Refined invariants for rational tropical curves in arbitrary dimension

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Online seminar, November 5th 2020

Rational tropical curves

Let Γ be a finite metric graph without cycle with some infinite edges called *ends*.

Definition

A parametrized rational tropical curve in $\mathbb{R}^m$ is a pair (Γ, h) where $h : \Gamma \to \mathbb{R}^m$ is such that

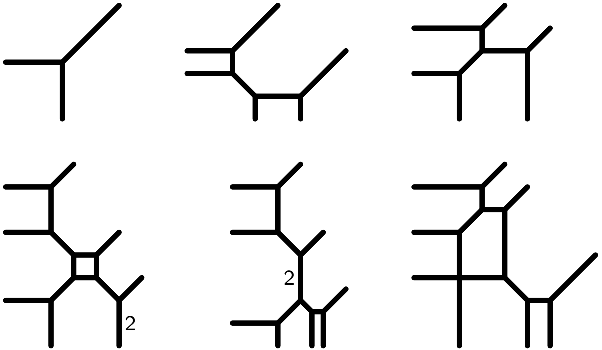
- h is affine with integer slope on the edges,
- at each vertex, one has the balancing condition:

$$\sum_{E \ni V} \frac{\partial h}{\partial E} = 0.$$

Definition

The collection (n_e) of slopes of the unbounded ends is called the *degree* Δ.

Example of planar tropical curves



Example of spacial tropical curve

Example

Some 3 dimensional example.

Enumerative problems

The space of parametrized tropical curve of degree Δ in $N_{\mathbb{R}} = \mathbb{R}^m$ has dimension: ($|\Delta|$ unbounded ends and thus $|\Delta| - 3$ bounded edges)

$$\boxed{\dim \mathcal{M}_0(\Delta, \mathbb{R}^m) = |\Delta| - 3 + m.}$$

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For e unbounded end, let $L_e \ni n_e$ be a linear subspace of $N_{\mathbb{R}}$ of codim l_e, with rational slope. $\mathcal{L}_e \subset N_{\mathbb{R}}$ affine subspace of slope L_e.

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Problem

How many paramatrized rational tropical curves satisfy $h(e) \subset \mathcal{L}_e$.

If $(\mathcal{L}_e)$ is chosen generically, solutions are trivalent curves.

theoretical resolution

- The space of curves $\mathcal{M}_0(\Delta, N_\mathbb{R})$ is a union of cones of the form $\mathbb{R}_+^{|\Delta|-3} \times N_\mathbb{R}$ over the possible shapes of trivalent graphs.
- Each cone is endowed with a linear evaluation map

$$\mathrm{ev} : \mathbb{R}_+^{|\Delta|-3} \times N_\mathbb{R} \longrightarrow \prod_e N_\mathbb{R}/L_e,$$

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  which sends a *curve* to the position of its unbounded ends e up to a translate by L_e.
- Solving the enumerative problem amounts to find the preimages of a point $\lambda \in \prod_e N_\mathbb{R}/L_e$. (space of choices of $(\mathcal{L}_e)$)
- This can be done as follows: for each cone, find the formal solution in $\mathbb{R}^{|\Delta|-3} \times N_\mathbb{R}$ and check that the first coordinates are positive. If the map is non invertible, there is no solution.

Example

In the planar case, $m = 2$, the dimension is $|\Delta| - 1$.

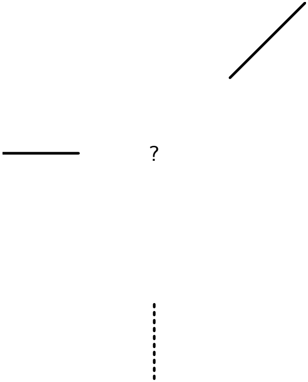
We look for rational curves which have all but one unbounded ends belonging to fixed lines.

Example

Take $\Delta = \{(-1, 0)^d, (0, -1)^d, (1, 1)^d\}$.

Example

Degree 1 curves with boundary constraints.



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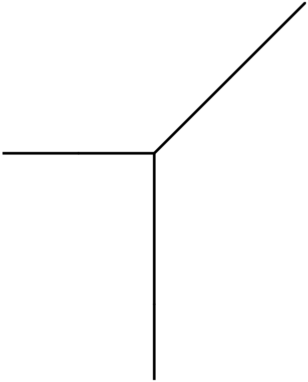

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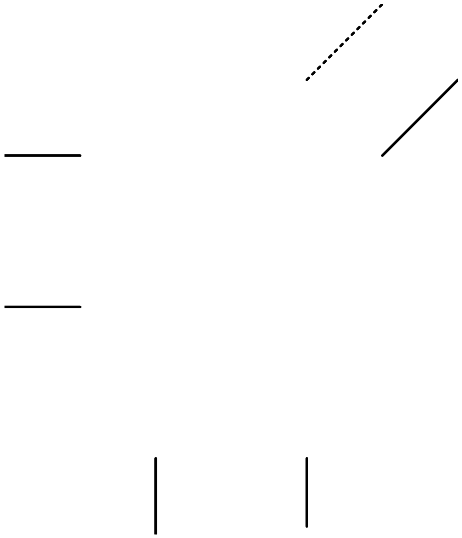
Degree 2 curves with boundary constraints.



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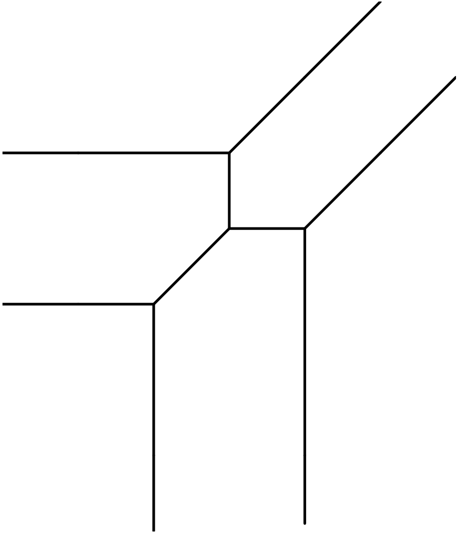

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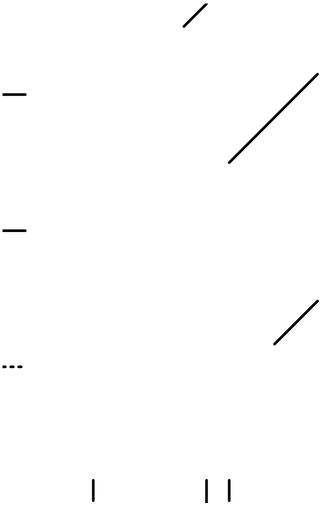

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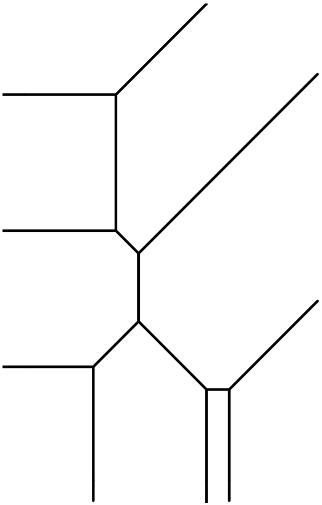

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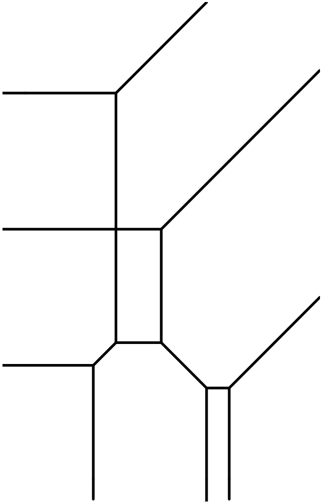

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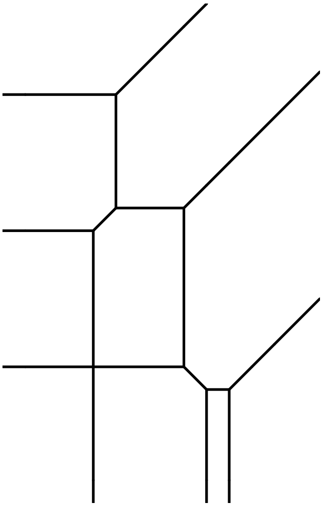

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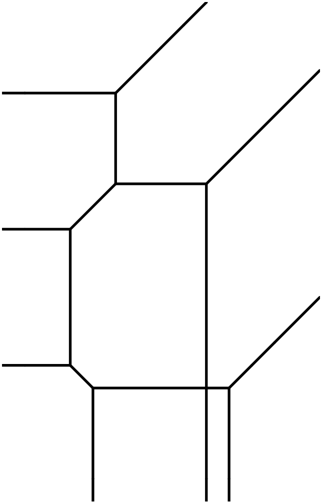

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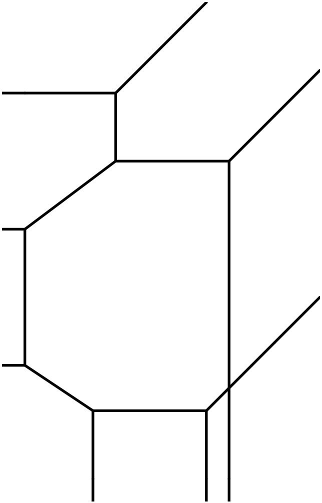

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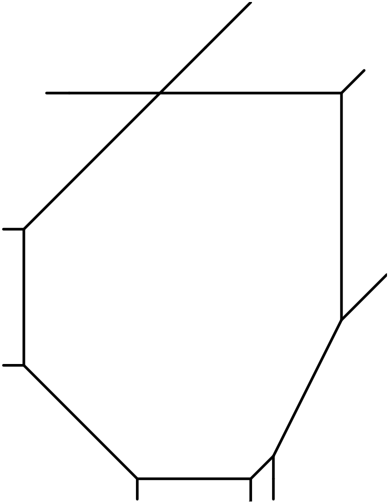

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Some 3-dimensional example.

Invariance Statements

The enumerative problem is related to a complex enumerative problem through a *correspondence theorem* (Mikhalkin, Shustin, Nishinou-Siebert, Tyomkin), providing a complex curve multiplicity $m_\Gamma^{\mathbb{C}}$.

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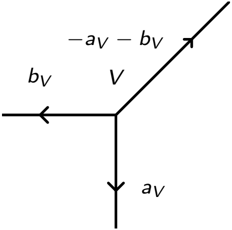
Proposition

The count $N_\Delta(\mathcal{L}_e)$ does not depend on the choice of $(\mathcal{L}_e)$ as long as it is generic. It only depends on the choice of (L_e).

In the **planar case**, the complex multiplicity (Mikhalkin) is given by



$$m_\Gamma^{\mathbb{C}} = \prod_V m_V, \text{ where } m_V = |\det(a_V \wedge b_V)|.$$

Now, consider the *refined multiplicity* (Block-Göttsche):

$$\boxed{B_\Gamma = \prod_V (q^{m_V} - q^{-m_V}) \in \mathbb{Z}[q^{\pm 1}].}$$

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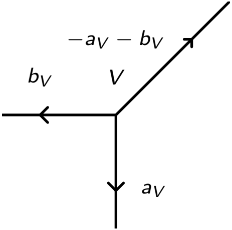

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Theorem (Itenberg-Mikhalkin, Göttsche-Schroeter)

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Theorem (B.)

The count $\mathcal{B}_\Delta(\mathcal{L}_e)$ does not depend on the choice of $(\mathcal{L}_e)$ as long as it is generic. It only depends on the choice of (L_e).

ω-problem

Inside $N_{\mathbb{R}}$, where $N = \mathbb{Z}^m$. Let Δ be a degree, e_0 directed by $n_{e_0} \in \Delta$ some end, $\omega \in \mathrm{Hom}(\Lambda^2 N, \mathbb{Z})$ be a generic 2-form, and $L_{e_0} = P \ni n_{e_0}$ be a plane.

For $e \neq e_0$, let

$$L_e = \langle n_e \rangle^{\perp\omega} \ni n_e,$$

so that

$$\sum_e l_e = |\Delta| - 1 + m - 2.$$

Problem

This particular enumerative problem is called the *ω-problem*.

For the ω-problem, up to a global scalar depending on ω and P, the complex multiplicity given by the correspondence theorem has the form

$$m_\Gamma^{\mathbb{C}} = \prod_V m_V, \text{ where } m_V = \omega(a_V \wedge b_V) > 0.$$

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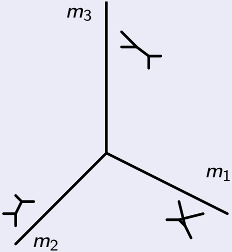
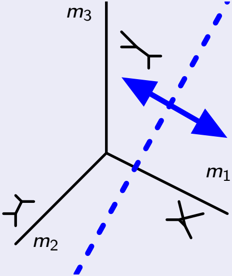
Proposition

The count of solutions to the ω-problem using previous multiplicities does not depend on the choice of $(\mathcal{L}_e)$ as long as it is generic. It only depends on the choice of ω and e_0.

Proof.

Proof is entirely similar to the planar case: $a+b+c+d=0$.





$$m_1 = m_2 + m_3$$

$$\omega(a,b)\omega(c,d) + \omega(a,c)\omega(b,d) = \omega(a,d)\omega(b,c)$$

□

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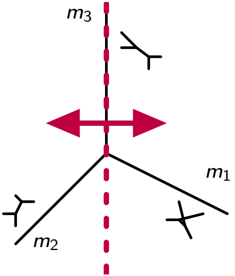
□

If ω is not generic, some combinatorial type might have complex multiplicity 0: it never provides a solution.



One needs to have $m_1 = m_2$, which can be done as before by quotienting the exponents by K_ω: space spanned by the $a_V \wedge b_V$ for some vertex of some curve, with $\omega(a_V \wedge b_V) = 0$.

$$\omega(a,b)\omega(c,d) + \cancel{\omega(a,c)\omega(b,d)} = \omega(a,d)\omega(b,c)$$

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Idea

Back to the **general case** where (L_e) are not defined by a 2-form.

- No magic recipe for the complex multiplicity. (*i.e.* as a product over the vertices)
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- The plan is to recycle and use the same multiplicity.

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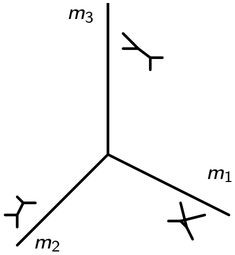
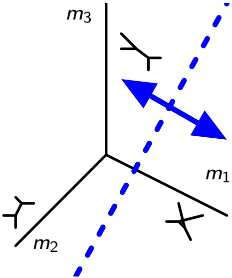
However, the invariance might fail.

Wall



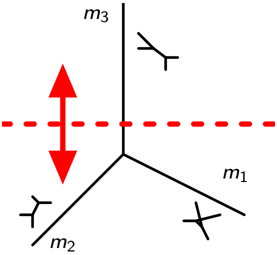
Invariance for B_Γ



$$m_1 = m_2 + m_3$$

(ω-problem)

Invariance for $m^{\mathbb{C},(L_e)}$



~~$m_1 + m_2 = m_3$~~

$$m_1 - m_2 = m_3$$

Nevertheless, adding some signs might just work.

Statement of the main result

Assume no combinatorial type has zero complex multiplicity. Let ω be a generic 2-form.

$$\boxed{B_\Gamma = \prod_V (q^{a_V \wedge b_V} - q^{-a_V \wedge b_V}) \in \mathbb{Z}[\Lambda^2 N]},$$

where $\omega(a_V, b_V) > 0$.

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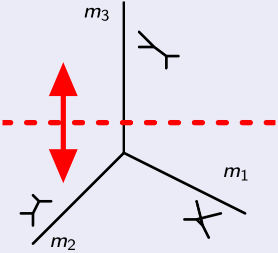
where $\omega(a_V, b_V) > 0$.

Theorem (B.)

There exists some signs $\varepsilon_\Gamma = \pm 1$ such that the count of solutions using multiplicity $\varepsilon_\Gamma B_\Gamma$ leads to an invariant.

First Proof.

1. Use the rule presented before the proof to propagate the definition of the signs.



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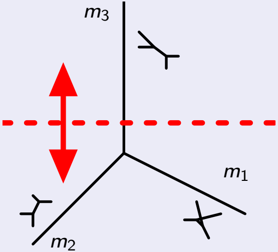

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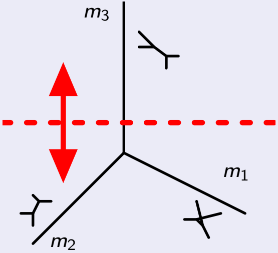

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2. Check that these are well-defined, *i.e.* the sign does not depend on the path from a combinatorial type to another.
3. The obtained condition does not depend on the problem, so it works because $B_\Gamma^{K_\omega}$ leads to an invariant in the ω-problem.

□

Second Proof.

The space $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$ is a fan, endowed with an evaluation map whose fibers are the solutions for a choice of $(\mathcal{L}_e)$:

$$\mathrm{ev} : \mathcal{M}_0(\Delta, N_{\mathbb{R}}) \to \prod_e \mathbb{R}^m / L_e \simeq \mathbb{R}^{|\Delta|+m-3}.$$

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For a cone $\Gamma \simeq \mathbb{R}_+^{|\Delta|-3} \times N_\mathbb{R}$, let $\mathfrak{or}_{\mathrm{ev}}(\Gamma)$ be the orientation induced by a fixed orientation of $\mathbb{R}^{|\Delta|+m-3}$. Then m_Γ leads to an invariant if and only if

$$\Xi = \sum_\Gamma m_\Gamma(\Gamma, \mathfrak{or}_{\mathrm{ev}}(\Gamma)) \in C_{|\Delta|+m-3}(\mathcal{M}_0(\Delta, N_\mathbb{R}))$$

is a cycle. (*i.e.* $\partial\Xi = 0$): at each wall "$m_1 + m_2 - m_3 = 0$".

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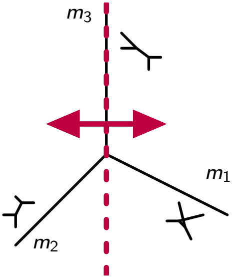
$$\sum_\Gamma \frac{\mathfrak{or}_\omega(\Gamma)}{\mathfrak{or}_{\mathrm{ev}}(\Gamma)} B_\Gamma(\Gamma, \mathfrak{or}_{\mathrm{ev}}(\Gamma)) = \sum_\Gamma B_\Gamma(\Gamma, \mathfrak{or}_\omega(\Gamma)) \text{ is a cycle.}$$

One takes $\varepsilon_\Gamma = \frac{\mathfrak{or}_\omega(\Gamma)}{\mathfrak{or}_{\mathrm{ev}}(\Gamma)}$. □

If some combinatorial type has complex multiplicity 0, it never provides a solution.



Let ω be such that $m_\Gamma^{\mathbb{C},(L_e)} = 0 \Rightarrow m_\Gamma^{\mathbb{C},\omega} = 0$.

$$\boxed{B_\Gamma^{K_\omega} = \prod_V (q^{a_V \wedge b_V} - q^{-a_V \wedge b_V}) \in \mathbb{Z}[\Lambda^2 N / K_\omega]}.$$

Continuity

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Theorem

There is a fan Ω_Δ in $\mathrm{Hom}(\Lambda^2 N_{\mathbb{R}}, \mathbb{R})$ such that:

- $\omega \mapsto K_\omega$ *is constant on the cones.*
- $\omega \mapsto \mathcal{B}_\Delta^{\omega,e_0} \in \mathbb{Z}[\Lambda^2 N/K_\omega]$ *is constant on the cones.*
- *If $\tau \prec \sigma$, then $K_\tau \supset K_\sigma$,*
- *If $\tau \prec \sigma$, then $\mathcal{B}_\tau = \pi_{\sigma\tau}(\mathcal{B}_\sigma)$.*

Sketch of proof : Use the implicit function theorem for the evaluation map.

There is a similar statement for $\mathcal{B}_\Delta(L_e)$.

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Problem

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Can one replace $\mathcal{L}_e$ with tropical cycles of the same dimension ? (for instance a line with a tropical curve)

Not really since there is new kind of "*walls*" that appear, and one does not have an evaluation map anymore. However,

Theorem (B.)

In the ω-problem, P can be replaced by $C \times \langle n_{e_0} \rangle$, where C is a tropical curve.

Constraints in the main strata

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Yes, by using the same proof and the following analog to the ω-problem:

$$\omega - \text{problem} \ \oplus \ \text{meeting some hyperplanes.}$$

However, multiplicities become more complicated and depend on the slope of the chosen hyperplanes.

Theorem (B.)

In the ω-problem, the P condition can be replaced by meeting some tropical curve C inside $N_{\mathbb{R}}$.

relation to classical invariants

- Using some correspondence theorem, the enumerative problem relates to a complex and a real classical enumerative problem:

Problem

Let Δ be a degree and $\mathbb{C}\Delta$ be some toric variety associated to a fan containing Δ. How many rational curves meet the toric divisors in some chosen suborbits under the actions of L_e ?

For instance $\Delta = \{-e_1^d, \ldots, -e_n^d, (\sum e_i)^d\}$ for degree d curves in $\mathbb{C}\Delta = \mathbb{C}P^n$.

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- In the planar case, the tropical invariant is equal to a refined classical invariant introduced by Mikhalkin. (refined according to the value of a "*quantum index*")
- Sadly, in higher dimension, such a refined classical invariant remains to be found. (Although there is already some quantum class generalizing the quantum index)

Thanks !