i.e.* weights of the ends).
- We have a Caporaso-Harris type formula and floor diagrams to compute invariants.
- You can prove regularity results: quasi-modularity of generating series, piecewise polynomiality of relative invariants.
- Interpretation of refined invariants remains open:
  - Generating series of GW invariants with λ-classes ? (Bousseau)
  - Refined counts for real curves ?
  - ?

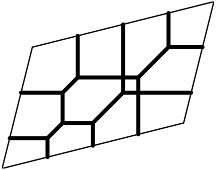
Degree of a curve

Definition

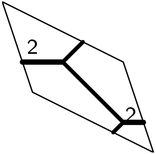
Let $\mathbb{T}A = \mathbb{R}^2/\Lambda$. The degree of Γ is the matrix $C : \Lambda^* \to \mathbb{Z}^2$ obtained by adding the slopes intersecting the right side and the top side respectively.



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$



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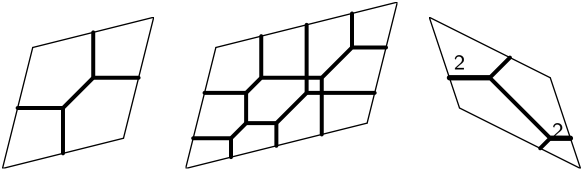

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix} \quad \begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}$$

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Let $S : \Lambda \to \mathbb{R}^2$ be the inclusion. The matrix C is the degree of a tropical curve if and only if CS^T is symmetric.

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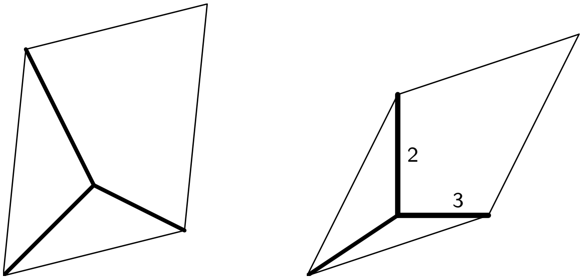

Dimension of moduli space

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Proposition

The dimension of the deformation space of genus g curves in class C is g.

Curves are *superabundant* but it matches the complex dimensions.

Problem

How many genus g curves in the class C pass through g points in general position ?

Correspondence theorem

Theorem (Nishinou)

Given a (Mumford) family of abelian surfaces $\mathbb{C}A_t$ with a point configuration $\mathcal{P}_t$ tropicalizing to $\mathcal{P} \subset \mathbb{T}A$, and $h : \Gamma \to \mathbb{T}A$ passing through $\mathcal{P}$, there are $m_\Gamma^{\mathbb{C}} = \cdots$ curves passing through $\mathcal{P}_t$ and tropicalizing to Γ. In particular, $N_{g,C}^{\mathbb{C}} = N_{g,C}^{\mathbb{T}}$, that does not depend on $\mathcal{P}$ nor $\mathbb{T}A$.

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- The multiplicity is given by $|\ker \Theta \otimes \mathbb{C}^*| \prod w_e$ where Θ is a map between some lattices.
- In the toric case, the lattices have the same dimension. We compute with the determinant and get $\prod m_V$.
- Here, domain is rank one less than codomain.

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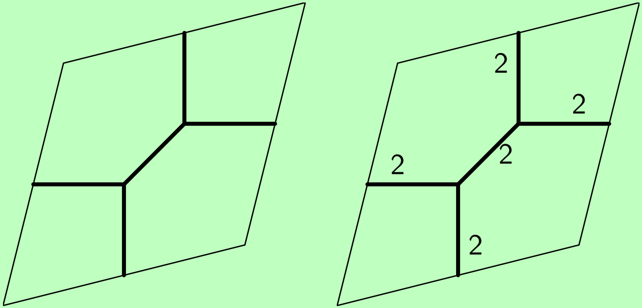
Theorem (B.)

The multiplicity expresses as $m_\Gamma^{\mathbb{C}} = \delta_\Gamma \prod m_V$, where $\delta_\Gamma = \gcd w_e$.

Example



The multiplicity is 1 on the left and $2 \cdot 4 \cdot 4 = 32$ on the right.

Refined invariants

Replace the complex multiplicity by the *refined multiplicity*:

$$(\delta_\Gamma)m_\Gamma^q = (\delta_\Gamma)\prod_V \frac{q^{m_V/2} - q^{-m_V/2}}{q^{1/2} - q^{-1/2}} \in \mathbb{Z}[q^{\pm 1/2}].$$

Let $BG_{g,C(,k)}(\mathbb{T}A, \mathcal{P})$ be the refined count of genus g, degree C (,gcd k) tropical curves passing through $\mathcal{P}$ with multiplicity $(\delta_\Gamma)m_\Gamma^q$.

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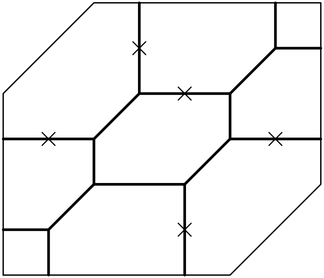
Notice that one has to check tropically the invariance: it does not come from a complex invariance.

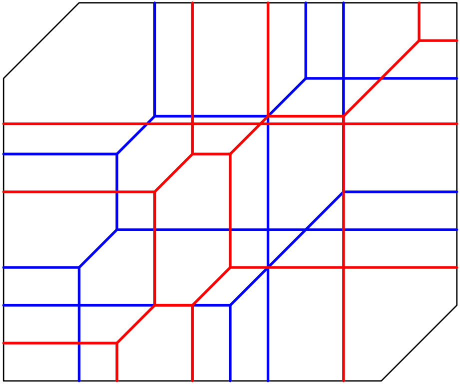
A few remarks

- We have circular floor diagrams to make computations.
- You can prove regularity statements such as the quasi-modularity of certain generating series.
- Interpretation of refined invariants remains open.

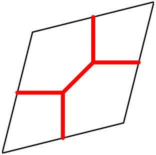
- Previously, curves of genus g through g points. Complement of marked points is without cycle.

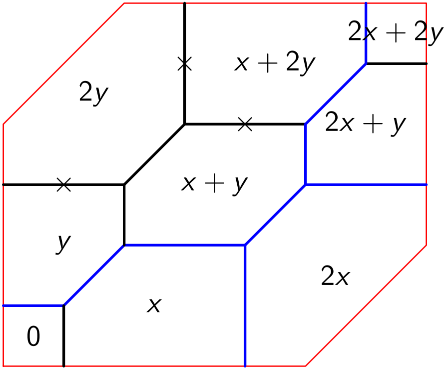


- Now, fix $g-2$ points and the linear system. ($\Rightarrow$ no translations anymore)
- In other words, you fix a curve Γ_0 of degree C, and ask for $\Gamma - \Gamma_0$ to be the corner locus of an piecewise affine function.



- You can assume that Γ_0 is of genus 2 and the boundary of a fundamental domain.





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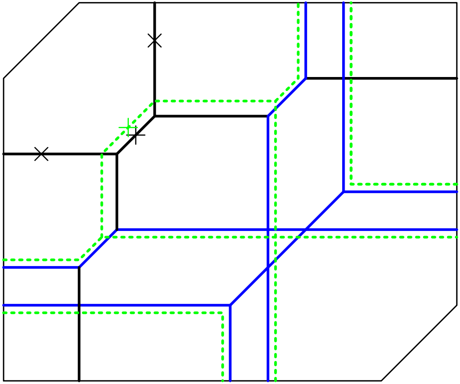

$$m_\Gamma^{\mathbb{C}} = 4$$

Deformation of the curve when moving a marked point.

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A few remarks

- We have circular floor diagrams to make computations.
- You can prove regularity statements such as the quasi-modularity of certain generating series.
- Interpretation of refined invariants remains open.
- It would be interesting if the new term Λ_Γ^Σ also had a refinement.

Thanks !