# Intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and the Gromov–Witten invariants of $\mathbb{CP}^1$ with an insertion of a Hodge class

Alexandr Buryak

National Research University Higher School of Economics and
Skolkovo Institute of Science and Technology
Russian Federation

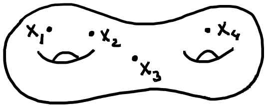
15 February 2024

# Outline

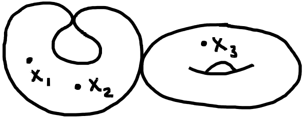
- The Kontsevich–Witten theorem for the intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$.
- Idea of the proof: a formula for the KdV Hamiltonians using the double ramification cycle and the degeneration formula for the Gromov–Witten invariants of $\mathbb{CP}^1$.
- Quantization of the KdV hierarchy and quantum intersection numbers.

# Stable algebraic curves

Smooth compact marked algebraic curves (Riemann surfaces) $(C; x_1, \ldots, x_n)$, $x_i \neq x_j$.




Marked nodal curves: all the singularities are nodes (locally, $\{xy = 0\} \subset \mathbb{C}^2$), and the marked points are smooth.



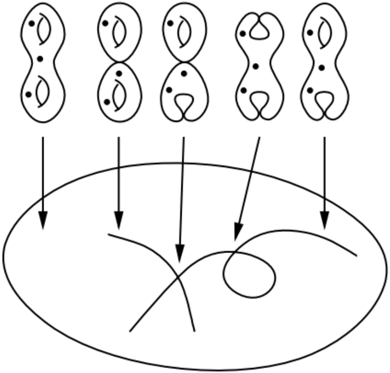

A stable curve is a marked nodal curve $(C; x_1, \ldots, x_n)$ such that $|\operatorname{Aut}(C; x_1, \ldots, x_n)| < \infty$.

# Moduli space of curves

$$\overline{\mathcal{M}}_{g,n} := \left\{ \begin{array}{c} \text{the isomorphism classes of stable algebraic} \\ \text{curves of genus } g \text{ with } n \text{ marked points} \end{array} \right\}$$

# Moduli space of curves

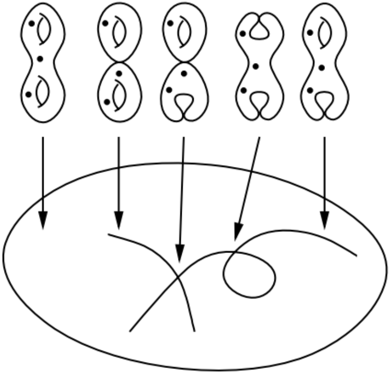
$$\overline{\mathcal{M}}_{g,n} := \left\{ \begin{array}{c} \text{the isomorphism classes of stable algebraic} \\ \text{curves of genus } g \text{ with } n \text{ marked points} \end{array} \right\}$$



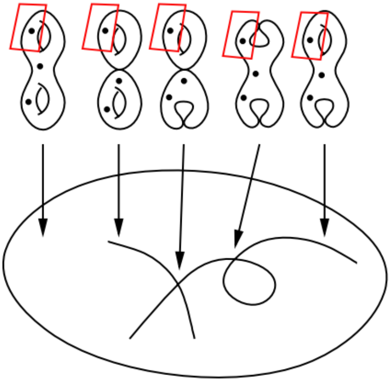
$\overline{\mathcal{M}}_{g,n}$ is a compact complex orbifold

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$$

# Intersection numbers



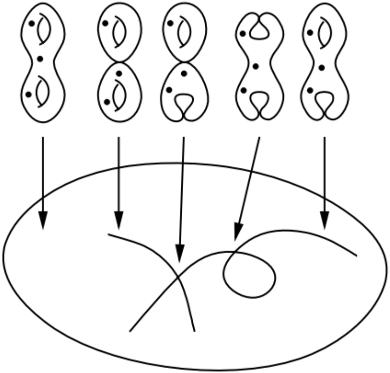
# Intersection numbers



Cotangent line bundles:

$$\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}, \quad i = 1, \ldots, n$$

# Intersection numbers

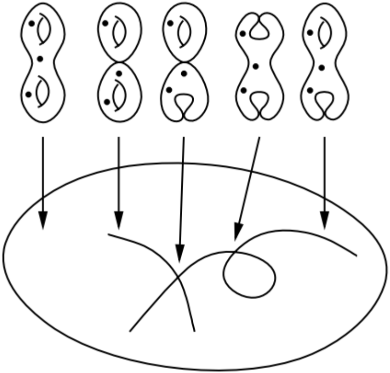


Cotangent line bundles:

$$\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}, \quad i = 1, \dots, n$$

$$\psi_i := e(\mathbb{L}_i) = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

# Intersection numbers



Cotangent line bundles:

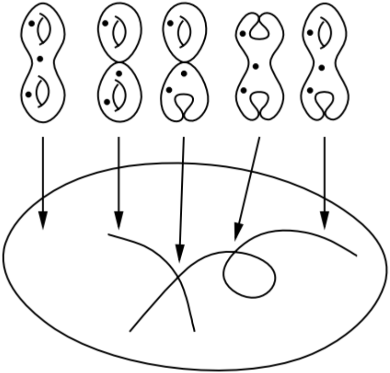
$$\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}, \quad i = 1, \ldots, n$$

$$\psi_i := e(\mathbb{L}_i) = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

Intersection numbers:

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

# Intersection numbers



Cotangent line bundles:

$$\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}, \quad i = 1, \ldots, n$$

$$\psi_i := e(\mathbb{L}_i) = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

Intersection numbers:

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

The generating series of intersection numbers:

$$\mathcal{F}(t_0, t_1, t_2, \ldots, \varepsilon) := \sum_{g,n \geq 0} \varepsilon^{2g} \sum_{d_1,\ldots,d_n \geq 0} \langle \tau_{d_1} \tau_{d_2} \ldots \tau_{d_n} \rangle_g \frac{t_{d_1} t_{d_2} \ldots t_{d_n}}{n!}$$

# Witten's conjecture

Witten's conjecture (1991, proved by Kontsevich in 1992)

- $u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the Korteweg–de Vries (KdV) equation (we identify $x = t_0$)

$$\frac{\partial u}{\partial t_1} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}.$$

# Witten's conjecture

Witten's conjecture (1991, proved by Kontsevich in 1992)

- $u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the Korteweg–de Vries (KdV) equation (we identify $x = t_0$)

$$\frac{\partial u}{\partial t_1} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}.$$

- Moreover, $u$ is a solution of the whole hierarchy of infinitesimal symmetries of the KdV equation

$$\frac{\partial u}{\partial t_2} = \frac{u^2 u_x}{2} + \varepsilon^2 \left( \frac{u u_{xxx}}{12} + \frac{u_x u_{xx}}{6} \right) + \varepsilon^4 \frac{u_{xxxxx}}{240},$$

$$\frac{\partial u}{\partial t_n} = \frac{u^n u_x}{n!} + \dots, \quad n \geq 3.$$

# Witten's conjecture

Witten's conjecture (1991, proved by Kontsevich in 1992)

- $u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the Korteweg–de Vries (KdV) equation (we identify $x = t_0$)
$$\frac{\partial u}{\partial t_1} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}.$$
- Moreover, $u$ is a solution of the whole hierarchy of infinitesimal symmetries of the KdV equation
$$\frac{\partial u}{\partial t_2} = \frac{u^2 u_x}{2} + \varepsilon^2 \left( \frac{u u_{xxx}}{12} + \frac{u_x u_{xx}}{6} \right) + \varepsilon^4 \frac{u_{xxxxx}}{240},$$
$$\frac{\partial u}{\partial t_n} = \frac{u^n u_x}{n!} + \dots, \quad n \geq 3.$$

Together with the string equation $\frac{\partial \mathcal{F}}{\partial t_0} = \sum_{k \geq 0} t_{k+1} \frac{\partial \mathcal{F}}{\partial t_k} + \frac{t_0^2}{2}$, this determines all the intersection numbers.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ I

For $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)$ the moduli space of stable relative maps to $(\mathbb{CP}^1, 0, \infty)$.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ I

For $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)$ the moduli space of stable relative maps to $(\mathbb{CP}^1, 0, \infty)$.

Let $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu) \to \mathbb{CP}^1$ be the evaluation maps.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ I

For $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)$ the moduli space of stable relative maps to $(\mathbb{CP}^1, 0, \infty)$.

Let $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu) \to \mathbb{CP}^1$ be the evaluation maps.

$\omega \in H^2(\mathbb{CP}^1)$ is the class dual to a point.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ I

For $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)$ the moduli space of stable relative maps to $(\mathbb{CP}^1, 0, \infty)$.

Let $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu) \to \mathbb{CP}^1$ be the evaluation maps.

$\omega \in H^2(\mathbb{CP}^1)$ is the class dual to a point.

Note that $\deg \left[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)\right]^{\mathrm{vir}} = 2(2g - 2 + l(\mu) + l(\nu) + n)$.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ I

For $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)$ the moduli space of stable relative maps to $(\mathbb{CP}^1, 0, \infty)$.

Let $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu) \to \mathbb{CP}^1$ be the evaluation maps.

$\omega \in H^2(\mathbb{CP}^1)$ is the class dual to a point.

Note that $\deg \left[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)\right]^{\mathrm{vir}} = 2(2g - 2 + l(\mu) + l(\nu) + n)$.

The space of holomorphic differentials on any nodal curve is $g$-dimensional. $\rightsquigarrow$ Rank $g$ vector bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu)$ called the Hodge bundle.

The Hodge classes $\lambda_i := c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, \mu, \nu))$.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ II

For $\overline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$, consider the integral

$$P_{g,\overline{d}}(a_1, \ldots, a_k) := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \ldots, a_k))]^{\mathrm{vir}}} \lambda_g \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega),$$

where $A := \sum a_i$.

The integral is nonzero only if $\sum d_i = g - 1 + k$.

# Relation with the Gromov–Witten invariants of $\mathbb{CP}^1$ II

For $\overline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$, consider the integral

$$P_{g,\overline{d}}(a_1, \ldots, a_k) := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \ldots, a_k))]^{\mathrm{vir}}} \lambda_g \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega),$$

where $A := \sum a_i$.

The integral is nonzero only if $\sum d_i = g - 1 + k$.

**Theorem (Blot–B., 2023)**

1. *$P_{g,\overline{d}}(a_1, \ldots, a_k)$ is a polynomial in $a_1, \ldots, a_k$, homogeneous of degree $2g + n - 1$.*
2. *For $\sum d_i = 3g - 2 + n$ and $k = 2g + n - 1$, we have*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \frac{1}{k!} \mathrm{Coef}_{a_1 \cdots a_k} P_{g,\overline{d}}.$$

# Conserved quantities of the KdV equation

KdV equation $u_t = uu_x + \frac{\varepsilon^2}{12} u_{xxx}$.

Suppose $\varepsilon \in \mathbb{R}$, $u = u(x,t)$ is a smooth function, and $u$ together with sufficient number of $x$-derivatives go to zero when $x \to \pm\infty$.

# Conserved quantities of the KdV equation

KdV equation $u_t = uu_x + \frac{\varepsilon^2}{12}u_{xxx}$.

Suppose $\varepsilon \in \mathbb{R}$, $u = u(x,t)$ is a smooth function, and $u$ together with sufficient number of $x$-derivatives go to zero when $x \to \pm\infty$.

$$\left(\int_{\mathbb{R}} u dx\right)_t = \int_{\mathbb{R}} \partial_x \left[\frac{u^2}{2} + \frac{\varepsilon^2}{12}u_{xx}\right] dx = 0,$$

$$\left(\int_{\mathbb{R}} u^2 dx\right)_t = \int_{\mathbb{R}} \partial_x \left[\frac{2u^3}{3} + \frac{\varepsilon^2}{6}uu_{xx} - \frac{\varepsilon^2}{12}u_x^2\right] dx = 0,$$

$$\left(\int_{\mathbb{R}} \left(u^3 + \frac{\varepsilon^2}{4}uu_{xx}\right) dx\right)_t = \int_{\mathbb{R}} \partial_x \left[\frac{3u^4}{4} + \frac{\varepsilon^2}{2}u^2u_{xx} + \frac{\varepsilon^4}{48}\left(uu_{xxxx} + u_{xx}^2 - u_{xxx}u_x\right)\right] dx = 0.$$

# Conserved quantities of the KdV equation

KdV equation $u_t = uu_x + \frac{\varepsilon^2}{12} u_{xxx}$.

Suppose $\varepsilon \in \mathbb{R}$, $u = u(x,t)$ is a smooth function, and $u$ together with sufficient number of $x$-derivatives go to zero when $x \to \pm\infty$.

$$\left(\int_{\mathbb{R}} u dx\right)_t = \int_{\mathbb{R}} \partial_x \left[\frac{u^2}{2} + \frac{\varepsilon^2}{12} u_{xx}\right] dx = 0,$$

$$\left(\int_{\mathbb{R}} u^2 dx\right)_t = \int_{\mathbb{R}} \partial_x \left[\frac{2u^3}{3} + \frac{\varepsilon^2}{6} uu_{xx} - \frac{\varepsilon^2}{12} u_x^2\right] dx = 0,$$

$$\left(\int_{\mathbb{R}} \left(u^3 + \frac{\varepsilon^2}{4} uu_{xx}\right) dx\right)_t = \int_{\mathbb{R}} \partial_x \left[\frac{3u^4}{4} + \frac{\varepsilon^2}{2} u^2 u_{xx} + \frac{\varepsilon^4}{48}\left(uu_{xxxx} + u_{xx}^2 - u_{xxx}u_x\right)\right] dx = 0.$$

For each $n \geq 0$, there is a conserved quantity $\overline{h}_n$ of the form

$$\overline{h}_n = \int_{\mathbb{R}} \left(\frac{u^{n+2}}{(n+2)!} + O(\varepsilon^2)\right) dx.$$

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n \geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n\geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

Space of local functionals $\widehat{\Lambda} := \widehat{\mathcal{A}}/\mathrm{Im}(\partial_{\mathrm{x}})$.

# Space of local functionals, Poisson structure

Algebra of <u>differential polynomials</u> $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n\geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

Space of <u>local functionals</u> $\widehat{\Lambda} := \widehat{\mathcal{A}}/\mathrm{Im}(\partial_\mathrm{x})$.

Projection $\widehat{\mathcal{A}} \to \widehat{\Lambda}$, $f \mapsto \int f dx$.

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n\geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

Space of local functionals $\widehat{\Lambda} := \widehat{\mathcal{A}}/\mathrm{Im}(\partial_{\mathrm{x}})$.

Projection $\widehat{\mathcal{A}} \to \widehat{\Lambda}$, $f \mapsto \int f dx$.

Conserved quantities of the KdV equation $\overline{h}_n \in \widehat{\Lambda}$.

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n\geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

Space of local functionals $\widehat{\Lambda} := \widehat{\mathcal{A}}/\mathrm{Im}(\partial_{\mathrm{x}})$.

Projection $\widehat{\mathcal{A}} \to \widehat{\Lambda}$, $f \mapsto \int f dx$.

Conserved quantities of the KdV equation $\overline{h}_n \in \widehat{\Lambda}$.

Variational derivative $\frac{\delta}{\delta u} := \sum_{n\geq 0} (-\partial_x)^n \circ \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n\geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

Space of local functionals $\widehat{\Lambda} := \widehat{\mathcal{A}}/\mathrm{Im}(\partial_\mathrm{x})$.

Projection $\widehat{\mathcal{A}} \to \widehat{\Lambda}$, $f \mapsto \int f dx$.

Conserved quantities of the KdV equation $\overline{h}_n \in \widehat{\Lambda}$.

Variational derivative $\frac{\delta}{\delta u} := \sum_{n\geq 0}(-\partial_x)^n \circ \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

$\mathrm{Im}(\partial_\mathrm{x}) \subset \mathrm{Ker}(\frac{\delta}{\delta \mathrm{u}}) \Rightarrow \frac{\delta}{\delta u} : \widehat{\Lambda} \to \widehat{\mathcal{A}}$ is well defined.

# Space of local functionals, Poisson structure

Algebra of differential polynomials $\widehat{\mathcal{A}} := \mathbb{C}[u, u_x, u_{xx}, \ldots, \varepsilon]$, $u_i := \partial_x^i u$.

$\partial_x := \sum_{n\geq 0} u_{n+1} \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

Space of local functionals $\widehat{\Lambda} := \widehat{\mathcal{A}}/\mathrm{Im}(\partial_{\mathrm{x}})$.

Projection $\widehat{\mathcal{A}} \to \widehat{\Lambda}$, $f \mapsto \int f dx$.

Conserved quantities of the KdV equation $\overline{h}_n \in \widehat{\Lambda}$.

Variational derivative $\frac{\delta}{\delta u} := \sum_{n\geq 0} (-\partial_x)^n \circ \frac{\partial}{\partial u_n} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$.

$\mathrm{Im}(\partial_{\mathrm{x}}) \subset \mathrm{Ker}(\frac{\delta}{\delta \mathrm{u}}) \Rightarrow \frac{\delta}{\delta u} : \widehat{\Lambda} \to \widehat{\mathcal{A}}$ is well defined.

Poisson bracket on $\widehat{\Lambda}$: $\{\overline{f}, \overline{g}\} := \int \frac{\delta \overline{f}}{\delta u} \partial_x \frac{\delta \overline{g}}{\delta u} dx$.

# The Hamiltonian structure of the KdV hierarchy

The KdV hierarchy is Hamiltonian:

$$\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\} = \partial_x \frac{\delta \overline{h}_n}{\delta u}, \qquad \{\overline{h}_m, \overline{h}_n\} = 0.$$

# The Hamiltonian structure of the KdV hierarchy

The KdV hierarchy is Hamiltonian:

$$\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\} = \partial_x \frac{\delta \overline{h}_n}{\delta u}, \qquad \{\overline{h}_m, \overline{h}_n\} = 0.$$

The first few Hamiltonians are

$$\overline{h}_0 = \int \frac{u^2}{2} dx,$$
$$\overline{h}_1 = \int \left( \frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx} \right) dx,$$
$$\overline{h}_2 = \int \left( \frac{u^4}{24} + \frac{\varepsilon^2}{48} u^2 u_{xx} + \frac{\varepsilon^4}{480} u_{xx}^2 \right).$$

# The Hamiltonian structure of the KdV hierarchy

The KdV hierarchy is Hamiltonian:

$$\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\} = \partial_x \frac{\delta \overline{h}_n}{\delta u}, \qquad \{\overline{h}_m, \overline{h}_n\} = 0.$$

The first few Hamiltonians are

$$\overline{h}_0 = \int \frac{u^2}{2} dx,$$
$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx}\right) dx,$$
$$\overline{h}_2 = \int \left(\frac{u^4}{24} + \frac{\varepsilon^2}{48} u^2 u_{xx} + \frac{\varepsilon^4}{480} u_{xx}^2\right).$$

The Hamiltonian $\overline{h}_1$ gives the KdV equation
$\{u, \overline{h}_1\} = \partial_x \frac{\delta \overline{h}_1}{\delta u} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}$.

# More on the intersection numbers

$u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the KdV hierarchy $\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\}$, with the initial condition $u|_{t_{\geq 1}=0} = x$ (recall that we identify $t_0 = x$).

# More on the intersection numbers

$u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the KdV hierarchy $\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\}$, with the initial condition $u|_{t_{\geq 1}=0} = x$ (recall that we identify $t_0 = x$).

$$\left.\frac{\partial^n u}{\partial t_{d_1} \dots \partial t_{d_n}}\right|_{t_*=0} = \left.\frac{\partial^{n+2} \mathcal{F}}{\partial t_0^2 \partial t_{d_1} \dots \partial t_{d_n}}\right|_{t_*=0} =$$
$$= \left\{\left\{\dots\left\{\left\{u, \overline{h}_{d_1}\right\}, \overline{h}_{d_2}\right\}, \dots\right\}, \overline{h}_{d_n}\right\}\Big|_{u_k=\delta_{k,1}},$$

# More on the intersection numbers

$u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the KdV hierarchy $\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\}$, with the initial condition $u|_{t_{\geq 1}=0} = x$ (recall that we identify $t_0 = x$).

$$\left.\frac{\partial^n u}{\partial t_{d_1} \dots \partial t_{d_n}}\right|_{t_*=0} = \left.\frac{\partial^{n+2} \mathcal{F}}{\partial t_0^2 \partial t_{d_1} \dots \partial t_{d_n}}\right|_{t_*=0} =$$

$$= \left\{\left\{\dots\left\{\left\{u, \overline{h}_{d_1}\right\}, \overline{h}_{d_2}\right\}, \dots\right\}, \overline{h}_{d_n}\right\}\Big|_{u_k=\delta_{k,1}},$$

or equivalently

$$\boxed{\left.\frac{\partial^{n+1} \mathcal{F}}{\partial t_0 \partial t_{d_1} \dots \partial t_{d_n}}\right|_{t_*=0} = \left\{\left\{\dots\left\{\left\{\frac{\delta \overline{h}_{d_1}}{\delta u}, \overline{h}_{d_2}\right\}, \overline{h}_{d_3}\right\}, \dots\right\}, \overline{h}_{d_n}\right\}\Bigg|_{u_k=\delta_{k,1}}}$$

# More on the Poisson structure on $\widehat{\Lambda}$

The Poisson structure on $\widehat{\Lambda}$ can be described in more familiar terms.

$\widehat{\mathcal{B}} := \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$.

Poisson bracket on $\widehat{\mathcal{B}}$: $\{P, Q\} := \sum_{k\in\mathbb{Z}} ik \frac{\partial P}{\partial p_k}\frac{\partial Q}{\partial p_{-k}}$, $P, Q \in \widehat{\mathcal{B}}$.

# More on the Poisson structure on $\widehat{\Lambda}$

The Poisson structure on $\widehat{\Lambda}$ can be described in more familiar terms.

$\widehat{\mathcal{B}} := \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$.

Poisson bracket on $\widehat{\mathcal{B}}$: $\{P, Q\} := \sum_{k\in\mathbb{Z}} ik \frac{\partial P}{\partial p_k} \frac{\partial Q}{\partial p_{-k}}$, $P, Q \in \widehat{\mathcal{B}}$.

Linear map $\phi$: $\widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$, $f \mapsto \mathrm{Coef}_{e^{i0x}} \left( f|_{u_k = \sum_{a\in\mathbb{Z}} (ia)^k p_a e^{iax}} \right)$ (informally, we consider the expansion of $u$ in the Fourier series "$u(x) = \sum_{a\in\mathbb{Z}} p_a e^{iax}$").

# More on the Poisson structure on $\widehat{\Lambda}$

The Poisson structure on $\widehat{\Lambda}$ can be described in more familiar terms.

$\widehat{\mathcal{B}} := \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$.

Poisson bracket on $\widehat{\mathcal{B}}$: $\{P, Q\} := \sum_{k\in\mathbb{Z}} ik \frac{\partial P}{\partial p_k}\frac{\partial Q}{\partial p_{-k}}$, $P, Q \in \widehat{\mathcal{B}}$.

Linear map $\phi\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$, $f \mapsto \mathrm{Coef}_{e^{i0x}}\left(f|_{u_k=\sum_{a\in\mathbb{Z}}(ia)^k p_a e^{iax}}\right)$ (informally, we consider the expansion of $u$ in the Fourier series "$u(x) = \sum_{a\in\mathbb{Z}} p_a e^{iax}$").

$\mathrm{Im}(\partial_x) \subset \mathrm{Ker}(\phi) \Rightarrow \phi\colon \widehat{\Lambda} \to \widehat{\mathcal{B}}$ is well defined.

$\phi\colon \widehat{\Lambda} \to \widehat{\mathcal{B}}$ is an injection.

$\phi(\{\overline{f}, \overline{g}\}) = \{\phi(\overline{f}), \phi(\overline{g})\}, \quad \overline{f}, \overline{g} \in \widehat{\Lambda}$.

We will denote $\phi(\overline{h}_n) \in \widehat{\mathcal{B}}$ by $\overline{h}_n$.

# More on the Poisson structure on $\widehat{\Lambda}$

The Poisson structure on $\widehat{\Lambda}$ can be described in more familiar terms.

$\widehat{\mathcal{B}} := \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$.

Poisson bracket on $\widehat{\mathcal{B}}$: $\{P, Q\} := \sum_{k\in\mathbb{Z}} ik \frac{\partial P}{\partial p_k}\frac{\partial Q}{\partial p_{-k}}, P, Q \in \widehat{\mathcal{B}}$.

Linear map $\phi\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$, $f \mapsto \mathrm{Coef}_{e^{i0x}}\left(f|_{u_k=\sum_{a\in\mathbb{Z}}(ia)^k p_a e^{iax}}\right)$ (informally, we consider the expansion of $u$ in the Fourier series "$u(x) = \sum_{a\in\mathbb{Z}} p_a e^{iax}$").

$\mathrm{Im}(\partial_x) \subset \mathrm{Ker}(\phi) \Rightarrow \phi\colon \widehat{\Lambda} \to \widehat{\mathcal{B}}$ is well defined.

$\phi\colon \widehat{\Lambda} \to \widehat{\mathcal{B}}$ is an injection.

$\phi(\{\overline{f}, \overline{g}\}) = \{\phi(\overline{f}), \phi(\overline{g})\}, \quad \overline{f}, \overline{g} \in \widehat{\Lambda}$.

We will denote $\phi(\overline{h}_n) \in \widehat{\mathcal{B}}$ by $\overline{h}_n$.

> We will give an explicit formula for the KdV Hamiltonians $\overline{h}_n$, as elements of $\widehat{\mathcal{B}}$, using the moduli space $\overline{\mathcal{M}}_{g,n}$.

# The double ramifiction cycles on $\overline{\mathcal{M}}_{g,n}$

$(a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\sum a_i = 0$.

The partition $\mu$ is formed by the positive numbers among the $a_i$-s. The partition $\nu$ is formed by the negatives of the negative numbers among the $a_i$-s. $n_0$ is the number of zeroes among the $a_i$-s.

# The double ramifiction cycles on $\overline{\mathcal{M}}_{g,n}$

$(a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\sum a_i = 0$.

The partition $\mu$ is formed by the positive numbers among the $a_i$-s. The partition $\nu$ is formed by the negatives of the negative numbers among the $a_i$-s. $n_0$ is the number of zeroes among the $a_i$-s.

The space $\overline{\mathcal{M}}^{\sim}_{g,n_0}(\mu,\nu)$ is the moduli space of stable relative maps to rubber $\mathbb{CP}^1$ (stable relative maps that differ by an automorphism of the target are considered as isomorphic).

# The double ramifiction cycles on $\overline{\mathcal{M}}_{g,n}$

$(a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\sum a_i = 0$.

The partition $\mu$ is formed by the positive numbers among the $a_i$-s. The partition $\nu$ is formed by the negatives of the negative numbers among the $a_i$-s. $n_0$ is the number of zeroes among the $a_i$-s.

The space $\overline{\mathcal{M}}^{\sim}_{g,n_0}(\mu,\nu)$ is the moduli space of stable relative maps to rubber $\mathbb{CP}^1$ (stable relative maps that differ by an automorphism of the target are considered as isomorphic).

The map $\mathrm{st}\colon \overline{\mathcal{M}}^{\sim}_{g,n_0}(\mu,\nu) \to \overline{\mathcal{M}}_{g,n}$ assigns to a stable relative map the source curve.

The class $\mathrm{DR}_g(a_1,\ldots,a_n) := \mathrm{st}_*([\overline{\mathcal{M}}^{\sim}_{g,n_0}(\mu,\nu)]^{\mathrm{vir}}) \in H^{2g}(\overline{\mathcal{M}}_{g,n})$ is called the double ramification (DR) cycle.

# The KdV Hamiltonians and the double ramification cycles

Here is an explicit geometric formula for the KdV Hamiltonians, as elements of $\widehat{\mathcal{B}}$.

**Theorem (B., 2015)**

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \frac{(-\varepsilon^2)^g}{n!} \sum_{\substack{a_1,\ldots,a_n\in\mathbb{Z} \\ \sum a_i=0}} \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \lambda_g \mathrm{DR}_g(0,a_1,\ldots,a_n)\right) \prod_{i=1}^n p_{a_i}.$$

# The KdV Hamiltonians and the double ramification cycles

Here is an explicit geometric formula for the KdV Hamiltonians, as elements of $\widehat{\mathcal{B}}$.

**Theorem (B., 2015)**

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \frac{(-\varepsilon^2)^g}{n!} \sum_{\substack{a_1,\ldots,a_n\in\mathbb{Z}\\ \sum a_i=0}} \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \lambda_g \mathrm{DR}_g(0,a_1,\ldots,a_n)\right) \prod_{i=1}^n p_{a_i}.$$

Example: consider

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx}\right) dx = \sum_{a_1+a_2+a_3=0} \frac{p_{a_1}p_{a_2}p_{a_3}}{6} - \varepsilon^2 \sum_{a\in\mathbb{Z}} a^2 \frac{p_a p_{-a}}{24}$$

# The KdV Hamiltonians and the double ramification cycles

Here is an explicit geometric formula for the KdV Hamiltonians, as elements of $\widehat{\mathcal{B}}$.

**Theorem (B., 2015)**

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \frac{(-\varepsilon^2)^g}{n!} \sum_{\substack{a_1,\ldots,a_n\in\mathbb{Z}\\ \sum a_i=0}} \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \lambda_g \mathrm{DR}_g(0,a_1,\ldots,a_n)\right) \prod_{i=1}^n p_{a_i}.$$

Example: consider

$$\overline{h}_1 = \int\left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx}\right)dx = \sum_{a_1+a_2+a_3=0} \frac{p_{a_1}p_{a_2}p_{a_3}}{6} - \varepsilon^2 \sum_{a\in\mathbb{Z}} a^2 \frac{p_a p_{-a}}{24}$$

Then

$$\sum_{a_1+a_2+a_3=0} \frac{p_{a_1}p_{a_2}p_{a_3}}{6} \quad \rightsquigarrow \quad \int_{\overline{\mathcal{M}}_{0,4}} \psi_1 \lambda_0 \mathrm{DR}_0(0,a_1,a_2,a_3) = 1$$

# The KdV Hamiltonians and the double ramification cycles

Here is an explicit geometric formula for the KdV Hamiltonians, as elements of $\widehat{\mathcal{B}}$.

**Theorem (B., 2015)**

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \frac{(-\varepsilon^2)^g}{n!} \sum_{\substack{a_1,\ldots,a_n\in\mathbb{Z}\\ \sum a_i=0}} \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \lambda_g \mathrm{DR}_g(0,a_1,\ldots,a_n) \right) \prod_{i=1}^n p_{a_i}.$$

Example: consider

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx}\right) dx = \sum_{a_1+a_2+a_3=0} \frac{p_{a_1}p_{a_2}p_{a_3}}{6} - \varepsilon^2 \sum_{a\in\mathbb{Z}} a^2 \frac{p_a p_{-a}}{24}$$

Then

$$\sum_{a_1+a_2+a_3=0} \frac{p_{a_1}p_{a_2}p_{a_3}}{6} \quad \leftrightsquigarrow \quad \int_{\overline{\mathcal{M}}_{0,4}} \psi_1 \lambda_0 \mathrm{DR}_0(0,a_1,a_2,a_3) = 1$$

$$\sum_{a\in\mathbb{Z}} a^2 \frac{p_a p_{-a}}{24} \quad \leftrightsquigarrow \quad \int_{\overline{\mathcal{M}}_{1,3}} \psi_1 \lambda_1 \mathrm{DR}_1(0,a,-a) = \frac{a^2}{12}$$

# Applying the degeneration formula

The theorem is proved using the formulas

$$\left.\frac{\partial^{n+1}\mathcal{F}}{\partial t_0 \partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \left.\left\{\left\{\ldots\left\{\left\{\frac{\delta\overline{h}_{d_1}}{\delta u},\overline{h}_{d_2}\right\},\overline{h}_{d_3}\right\},\ldots\right\},\overline{h}_{d_n}\right\}\right|_{u_k=\delta_{k,1}}$$

and

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2}\frac{(-\varepsilon^2)^g}{n!}\sum_{\substack{a_1,\ldots,a_n\in\mathbb{Z}\\ \sum a_i=0}}\left(\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_1^d\lambda_g \mathrm{DR}_g(0,a_1,\ldots,a_n)\right)\prod_{i=1}^n p_{a_i}$$

together with the degeneration formula in the Gromov–Witten theory of $\mathbb{CP}^1$.

# Towards quantum KdV I

We want to describe a certain quantization of the KdV hierarchy. What do we mean by this?

# Towards quantum KdV I

We want to describe a certain quantization of the KdV hierarchy. What do we mean by this?

Baby example: Weyl algebra $W = \mathbb{C}[x, p, \hbar] = \{\sum_i f_i(x, \hbar) p^i\}$.

Viewing "$p = \hbar \partial_x$", noncommutative multiplication $\star$ is given by the composition of operators. For example, $p \star x = x \star p + \hbar$.

Equivalently, $f \star g = \sum_{k \geq 0} \frac{\hbar^k}{k!} \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial x^k} = f \left( e^{\hbar \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial x}}} \right) g$, $f, g \in W$.

# Towards quantum KdV I

We want to describe a certain quantization of the KdV hierarchy. What do we mean by this?

Baby example: Weyl algebra $W = \mathbb{C}[x, p, \hbar] = \{\sum_i f_i(x, \hbar) p^i\}$.

Viewing "$p = \hbar\partial_x$", noncommutative multiplication $\star$ is given by the composition of operators. For example, $p \star x = x \star p + \hbar$.

Equivalently, $f \star g = \sum_{k \geq 0} \frac{\hbar^k}{k!} \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial x^k} = f \left( e^{\hbar \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial x}}} \right) g$, $f, g \in W$.

Commutator $[f, g] = f \star g - g \star f$.

Poisson structure on $\mathbb{C}[x, p]$: $\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}$.

For $f = \underbrace{f_0}_{\in \mathbb{C}[x,p]} + O(\hbar)$, $g = \underbrace{g_0}_{\in \mathbb{C}[x,p]} + O(\hbar)$, we have

$$\boxed{[f, g] = \hbar\{f_0, g_0\} + O(\hbar^2)}.$$

This gives a (deformation) quantization of the Poisson bracket on $\mathbb{C}[x, p]$.

# Towards quantum KdV II

Recall that $\widehat{\mathcal{B}} = \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$ with bracket

$$\{P, Q\} = \sum_{k \in \mathbb{Z}} ik \frac{\partial P}{\partial p_k} \frac{\partial Q}{\partial p_{-k}}.$$

# Towards quantum KdV II

Recall that $\widehat{\mathcal{B}} = \mathbb{C}[p_1, p_2, \dots][[p_0, p_{-1}, p_{-2}, \dots, \varepsilon]]$ with bracket

$$\{P, Q\} = \sum_{k \in \mathbb{Z}} ik \frac{\partial P}{\partial p_k} \frac{\partial Q}{\partial p_{-k}}.$$

Introduce

$$f \star g := f \left( e^{\sum_{k>0} i\hbar k \overleftarrow{\frac{\partial}{\partial p_k}} \overrightarrow{\frac{\partial}{\partial p_{-k}}}} \right) g, \quad f, g \in \widehat{\mathcal{B}}[[\hbar]].$$

For $f = \underbrace{f_0}_{\in \widehat{\mathcal{B}}} + O(\hbar)$, $g = \underbrace{g_0}_{\in \widehat{\mathcal{B}}} + O(\hbar)$, we have $[f, g] = \hbar\{f_0, g_0\} + O(\hbar^2)$.

# Towards quantum KdV II

Recall that $\widehat{\mathcal{B}} = \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$ with bracket

$$\{P, Q\} = \sum_{k \in \mathbb{Z}} ik \frac{\partial P}{\partial p_k} \frac{\partial Q}{\partial p_{-k}}.$$

Introduce

$$f \star g := f \left( e^{\sum_{k>0} i\hbar k \overleftarrow{\frac{\partial}{\partial p_k}} \overrightarrow{\frac{\partial}{\partial p_{-k}}}} \right) g, \quad f, g \in \widehat{\mathcal{B}}[[\hbar]].$$

For $f = \underbrace{f_0}_{\in \widehat{\mathcal{B}}} + O(\hbar)$, $g = \underbrace{g_0}_{\in \widehat{\mathcal{B}}} + O(\hbar)$, we have $[f, g] = \hbar\{f_0, g_0\} + O(\hbar^2)$.

By definition, a quantization of the KdV hierarchy is a collection of elements $\overline{H}_n = \overline{h}_n + O(\hbar) \in \widehat{\mathcal{B}}[\hbar]]$ such that $[\overline{H}_m, \overline{H}_n] = 0$.

**The moduli space $\overline{\mathcal{M}}_{g,n}$ gives a beautiful construction!**

# Quantum KdV hierarchy I

Recall

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \sum_{\substack{\overline{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d (-\varepsilon^2)^g \lambda_g \mathrm{DR}_g(0,\overline{a}) \right)$$

# Quantum KdV hierarchy I

Recall

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \sum_{\substack{\overline{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d (-\varepsilon^2)^g \lambda_g \mathrm{DR}_g(0,\overline{a}) \right)$$

Define

$$\widehat{\mathcal{B}}[[\hbar]] \ni \overline{H}_d := \sum_{g\geq 0,\, n\geq 1} \sum_{\substack{\overline{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \times$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \Big( \underbrace{(-\varepsilon^2)^g \lambda_g + (-\varepsilon^2)^{g-1} i\hbar \lambda_{g-1} + \ldots + (i\hbar)^g}_{=(-\varepsilon^2)^g\lambda_g + O(\hbar)} \Big) \mathrm{DR}_g(0,\overline{a})$$

# Quantum KdV hierarchy I

Recall

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \sum_{\substack{\overline{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d (-\varepsilon^2)^g \lambda_g \mathrm{DR}_g(0,\overline{a}) \right)$$

Define

$$\widehat{\mathcal{B}}[[\hbar]] \ni \overline{H}_d := \sum_{g\geq 0,\, n\geq 1} \sum_{\substack{\overline{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \times$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \Big( \underbrace{(-\varepsilon^2)^g \lambda_g + (-\varepsilon^2)^{g-1} i\hbar \lambda_{g-1} + \ldots + (i\hbar)^g}_{=(-\varepsilon^2)^g\lambda_g + O(\hbar)} \Big) \mathrm{DR}_g(0,\overline{a})$$

Obviously, $\overline{H}_n = \overline{h}_n + O(\hbar)$.

# Quantum KdV hierarchy I

Recall

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \sum_{\substack{\overline{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d (-\varepsilon^2)^g \lambda_g \mathrm{DR}_g(0,\overline{a}) \right)$$

Define

$$\widehat{\mathcal{B}}[[\hbar]] \ni \overline{H}_d := \sum_{g\geq 0,\, n\geq 1} \sum_{\substack{\overline{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \times$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \Big( \underbrace{(-\varepsilon^2)^g \lambda_g + (-\varepsilon^2)^{g-1} i\hbar \lambda_{g-1} + \ldots + (i\hbar)^g}_{=(-\varepsilon^2)^g\lambda_g + O(\hbar)} \Big) \mathrm{DR}_g(0,\overline{a})$$

Obviously, $\overline{H}_n = \overline{h}_n + O(\hbar)$.

**Theorem (B.–Rossi, 2016)**

$$[\overline{H}_m, \overline{H}_n] = 0.$$

# Quantum KdV hierarchy II

The integral

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \big((-\varepsilon^2)^g \lambda_g + (-\varepsilon^2)^{g-1} i\hbar \lambda_{g-1} + \ldots + (i\hbar)^g\big) \mathrm{DR}_g(0, a_1, \ldots, a_n)$$

is a polynomial in $a_1, \ldots, a_n$.

# Quantum KdV hierarchy II

The integral

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d\big((-\varepsilon^2)^g\lambda_g + (-\varepsilon^2)^{g-1} i\hbar\lambda_{g-1} + \ldots + (i\hbar)^g\big)\mathrm{DR}_g(0, a_1, \ldots, a_n)$$

is a polynomial in $a_1, \ldots, a_n$.

This implies that $\overline{H}_n = \int H_n dx$ for some $H_n \in \widehat{\mathcal{A}}[\hbar]$.

# Quantum KdV hierarchy II

The integral

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d\big((-\varepsilon^2)^g\lambda_g + (-\varepsilon^2)^{g-1} i\hbar\lambda_{g-1} + \ldots + (i\hbar)^g\big)\mathrm{DR}_g(0,a_1,\ldots,a_n)$$

is a polynomial in $a_1,\ldots,a_n$.

This implies that $\overline{H}_n = \int H_n dx$ for some $H_n \in \widehat{\mathcal{A}}[\hbar]$.

For example,

$$\overline{H}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24}uu_{xx} - \frac{i\hbar}{24}u\right)dx,$$

$$\overline{H}_2 = \int \left(\frac{u^4}{24} + \varepsilon^2\frac{u^2u_{xx}}{48} + \varepsilon^4\frac{uu_{xxxx}}{480} - i\hbar\frac{2uu_{xx}+u^2}{48} - i\hbar\varepsilon^2\frac{u}{2880}\right)dx.$$

# Quantum intersection numbers

Recall

$$\frac{\partial^{n+1}\mathcal{F}}{\partial t_0 \partial t_{d_1} \ldots \partial t_{d_n}}\Big|_{t_*=0} = \left\{ \left\{ \ldots \left\{ \left\{ \frac{\delta \overline{h}_{d_1}}{\delta u}, \overline{h}_{d_2} \right\}, \overline{h}_{d_3} \right\}, \ldots \right\}, \overline{h}_{d_n} \right\}\Big|_{u_k=\delta_{k,1}}.$$

# Quantum intersection numbers

Recall

$$\left.\frac{\partial^{n+1}\mathcal{F}}{\partial t_0 \partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \left.\left\{\left\{\ldots\left\{\left\{\frac{\delta \overline{h}_{d_1}}{\delta u}, \overline{h}_{d_2}\right\}, \overline{h}_{d_3}\right\}, \ldots\right\}, \overline{h}_{d_n}\right\}\right|_{u_k=\delta_{k,1}}.$$

Define $\mathcal{F}^{(q)} \in \mathbb{C}[[t_0, t_1, \ldots, \varepsilon, \hbar]]$ by the relations

$$\left.\frac{\partial^{n+1}\mathcal{F}^{(q)}}{\partial t_0 \partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \hbar^{1-n} \left.\left[\left[\ldots\left[\left[\frac{\delta \overline{H}_{d_1}}{\delta u}, \overline{H}_{d_2}\right], \overline{H}_{d_3}\right], \ldots\right], \overline{H}_{d_n}\right]\right|_{u_k=\delta_{k,1}},$$

$$\frac{\partial \mathcal{F}^{(q)}}{\partial t_0} = \sum_{i\geq 0} t_{i+1}\frac{\partial \mathcal{F}^{(q)}}{\partial t_i} + \frac{t_0^2}{2} - \frac{i\hbar}{24}.$$

# Quantum intersection numbers

Recall

$$\left.\frac{\partial^{n+1}\mathcal{F}}{\partial t_0\partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0}=\left\{\left\{\ldots\left\{\left\{\frac{\delta\overline{h}_{d_1}}{\delta u},\overline{h}_{d_2}\right\},\overline{h}_{d_3}\right\},\ldots\right\},\overline{h}_{d_n}\right\}\Big|_{u_k=\delta_{k,1}}.$$

Define $\mathcal{F}^{(q)}\in\mathbb{C}[[t_0,t_1,\ldots,\varepsilon,\hbar]]$ by the relations

$$\left.\frac{\partial^{n+1}\mathcal{F}^{(q)}}{\partial t_0\partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0}=\hbar^{1-n}\left[\left[\ldots\left[\left[\frac{\delta\overline{H}_{d_1}}{\delta u},\overline{H}_{d_2}\right],\overline{H}_{d_3}\right],\ldots\right],\overline{H}_{d_n}\right]\Big|_{u_k=\delta_{k,1}},$$

$$\frac{\partial\mathcal{F}^{(q)}}{\partial t_0}=\sum_{i\geq 0}t_{i+1}\frac{\partial\mathcal{F}^{(q)}}{\partial t_i}+\frac{t_0^2}{2}-\frac{i\hbar}{24}.$$

Obviously, $\mathcal{F}^{(q)}=\mathcal{F}+O(\hbar)$.

# Quantum intersection numbers

Recall

$$\left.\frac{\partial^{n+1}\mathcal{F}}{\partial t_0 \partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \left.\left\{\left\{\ldots\left\{\left\{\frac{\delta \overline{h}_{d_1}}{\delta u}, \overline{h}_{d_2}\right\}, \overline{h}_{d_3}\right\}, \ldots\right\}, \overline{h}_{d_n}\right\}\right|_{u_k=\delta_{k,1}}.$$

Define $\mathcal{F}^{(q)} \in \mathbb{C}[[t_0, t_1, \ldots, \varepsilon, \hbar]]$ by the relations

$$\left.\frac{\partial^{n+1}\mathcal{F}^{(q)}}{\partial t_0 \partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \hbar^{1-n}\left.\left[\left[\ldots\left[\left[\frac{\delta \overline{H}_{d_1}}{\delta u}, \overline{H}_{d_2}\right], \overline{H}_{d_3}\right], \ldots\right], \overline{H}_{d_n}\right]\right|_{u_k=\delta_{k,1}},$$

$$\frac{\partial \mathcal{F}^{(q)}}{\partial t_0} = \sum_{i\geq 0} t_{i+1}\frac{\partial \mathcal{F}^{(q)}}{\partial t_i} + \frac{t_0^2}{2} - \frac{i\hbar}{24}.$$

Obviously, $\mathcal{F}^{(q)} = \mathcal{F} + O(\hbar)$.

Introduce quantum intersection numbers:

$$\langle \tau_{d_1}\ldots\tau_{d_n}\rangle_{l,g-l} := i^{\sum d_j - 3g - n + 3}\mathrm{Coef}_{\varepsilon^{2l}\hbar^{g-l}}\left.\frac{\partial^n \mathcal{F}^{(q)}}{\partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} \in \mathbb{Q}.$$

# Quantum intersection numbers

Recall

$$\left.\frac{\partial^{n+1}\mathcal{F}}{\partial t_0 \partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \left.\left\{\left\{\ldots\left\{\left\{\frac{\delta\overline{h}_{d_1}}{\delta u},\overline{h}_{d_2}\right\},\overline{h}_{d_3}\right\},\ldots\right\},\overline{h}_{d_n}\right\}\right|_{u_k=\delta_{k,1}}.$$

Define $\mathcal{F}^{(q)} \in \mathbb{C}[[t_0,t_1,\ldots,\varepsilon,\hbar]]$ by the relations

$$\left.\frac{\partial^{n+1}\mathcal{F}^{(q)}}{\partial t_0\partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} = \hbar^{1-n}\left.\left[\left[\ldots\left[\left[\frac{\delta\overline{H}_{d_1}}{\delta u},\overline{H}_{d_2}\right],\overline{H}_{d_3}\right],\ldots\right],\overline{H}_{d_n}\right]\right|_{u_k=\delta_{k,1}},$$

$$\frac{\partial\mathcal{F}^{(q)}}{\partial t_0} = \sum_{i\ge 0} t_{i+1}\frac{\partial\mathcal{F}^{(q)}}{\partial t_i} + \frac{t_0^2}{2} - \frac{i\hbar}{24}.$$

Obviously, $\mathcal{F}^{(q)} = \mathcal{F} + O(\hbar)$.

Introduce quantum intersection numbers:

$$\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{l,g-l} := i^{\sum d_j - 3g - n + 3}\,\mathrm{Coef}_{\varepsilon^{2l}\hbar^{g-l}}\left.\frac{\partial^n\mathcal{F}^{(q)}}{\partial t_{d_1}\ldots\partial t_{d_n}}\right|_{t_*=0} \in \mathbb{Q}.$$

Relation with the classical intersection numbers:

$\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,0} = \langle\tau_{d_1}\ldots\tau_{d_n}\rangle_g$.

# Relation with stationary Gromov–Witten invariants of $\mathbb{CP}^1$

**Theorem (Blot–B., 2023)**

*Let* $g, l \geq 0$, $n \geq 1$, *and* $\overline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$.

*1. For any* $k \geq 1$, *the integral*

$P_{g,l,\overline{d}}(a_1, \ldots, a_k) :=$

$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1,\ldots,a_k))]^{\mathrm{vir}}} \lambda_l \prod_{j=1}^n \psi_j^{d_j} \mathrm{ev}_j^*(\omega), \quad a_1, \ldots, a_k \in \mathbb{Z}_{\geq 1},$

*is a polynomial in* $a_1, \ldots, a_k$. *Here* $A := \sum a_i$.

# Relation with stationary Gromov–Witten invariants of $\mathbb{CP}^1$

**Theorem (Blot–B., 2023)**

*Let* $g, l \geq 0$, $n \geq 1$, *and* $\bar{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n_{\geq 0}$.

*1. For any* $k \geq 1$, *the integral*

$$P_{g,l,\bar{d}}(a_1, \dots, a_k) := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1,\dots,a_k))]^{\mathrm{vir}}} \lambda_l \prod_{j=1}^n \psi_j^{d_j} \mathrm{ev}_j^*(\omega), \quad a_1, \dots, a_k \in \mathbb{Z}_{\geq 1},$$

*is a polynomial in* $a_1, \dots, a_k$. *Here* $A := \sum a_i$.

*2. Let* $k := \sum d_j - 2g + l + 1$. *Then*

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l, g-l} = \begin{cases} \boxed{\dfrac{1}{k!} \mathrm{Coef}_{a_1 \cdots a_k} P_{g,l,\bar{d}}}, & \text{if } k \geq 1, \\ (-1)^g \int_{\overline{\mathcal{M}}_{g,2}} \lambda_g \lambda_l \psi_1^{d_1}, & \text{if } k = 0 \text{ and } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

# Theorem of X. Blot

Denote $S(z) := \frac{e^{z/2}-e^{-z/2}}{z}$.

## Theorem (X. Blot, 2022)

*For* $g, n \geq 0$ *satisfying* $2g-2+n>0$, *we have*

$$\sum_{d_1,\ldots,d_n\geq 0} \langle \tau_{d_1}\ldots\tau_{d_n}\rangle_{0,g}\, \mu_1^{d_1}\cdots\mu_n^{d_n} =$$

$$= \left(\sum \mu_j\right)^{2g-3+n} \mathrm{Coef}_{z^{2g}} \left(\frac{\prod_{j=1}^n S(\mu_j z)}{S(z)}\right).$$

# Relation with one-part double Hurwitz numbers

X. Blot interpreted the theorem in the following way.

For two tuples $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $H^g_{\mu,\nu}$ the double Hurwitz number.

# Relation with one-part double Hurwitz numbers

X. Blot interpreted the theorem in the following way.

For two tuples $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $H^g_{\mu,\nu}$ the double Hurwitz number.

Goulden–Jackson–Vakil (2005):

$$H^g_{\sum \mu_j, (\mu_1, \ldots, \mu_n)} = r! \left(\sum \mu_j\right)^{r-1} \mathrm{Coef}_{z^{2g}} \left(\frac{\prod_{j=1}^n S(\mu_j z)}{S(z)}\right),$$

where $r := 2g - 1 + n$.

# Relation with one-part double Hurwitz numbers

X. Blot interpreted the theorem in the following way.

For two tuples $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $H^g_{\mu,\nu}$ the double Hurwitz number.

Goulden–Jackson–Vakil (2005):

$$H^g_{\sum \mu_j, (\mu_1, \ldots, \mu_n)} = r! \left(\sum \mu_j\right)^{r-1} \mathrm{Coef}_{z^{2g}} \left( \frac{\prod_{j=1}^n S(\mu_j z)}{S(z)} \right),$$

where $r := 2g - 1 + n$.

Equivalent formulation of the theorem:

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \mathrm{Coef}_{\mu_1^{d_1} \cdots \mu_n^{d_n}} \left( \frac{H^g_{\sum \mu_j, (\mu_1, \ldots, \mu_n)}}{r! \sum \mu_j} \right).$$

# More on the theorem of X. Blot

In the case $l = 0$, our theorem says that

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} =$$

$$= \frac{1}{k!} \mathrm{Coef}_{a_1 \cdots a_k} \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \ldots, a_k))]^{\mathrm{vir}}} \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega),$$

where $k = \sum d_j - 2g + 1 \geq 1$.

# More on the theorem of X. Blot

In the case $l = 0$, our theorem says that

$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} =$

$$= \frac{1}{k!} \mathrm{Coef}_{a_1 \cdots a_k} \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \dots, a_k))]^{\mathrm{vir}}} \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega),$$

where $k = \sum d_j - 2g + 1 \geq 1$.

The integral $\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \dots, a_k))]^{\mathrm{vir}}} \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega)$ is a stationary relative Gromov–Witten invariant of $\mathbb{CP}^1$, for which Okounkov and Pandharipande (in 2006) presented an explicit formula using the infinite wedge technique.

# Infinite wedge formalism

$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

# Infinite wedge formalism

$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \dots \middle| a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

# Infinite wedge formalism

$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \ldots \middle| a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

# Infinite wedge formalism

$V = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \dots \,\middle|\, a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \dots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \dots \wedge \widehat{\underline{a_i}} \wedge \dots$,

# Infinite wedge formalism

$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \ldots \middle| a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \ldots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \ldots \wedge \widehat{\underline{a_i}} \wedge \ldots$,

$$: \psi_i \psi_j^* := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0, \\ -\psi_j^* \psi_i, & \text{if } j < 0, \end{cases}$$

# Infinite wedge formalism

$V = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \ldots \,\middle|\, a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \ldots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \ldots \wedge \widehat{\underline{a_i}} \wedge \ldots$,

$$: \psi_i \psi_j^* : \; := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0, \\ -\psi_j^* \psi_i, & \text{if } j < 0, \end{cases}$$

$\mathcal{E}_r(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} : \psi_{k-r}\psi_k^* : + \frac{\delta_{r,0}}{\zeta(z)}$, where $r \in \mathbb{Z}$ and

$\zeta(z) = e^{\frac{z}{2}} - e^{-\frac{z}{2}}$,

# Infinite wedge formalism

$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \dots \middle| a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \dots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \dots \wedge \widehat{\underline{a_i}} \wedge \dots$,

$$: \psi_i \psi_j^* := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0, \\ -\psi_j^* \psi_i, & \text{if } j < 0, \end{cases}$$

$\mathcal{E}_r(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} : \psi_{k-r} \psi_k^* : + \frac{\delta_{r,0}}{\zeta(z)}$, where $r \in \mathbb{Z}$ and

$\zeta(z) = e^{\frac{z}{2}} - e^{-\frac{z}{2}}$,

Commutation relation $[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta(aw - bz) \mathcal{E}_{a+b}(z+w)$,

# Infinite wedge formalism

$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \dots \,\middle|\, a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \dots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \dots \wedge \widehat{\underline{a_i}} \wedge \dots$,

$$: \psi_i \psi_j^* := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0, \\ -\psi_j^* \psi_i, & \text{if } j < 0, \end{cases}$$

$\mathcal{E}_r(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} : \psi_{k-r} \psi_k^* : + \frac{\delta_{r,0}}{\zeta(z)}$, where $r \in \mathbb{Z}$ and

$\zeta(z) = e^{\frac{z}{2}} - e^{-\frac{z}{2}}$,

Commutation relation $[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta(aw - bz) \mathcal{E}_{a+b}(z+w)$,

$\alpha_k = \mathcal{E}_k(0)$, $k \neq 0$,

# Infinite wedge formalism

$V = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \dots \middle| a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \dots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \dots \wedge \widehat{\underline{a_i}} \wedge \dots$,

$$: \psi_i \psi_j^* : = \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0, \\ -\psi_j^* \psi_i, & \text{if } j < 0, \end{cases}$$

$\mathcal{E}_r(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} : \psi_{k-r}\psi_k^* : + \frac{\delta_{r,0}}{\zeta(z)}$, where $r \in \mathbb{Z}$ and

$\zeta(z) = e^{\frac{z}{2}} - e^{-\frac{z}{2}}$,

Commutation relation $[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta(aw - bz)\mathcal{E}_{a+b}(z+w)$,

$\alpha_k = \mathcal{E}_k(0),\ k \neq 0$,

$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \dots$,

# Infinite wedge formalism

$V = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}\underline{k}$,

$\Lambda^{\frac{\infty}{2}} V = \left\langle \underline{a_1} \wedge \underline{a_2} \wedge \dots \,\middle|\, a_i = -i + \frac{1}{2} + c \text{ for some } c \text{ and } i \text{ big enough} \right\rangle$,

$\psi_k \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k(v) := \underline{k} \wedge v$,

$\psi_k^* \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \psi_k^*(\underline{a_1} \wedge \underline{a_2} \wedge \dots) := \sum_{i=1}^{\infty} (-1)^{i-1} \delta_{a_i,k} \underline{a_1} \wedge \dots \wedge \widehat{\underline{a_i}} \wedge \dots$,

$$: \psi_i \psi_j^* := \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0, \\ -\psi_j^* \psi_i, & \text{if } j < 0, \end{cases}$$

$\mathcal{E}_r(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})} : \psi_{k-r}\psi_k^* : + \frac{\delta_{r,0}}{\zeta(z)}$, where $r \in \mathbb{Z}$ and

$\zeta(z) = e^{\frac{z}{2}} - e^{-\frac{z}{2}}$,

Commutation relation $[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta(aw - bz)\mathcal{E}_{a+b}(z + w)$,

$\alpha_k = \mathcal{E}_k(0)$, $k \neq 0$,

$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \dots$,

For an operator $A \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ denote by $\langle A \rangle$ the coefficient of $v_\emptyset$ in $Av_\emptyset$.

# Okounkov–Pandharipande formula

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,\mu,\nu)^\bullet]^{\mathrm{vir}}} \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega) =$$

$$\frac{1}{\prod_{i=1}^{l(\mu)} \mu_i \prod_{j=1}^{l(\nu)} \nu_j} \mathrm{Coef}_{z_1^{d_1+1} \cdots z_n^{d_n+1}} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{E}_0(z_j) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle.$$

In a joint work with X. Blot (will appear in arXiv soon), using the Okounkov–Pandharipande formula, we give a short proof of the theorem of X. Blot.