Refined Gromov-Witten invariants (w/ Schuler)

[Take-home msg: Refined curve counts on CY3 $\subset$ Equivariant curve counts on CY5]

$$\circ \longrightarrow \bigcirc \longrightarrow \circ$$

Context:
- X smooth (quasi-)proj. 3-fold
- $\omega_X \simeq \mathcal{O}_X$
- $g \geq 0$, $\beta \in H_2(X, \mathbb{Z})$

Want: $\#\{\text{genus-}g\ C \subset X,\ [C] = \beta\}$

(exp #: $< \infty$ (cf Clemens))

What GW is : "curve" = "param'd curve"

- $\mathcal{M} := \overline{\mathcal{M}}_g(X,\beta) := \{ f: C \to X,\ g(C)=g,\ f_*[C]=\beta \}/\sim$

  - C at worst nodal
  - $|\mathrm{Aut}(f)| < \infty$

- $[\mathcal{M}]^{vir} \in A_0(\mathcal{M}, \mathbb{Q})$; $GW_{g,\beta} := \deg [\mathcal{M}]^{vir}$
- $GW(\varepsilon_-, Q) := \sum_{g,\beta} Q^\beta (-\varepsilon_-^2)^{g-1} \cdot GW_{g,\beta}$

What GV does : "F-terms of M-th on $X \times S^1$ "

$\hookrightarrow$ Part 1) $\longleftrightarrow$ sheaf-th curve counts on X

- $\$:=$ moduli of pure 1-dim $\frac{1}{2}$ stable sheaves on X (fixed pol., χ), supp $= \beta$

[Katz, Hosono-Saito-Takahashi, Maulik-Toda] $H^{\bullet}(\$) \simeq \bigoplus_{\bar{\partial}_+, \bar{\partial}_-} H^{\bar{\partial}_+, \bar{\partial}_-}(\$)$

perverse deg

cohom degree

- $\Omega_\beta(q_-, q_+) := \sum_{\bar{\partial}_+, \bar{\partial}_-} (-q_+)^{\bar{\partial}_+} q_-^{\bar{\partial}_-} \dim H^{\bar{\partial}_+, \bar{\partial}_-}(\$) \in \mathbb{Z}[q_\pm^{\pm 1}]$

- $GV(q_\pm, Q) := \sum_{\beta \in H_2(X, \mathbb{Z})} Q^\beta \dfrac{\Omega_\beta(q_-, q_+)}{[q_-]_{q_+} [q_- q_+]}$

$\left[X^{\circlearrowleft \mathbb{C}^\times}, \text{rigid (for e.g. toric)} \right]$

GW/GV: $\quad \underline{Q}: \exp(G\underline{W}) \overset{?}{\doteq} \mathrm{Exp}(GV)\Big|_{q_- = e^{\varepsilon_\pm}}$

- $\mathbb{P} :=$ Pandharipande-Thomas mod. space of stable pairs on X, supp $= \beta$, $\chi = n$

- $[\mathbb{P}]^{vir} \in A_0(\mathbb{P}, \mathbb{Z})$ $\quad$; $\quad PT_{n,\beta} := \chi_{\mathbb{C}^*}(\mathbb{P}, \hat{\mathcal{O}}_{\mathbb{P}}) \in \mathbb{Z}[q_+^{\pm 1}]$

$\mathbb{C}^*$-char on ω_X^{-1}

$$PT(q_\pm, Q) := \sum_{n,\beta} (-q_-)^n Q^\beta PT_{n,\beta}(q_+)$$

GW/PT $\qquad \exp(GW) \overset{?}{\doteq} PT\Big|_{q_\pm = e^{\varepsilon_\pm}}$

Part 2: other theories of inv'ts / gen. functions

- gauge theory / instanton counts on surfaces $\rightsquigarrow$ Ω-background
- topological vertex $\rightsquigarrow$ refined vertex
- low-dim topol. (CS/RTW) $\rightsquigarrow$ refined CS / RTW
- mirror symmetry $\rightsquigarrow$ refined BCOV (HK)

Q: GW $\rightsquigarrow$ refined GW?

A: ref GW inv'ts of CY3 = equiv. inv'ts of CY5

<u>Setup</u>: $\cdot\ Z = X \times \mathbb{A}^2_{\mathbb{C}}$

$\cdot\ (\mathbb{C}^\times)^2 =: T$ acting linearly on $\mathbb{A}^2_{\mathbb{C}}$, char's q_1, q_2

$\cdot$ char q_t^{-2} on ω_X, $q_t^2 = q_1 q_2$

$\cdot\ X^T \simeq Z^T$ proper

$\boxed{\cdot\ \mathbb{P}, \mathbb{M} \text{ proper } \forall \beta \neq 0}$

$$GW(\varepsilon_1, \varepsilon_2, Q) := \sum_\beta Q^\beta \sum_{g \geq 0} \langle 1 \rangle^{Z,T}_{g,\beta}$$

$$\left(\langle 1 \rangle^{Z,T}_{g,\beta} := {}^T\!\!\int_{[\overline{M}_g^\bullet(Z,\beta)]^{vir}} \frac{1}{e(N_T^{vir})} \in \mathbb{Q}^{[2g-2]}(\varepsilon_1, \varepsilon_2) \right)$$

$\varepsilon_\cdot = c_1^T(q_\cdot)$

$\varepsilon_\pm = \dfrac{\varepsilon_1 \pm \varepsilon_2}{2}$

$$\rightsquigarrow GW \in \frac{1}{\varepsilon_1\varepsilon_2}\mathbb{Q}[[\varepsilon_+^2, \varepsilon_1\varepsilon_2, Q]]$$

$$= \sum_{k,g} GW_{k,g}(Q)\, \varepsilon_+^{2k} (\varepsilon_1\varepsilon_2)^{g-1}$$

This any good?

T: CY action on X

$\lambda \mathbb{C}^2$

0) unrefined limit ($q_+ \to 1$ / $\varepsilon_+ \to 0$)

A) match w/ refined GV/PT?

B) " " " MS?

0) ✓

$$[q] := q^{1/2} - q^{-1/2}$$

Pf: $\displaystyle\int_{[\overline{M}(Z)]^{vir}} (\quad) = \int_{[\overline{M}(X)]^{vir}} (\quad) \cdot \Lambda^{\vee}(\varepsilon_1)\Lambda^{\vee}(\varepsilon_2)$

$$\Lambda^{\vee}(\varepsilon) = \sum_{k=0}^{g} (-1)^k \lambda_k \, \varepsilon^{g-k-1} \qquad \lambda_k = c_k(\overset{\text{Hodge}}{\mathbb{E}})$$

$\varepsilon_+ = 0$: $\quad \Lambda^{\vee}(\varepsilon_-)\Lambda^{\vee}(-\varepsilon_-) = (-\varepsilon_-^2)^{g-1}$ (Mumford)

A) Thm A1: let $X = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$. Then.

$$\exp(GW) = \exp\left(\sum_{d=1}^{\infty} \frac{Q^d}{d[q_1^d][q_2^d]}\right) = \mathrm{Exp}(GV) = PT$$

Thm A2 (NS limit for local dPezzo): let S be a del Pezzo surface, $D \in |-K_S|$ smooth, $X := \mathrm{Tot}_S(\mathcal{O}(-D))$. Then

$$\boxed{\text{i)} \quad \lim_{\varepsilon_2 \to 0} \varepsilon_2 GW = GW^{rel}(S|D)}$$

ii) $S = \mathbb{P}^2$, $D = E$:

$$\lim_{\varepsilon_2 \to 0} \varepsilon_2 GW = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathrm{Exp}(GV)\Big|_{u_1 = e^{\varepsilon_1}}$$

(Bousseau)

B) Refined mirror symmetry $[X = K_{\mathbb{P}^2}]$ (Huang–Klemm)

$GW_{(K,g)}$ def'd recursively:

from (P1) $GW_{(K,g)} \in QMod^{[0, 3(g+K-1))}(\Gamma_1(3))$

(P2) refined HAE

(P3) refined CTC $\pi : K_{\mathbb{P}^2} \times \mathbb{C}^2 \to \mathbb{C}^5/\mu_3$

(P4) ~~at/at~~ "conifold gap condition"

<u>Thm B)</u>: P1-4 hold for $X = \mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-3))$

<u>Pf</u>: Givental-Teleman group action on 1/2-simple CohFTs

$$GW_g(t_{cf}) \overset{(!)}{=} \frac{B_{2g}}{2g(2g-2)} \frac{1}{t_{cf}^{2g-2}} + 0 + 0 + 0 + \ldots + O(1)$$

$t_{cf} = t_{cf}(Q)$

$\frac{B_{2g}}{2g(2g-2)} \rightsquigarrow c_g(\varepsilon_+)$