Gromov–Witten theory of complete intersections

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Introduction

- Talk based on arXiv:2109.13323, joint work with Hülya Argüz, Rahul Pandharipande, Dimitri Zvonkine.
- Main result: an inductive algorithm computing Gromov-Witten invariants in all genera with arbitrary insertions of all smooth complete intersections of hypersurfaces in projective space.
- Application: all Gromov–Witten classes of complete intersections in projective space are tautological elements in the cohomology of the moduli space of stable curves.
- Main technical tool: nodal Gromov–Witten theory, working with domain curves with prescribed nodes.

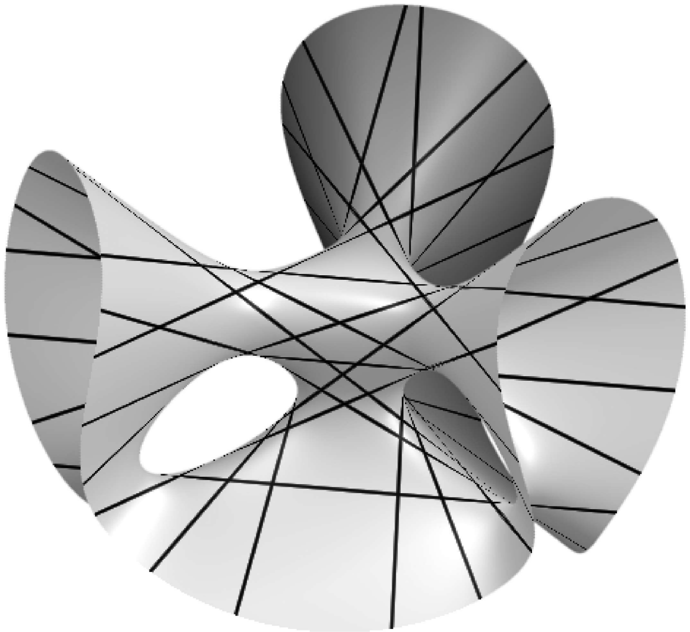
Plan

- Gromov–Witten invariants and Gromov–Witten classes.
- Complete intersections in projective space.
- The main issue: degeneration versus vanishing cycles.
- The main idea: trading vanishing cycles against nodes.
- Foundational results in nodal Gromov–Witten theory

Gromov–Witten invariants

Part of complex enumerative geometry: counts of complex algebraic curves in complex algebraic varieties.

- How many lines on a cubic surface? 27
- How many lines on a general quintic 3-fold hypersurface? 2875
- How many lines in $\mathbb{P}^3$ meeting 4 given lines in general position? 2
- How many rational cubic curves in $\mathbb{P}^2$ passing through 8 points in general position? 12



Gromov–Witten invariants

- Gromov–Witten invariants of a smooth projective variety X: numbers defined by intersection theory on the moduli space of stable maps to X.
- In nice cases, agree with counts of curves passing through general constraints. But not always: “virtual counts" in general.
- Key property: invariant under deformation of the complex structure on X and of the constraints.

Fix X a smooth projective variety over $\mathbb{C}$

Definition (Kontsevich, 1994)

An n-pointed genus g stable map to X of class β is a morphism

$$f : (C, x_1, \dots, x_n) \longrightarrow X\,,$$

where

- C: nodal projective curve of arithmetic genus g.
- $x_1, \dots, x_n$: n (ordered) smooth marked points on C.
- $f_*[C] = \beta \in H_2(X, \mathbb{Z})$.
- (stability) there are finitely automorphisms of $(C, x_1, \dots, x_n)$ commuting with f.

- $\overline{\mathcal{M}}_{g,n,\beta}(X)$: moduli space of n-pointed genus g stable maps to X of class β. Proper Deligne-Mumford stack.

- $\overline{\mathcal{M}}_{g,n,\beta}(X)$: moduli space of n-pointed genus g stable maps to X of class β. Proper Deligne-Mumford stack.
- Virtual dimension:

$$vd_{g,n,\beta} := (1-g)(\dim X - 3) + \beta \cdot c_1(X) + n$$

- Virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \in H_{2\, vd_{g,n,\beta}}(\overline{\mathcal{M}}_{g,n,\beta}(X), \mathbb{Q})$$

- $ev_i : \overline{\mathcal{M}}_{g,n,\beta}(X) \longrightarrow X$ evaluation at the i-th marked point.
- L_i: line bundle on $\overline{\mathcal{M}}_{g,n,\beta}(X)$, cotangent line bundle at the i-th marked point,

$$\psi_i := c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n,\beta}(X), \mathbb{Q})\,.$$

- For $X = \mathrm{Spec}\, \mathbb{C}$ (and so $\beta = 0$), get the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed genus g stable curves.

- Fix

$$g, n \in \mathbb{Z}_{\geq 0}\,, \quad \beta \in H_2(X, \mathbb{Z})\,, \quad \alpha_1, \dots, \alpha_n \in H^\star(X, \mathbb{Q})$$

$$\text{and} \quad k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$$

- Gromov–Witten invariants of X:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\alpha_i) \right\rangle_{g,n,\beta}^X := \deg \left(\prod_{i=1}^n \psi_i^{k_i} \operatorname{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in \mathbb{Q}\,.$$

- For $k_i = 0$, virtual count of genus g curves in X of class β with n marked points constrained to land on fixed submanifolds of X Poincaré duals to the class α_i.
- Forgetful morphism $\pi\colon \overline{\mathcal{M}}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n}$.
- Gromov–Witten classes

$$\left[\prod_{i=1}^n \tau_{k_i}(\alpha_i) \right]_{g,n,\beta}^X := \pi_* \left(\prod_{i=1}^n \psi_i^{k_i} \operatorname{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in H^\star(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})\,.$$

Gromov–Witten invariants

Problem

Given a smooth projective variety X, "compute" all Gromov–Witten invariants of X.

Known cases:

- X the point (Kontsevich, Witten's conjecture, 1992)
- X a projective space, or more generally an homogeneous variety (localization, Graber-Pandharipande, 1999)
- X a curve (Okounkov-Pandharipande, 2003)
- X a quintic 3-fold hypersurface in $\mathbb{P}^4$ (Maulik-Pandharipande, 2006)

Main result (Argüz-B-Pandharipande-Zvonkine, 2021)

An algorithm computing all Gromov–Witten invariants of all complete intersections in projective space.

Gromov–Witten invariants

Tautological ring $RH^\star(\overline{\mathcal{M}}_{g,n}) \subset H^\star(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Smallest system of subrings containing 1 and preserved by pullback-pushforward along the natural maps $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$.

Conjecture

For every smooth projective variety X, the Gromov–Witten classes of X are tautological.

Known cases:

- X a projective space, or more generally an homogeneous variety (localization, Graber-Pandharipande, 1999)
- X a curve (Janda, 2013)

Theorem (Argüz-B-Pandharipande-Zvonkine, 2021)

All Gromov–Witten classes of all complete intersections in projective space are tautological.

Degeneration of complete intersections

- X: m-dimensional smooth complete intersection of r hypersurfaces of degrees $(d_1, \dots, d_r)$ in $\mathbb{P}^{m+r}$,

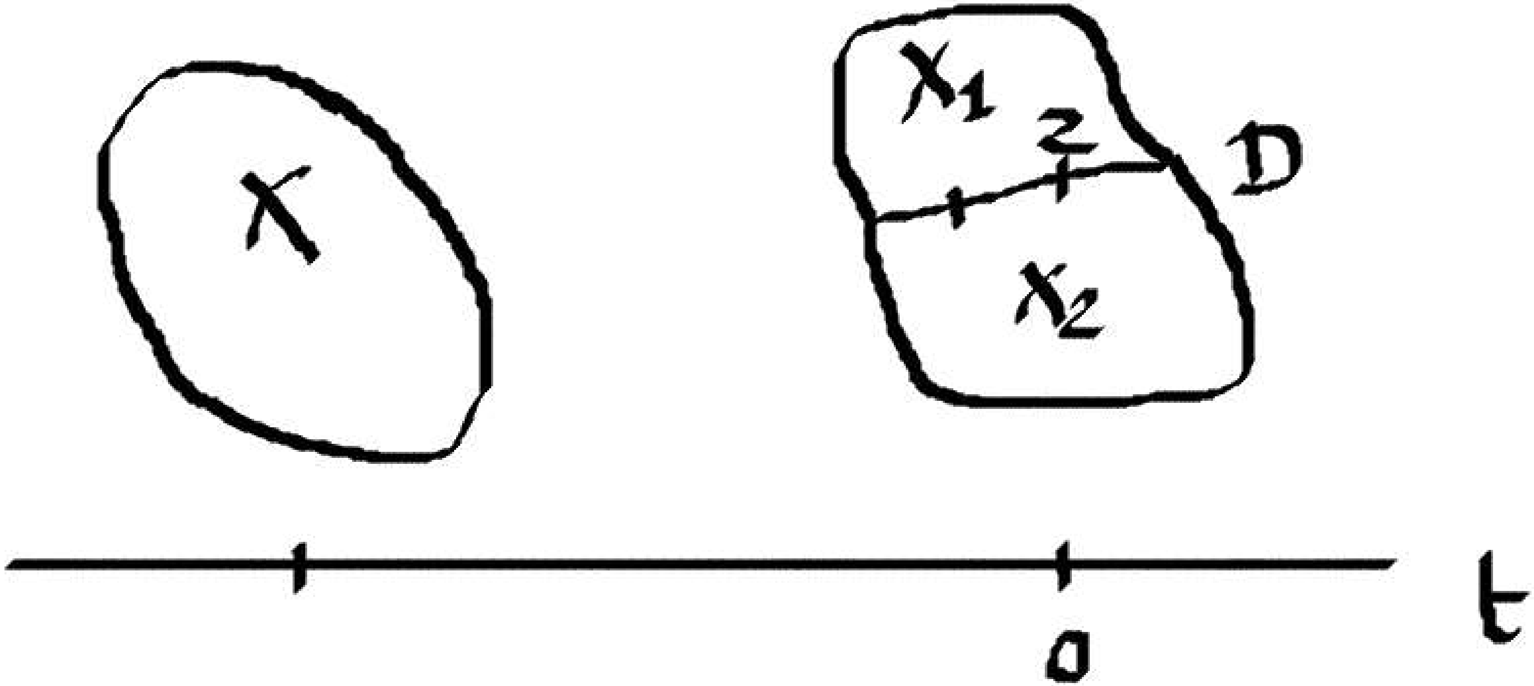
$$f_1 = \cdots = f_r = 0\,.$$

- Gromov–Witten invariants only depend on the dimension m and the degrees $(d_1, \dots, d_r)$ by deformation invariance.
- Main idea to study Gromov–Witten invariants of X: degeneration.
- Decompose $d_r = d_{r,1} + d_{r,2}$ and pick general polynomials $f_{r,1}$ and $f_{r,2}$ of degree $d_{r,1}$ and $d_{r,2}$.
- Deform f_r to the product $f_{r,1} f_{r,2}$, one-parameter family:

$$t f_r + f_{r,1} f_{r,2} = 0\,.$$

- Family over $\mathbb{A}^1$, with general fiber (deformation equivalent) to X, and fiber over $t = 0$: $X_1 \cup_D X_2$.

Degeneration of complete intersections



Degeneration of complete intersections

- X_1: m-dimensional smooth complete intersection of degree $(d_1, \dots, d_{r-1}, d_{r,1})$, of equations $f_1 = \dots = f_{r-1} = f_{r,1} = 0$.
- X_2: m-dimensional smooth complete intersection of degree $(d_1, \dots, d_{r-1}, d_{r,2})$, of equations $f_1 = \dots = f_{r-1} = f_{r,2} = 0$.
- D: $(m-1)$-dimensional smooth complete intersection of degree $(d_1, \dots, d_{r-1}, d_{r,1}, d_{r,2})$, of equations $f_1 = \dots = f_{r-1} = f_{r,1} = f_{r,2} = 0$.
- The total space of the family is singular. Resolve the singularities: new degeneration of X to $X_1 \cup_D \widetilde{X_2}$.
- $\widetilde{X_2}$: blow-up of X_2 along Z.
- Z: $(m-2)$-dimensional smooth complete intersection of degree $(d_1, \dots, d_{r-1}, d_r, d_{r,1}, d_{r,2})$, of equations $f_1 = \dots = f_{r-1} = f_r = f_{r,1} = f_{r,2} = 0$.

Theorem (Argüz-B-Pandharipande-Zvonkine, 2021)

Let X be an m-dimensional smooth complete intersection in $\mathbb{P}^{m+r}$ of degrees $(d_1, \ldots, d_r)$. Then, for every decomposition

$$d_r = d_{r,1} + d_{r,2} \quad \text{with} \quad d_{r,1}, d_{r,2} \in \mathbb{Z}_{\geq 1}\,,$$

then $GW(X)$ can be effectively reconstructed from:

(i) $GW(X_1)$, where $X_1 \subset \mathbb{P}^{m+r}$ is an m-dimensional smooth complete intersection $X_1 \subset \mathbb{P}^{m+r}$ of degrees $(d_1, \ldots, d_{r-1}, d_{r,1})$.

(ii) $GW(X_2)$, where $X_2 \subset \mathbb{P}^{m+r}$ is an m-dimensional smooth complete intersection of degrees $(d_1, \ldots, d_{r-1}, d_{r,2})$.

(iii) $GW(D)$, where $D \subset \mathbb{P}^{m+r}$ is an $(m-1)$-dimensional smooth complete intersection of degrees $(d_1, \ldots, d_{r-1}, d_{r,1}, d_{r,2})$.

(iv) $GW(Z)$, where $Z \subset \mathbb{P}^{m+r}$ is an $(m-2)$-dimensional smooth complete intersection of degrees $(d_1, \ldots, d_{r-1}, d_r, d_{r,1}, d_{r,2})$.

Degeneration of complete intersections

- Naïve idea: apply a "degeneration formula" to the degeneration $W \to \mathbb{A}^1$ with general fiber X and central fiber $X_1 \cup_D \widetilde{X}_2$.
- Problem: Jun Li's degeneration formula computes the Gromov–Witten invariants

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\alpha_i) \right\rangle_{g,n,\beta}^{X}$$

of X in terms of relative Gromov–Witten invariants of (X_1, D) and $(\widetilde{X}_2, D)$ only if the insertions α_i are in the image of the restriction map

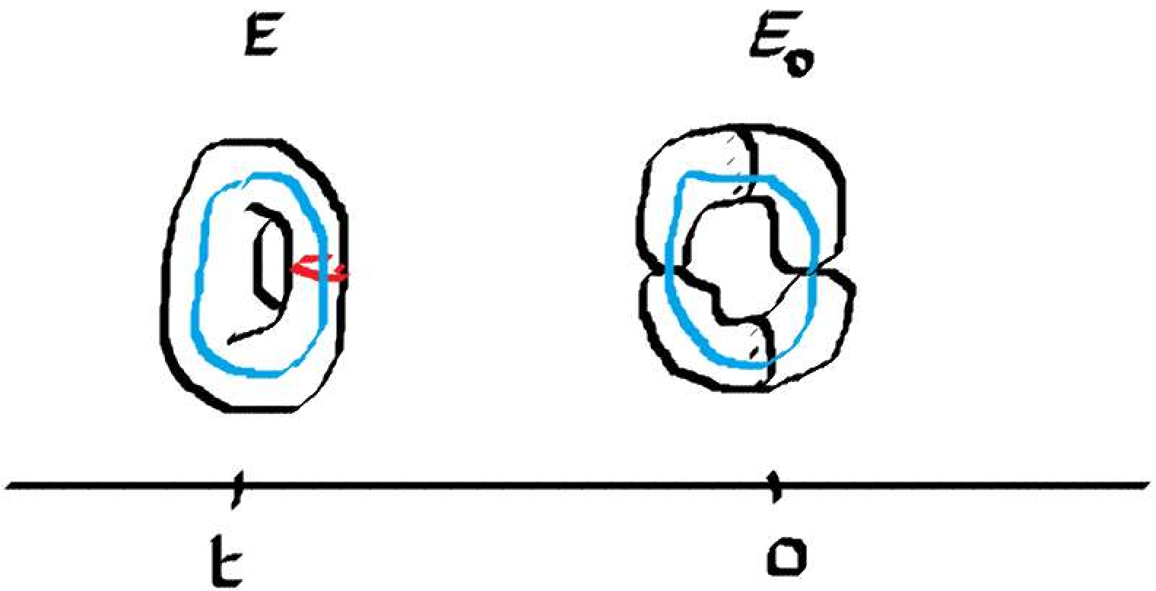
$$H^\star(W) \to H^\star(X)\,.$$

Not surjective in general!

- Monodromy action on $H^\star(X)$. The image of $H^\star(W) \to H^\star(X)$ is exactly the monodromy invariant part of $H^\star(X)$ (local cycle invariant theorem).
- Dually, $H_\star(X) \to H_\star(W)$ not injective in general, kernel: vanishing cycles.

- Example: degeneration of a smooth elliptic curve E to a nodal elliptic curve E_0. Vanishing cycle.
- $\dim H^1(E) = 2$, whereas $\dim H^1(E_0) = 1$.
- The restriction map $H^1(E_0) \to H^1(E)$ is not surjective.
- Monodromy action on $H^1(E)$:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} .$$



Cohomology of complete intersections

- $X \subset \mathbb{P}^{m+r}$, m-dimensional smooth complete intersection of degrees $(d_1, \dots, d_r)$.
- Lefschetz hyperplane theorem:

$$H^i(\mathbb{P}^{m+r}, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

  is an isomorphism for $0 \leq i \leq 2m$ and $i \neq m$.
- Middle degree cohomology:

$$H^m(X, \mathbb{Q}) = H^m(\mathbb{P}^{m+r}, \mathbb{Q}) \oplus H^m(X, \mathbb{Q})_{prim}$$

  Simple cohomology. Primitive cohomology.
- Monodromy action on the part of the simple cohomology is always trivial. Monodromy action possibly non-trivial on the primitive cohomology.
- Problem: the degeneration formula cannot be applied in general to compute Gromov–Witten invariants with insertions of primitive cohomology classes.

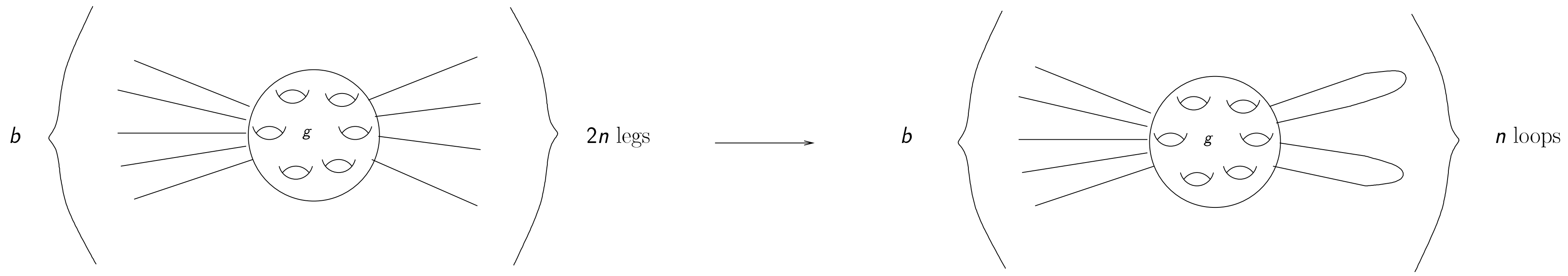
- Example: $E \subset \mathbb{P}^2$ smooth cubic (elliptic curve).
- $H^1(\mathbb{P}^2) = 0$, but $\dim H^1(E) = 2$.

$$H^1(E)_{prim} = H^1(E)\,.$$

Main idea: trading primitive insertions against nodes

- Starting point: Künneth decomposition of the class of the diagonal $\Delta \subset X \times X$ in $H^\star(X \times X) = H^\star(X) \otimes H^\star(X)$: for any basis $(\gamma_i)_i$ of $H^\star(X)$,
$$[\Delta] = \sum_i \gamma_i \otimes \gamma_i^\vee$$
where $(\gamma_i^\vee)$ is the Poincaré dual basis ($\int_X \gamma_i \cup \gamma_j^\vee = \delta_{ij}$).
- Splitting formula in Gromov–Witten theory:
$$\left\langle \left(\prod_{i=1}^n \tau_{k_i}(\alpha_i) \right) \tau_{k_{h_1}} \tau_{k_{h_2}} \right\rangle_{\Gamma,g,n,\beta}^X$$
$$= \sum_j \left\langle \left(\prod_{i=1}^n \tau_{k_i}(\alpha_i) \right) \tau_{k_{h_1}}(\gamma_j) \tau_{k_{h_2}}(\gamma_j^\vee) \right\rangle_{g-1,n+2,\beta}^X$$
where Γ is the graph with one vertex and one loop imposing a self-intersecting node.

Main idea: trading primitive insertions against nodes



- Γ: X-valued stable graph.
- Nodal Gromov–Witten invariants of X of type Γ are

$$\left\langle \prod_{i=1}^{n_\Gamma} \tau_{k_i}(\alpha_i) \prod_{h \in H_\Gamma \setminus L_\Gamma} \tau_{k_h} \right\rangle_\Gamma^X := \deg \left(\prod_{i=1}^{n_\Gamma} \psi_i^{k_i} \, \mathrm{ev}_i^*(\alpha_i) \prod_{h \in H_\Gamma \setminus L_\Gamma} \psi_h^{k_h} \cap [\overline{\mathcal{M}}_\Gamma(X)]^{\mathrm{virt}} \right)$$

Main idea: trading primitive insertions against nodes

- Example: E elliptic curve.
- Basis $a, b \in H^1(E)$ of primitive cohomology.
- GW invariants with two primitive insertions:

$$< x, a, b >^E_{g,n,\beta} = - < x, b, a >^E_{g,n,\beta} ,$$

- Diagonal $\Delta \subset E \times E$:

$$[\Delta] = 1 \otimes p + p \otimes 1 + a \otimes b - b \otimes a$$

- Γ: graph with one loop imposing a non-separating node,

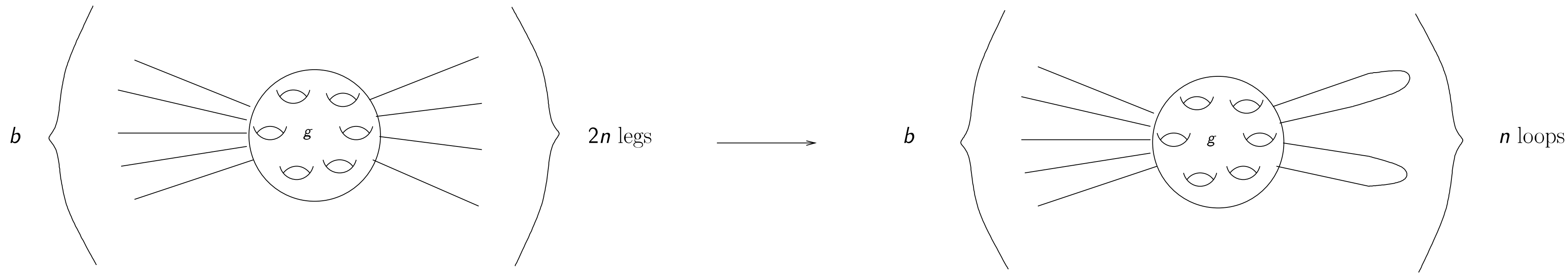
$$< x >^X_{\Gamma, g+1, n-2, \beta}$$

$$=< x, 1, p >^E_{g,n,\beta} + < x, p, 1 >^E_{g,n,\beta} + < x, a, b >^E_{g,n,\beta} - < x, b, a >^E_{g,n,\beta}$$

$$=< x, 1, p >^E_{g,n,\beta} + < x, p, 1 >^E_{g,n,\beta} + 2 < x, a, b >^E_{g,n,\beta}$$

Theorem (Argüz-B-Pandharipande-Zvonkine, 2021)

Let X be a complete intersection in projective space which is not a cubic surface or an even dimensional complete intersection of two quadrics. Then, the Gromov–Witten invariants of X can be effectively reconstructed from the nodal Gromov–Witten invariants of X with only insertions of simple cohomology classes.



Main idea: trading primitive insertions against nodes

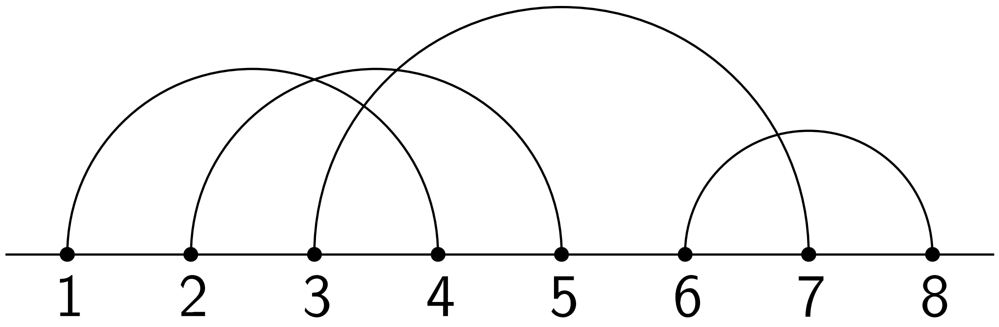
Idea of proof:

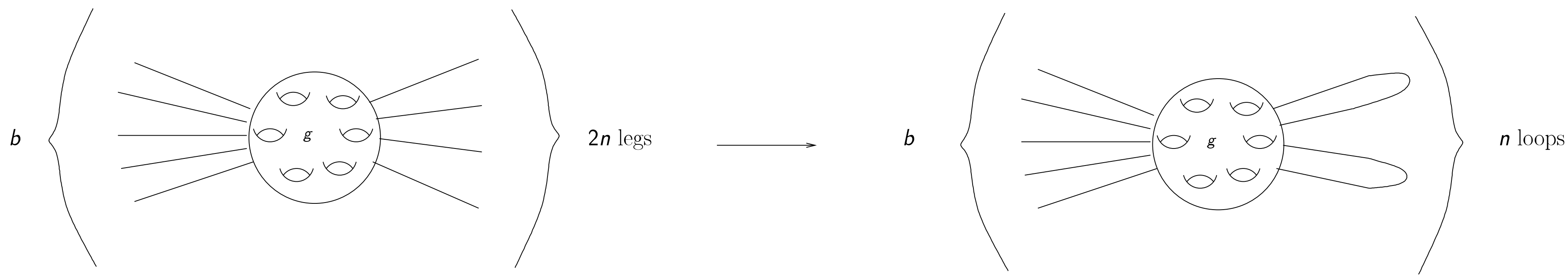
- $V := H^m(X, \mathbb{C})_{prim}$
- Deformation of X in the universal family of smooth complete intersections: monodromy action on V.
- G: (algebraic) monodromy group, Zariski closure of the monodromy group in $GL(V)$.
- $G \subset O(V)$ if m even, $G \subset Sp(V)$ if V odd.
- Deformation invariance: Gromov–Witten invariants are invariant under the action of G on V.
- If X is not a cubic surface, or an even dimensional complete intersection of two quadrics, G is as large as possible: $G = O(V)$ or $Sp(V)$.
- Use the invariant theory of $O(V)$ and $Sp(V)$ to constrain the form of Gromov–Witten invariants.

Main idea: trading primitive insertions against nodes

Idea of proof:

- Invariant theory: Gromov–Witten invariants are zero for a odd number of primitive insertions.
- Even number $2n$ of primitive insertions.
- Equations indexed by n-pairings (ways to produce n nodes).
- Unknowns indexed by n-pairing (invariant tensors).





Idea of proof:

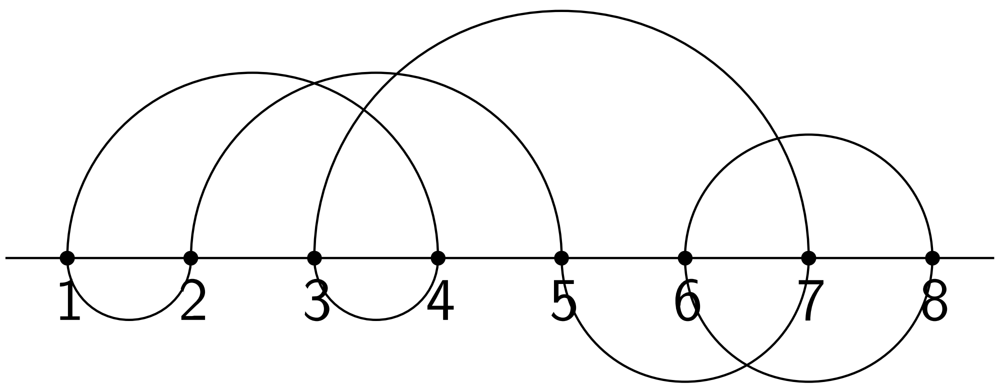
- Loop matrix: $(2n-1)!! \times (2n-1)!!$ matrix

$$M(n,x)_{P,P'} = x^{L(P,P')}$$

- $L(P,P')$: loop number of the n-pairings P and P'.
- $x = \dim V$ when m even, $x = -\dim V$ when m odd.

$$M(2,x) = \begin{pmatrix} x^2 & x & x \\ x & x^2 & x \\ x & x & x^2 \end{pmatrix},$$

- Result follows from the study of eigenvalues and eigenvectors of $M(n,x)$.



Main idea: trading primitive insertions against nodes

Theorem (Argüz-B-Pandharipande-Zvonkine, 2021)

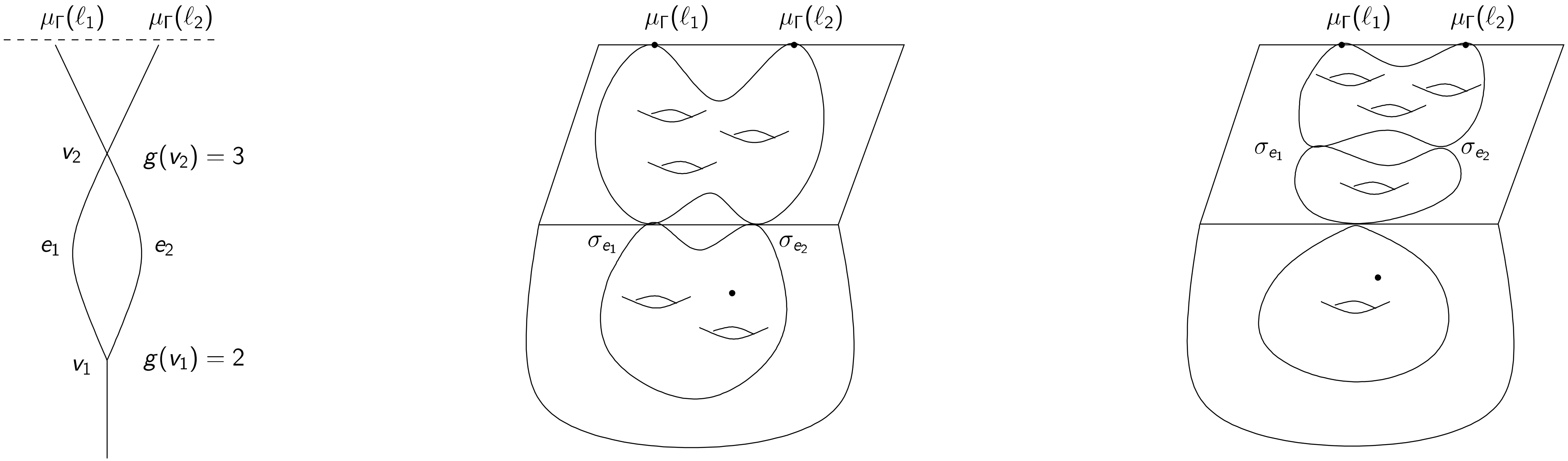
Let X be a complete intersection in projective space which is not a cubic surface or an even dimensional complete intersection of two quadrics. Then, the Gromov–Witten invariants of X can be effectively reconstructed from the nodal Gromov–Witten invariants of X with only insertions of simple cohomology classes.

Clear progress:

- General Gromov–Witten invariants of X, with primitive insertions, cannot be computed by the degeneration of X to $X_1 \cup_D \widetilde{X}_2$.
- Nodal Gromov–Witten invariants of X with only simple insertions can be computed by the degeneration of X to $X_1 \cup_D \widetilde{X}_2$.

Nodal Gromov–Witten theory

- Jun Li's degeneration formula applied to the degeneration of X to $X_1 \cup_D \widetilde{X}_2$ involves relative Gromov–Witten invariants of (X_1, D) and $(\widetilde{X}_2, D)$.
- Nodal version: one needs a notion of nodal relative Gromov–Witten invariant.



Nodal Gromov–Witten theory

- We prove a nodal degeneration formula.
- Nodal Gromov–Witten invariants of X can be computed in terms of nodal relative Gromov–Witten invariants of (X_1, D) and $(\widetilde{X}_2, D)$.
- How to compute these nodal relative Gromov–Witten invariants?
- We prove a splitting formula computing them in terms of ordinary relative Gromov–Witten invariants of (X_1, D) and $(\widetilde{X}_2, D)$.
- Relative Gromov–Witten invariants of (X_1, D) and $(\widetilde{X}_2, D)$ can be computed in terms of Gromov–Witten invariants of X_1, X_2, D, and Z by Maulik-Pandharipande (2006).

Summary of the algorithm

- Goal:

$$GW(X) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z),$$

  where X_1, X_2, D, Z are complete intersections of either smaller degree or smaller dimension.

- Step 1: trade primitive insertions for nodes, reduce

$$GW(X) \leftarrow sNGW(X)$$

- Step 2: apply the nodal degeneration formula

$$sNG(X) \leftarrow NGX(X_1, D), NGW(\widetilde{X}_2, D)$$

- Step 3: apply the splitting formula for nodal relative invariants

$$NGX(X_1, D), NGW(\widetilde{X}_2, D) \leftarrow GW(X_1, D), GW(\widetilde{X}_2, D)$$

- Step 4: apply previous results of Maulik-Pandharipande

$$GW(X_1, D), GW(\widetilde{X}_2, D) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z)$$

End

Thank you for your attention!