Fully simple maps and $x - y$ duality in topological recursion

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May 15, 2025

arXiv: 2106.08368, 2212.00320

Maps

Definition

Map is a graph on a 2-dim surface (compact, oriented, without boundary) such that the complement is homeomorphic to the disjoint union of open disks.

Fully simple maps

Definition

Fully simple map is a map with a fixed subset of faces (will call them boundary) with a coinditon:
each vertex can be incident to only one boundary face.

Permutational model

Maps

Consider the set of $2|E|$ oriented edges and 3 permutations:

- σ_0 rotates each edge anticlockwise around its end point;
- σ_1 rotates each edge around its middle point, i.e swaps orientation of each edge;
- σ_2 rotates each edge anticlockwise around the face to its left.

We obtain 3 permuations $\sigma_0, \sigma_1, \sigma_2 \in S_{2|E|}$, such that $\sigma_1\sigma_0\sigma_2 = \mathrm{Id}$.

There is a one to one correspondence between the equivalence classes of maps and equivalence classes of triples of permuations.

Permutational model

Fully simple maps

For each boundary face fix an oriented edge which is incident to it. Denote this set of oriented edges by R.
Then we have one more condition:

- no two elements of R lie in the same cycle of σ_2.

FSmaps then characterized by the following condition:

- The union of σ_2 orbits of R has to lie in different σ_0 orbits.

Enumeration

Definition

Two FSmaps are equivalent if there is an orientation preserving homeomorphism of the underlying surfaces such that all data is preserved and boundary faces are fixed.

- Let $\mu_1, \ldots, \mu_n$ be the degrees of the boundary faces;
- Let $\Gamma_{\mu_1,\ldots,\mu_n}$ the set of all maps;
- Let $\mathrm{FS}\Gamma_{\mu_1,\ldots,\mu_n}$ the set of fully simple maps;
- Let $f_i(M)$ be the number of internal faces of degree i;
- $\chi(M) = V - E + F_{int} = 2 - 2g - n$;

Definition

$$Map(\mu_1, \ldots, \mu_n) = \sum_{M \in \Gamma_{\mu_1,\ldots,\mu_n}} \hbar^{-\chi(M)} t_1^{f_1(M)} t_2^{f_2(M)} \ldots$$

$$FSMap(\mu_1, \ldots, \mu_n) = \sum_{M \in \mathrm{FS}\Gamma_{\mu_1,\ldots,\mu_n}} \hbar^{-\chi(M)} t_1^{f_1(M)} t_2^{f_2(M)} \ldots$$

Monotone Hurwitz numbers

- $\rho_\lambda, \tau_1, \ldots, \tau_m, \rho_\mu$ — permutations from S_d;
- $\tau_1, \ldots, \tau_m$ — transpositions;
- ρ_λ, ρ_μ — permutations of cyclic types λ and μ;
- Monotonicity condition — if $\tau_i = (a_i, b_i),\ a_i < b_i$, then $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_m$.

Definition

$$H_m^{\leqslant}(\lambda,\mu) = \frac{1}{d!}\#\{(\rho_\lambda, \tau_1, \ldots, \tau_m, \rho_\mu) | \rho_\lambda \tau_1 \ldots \tau_n \rho_\mu = \mathrm{Id}\},$$

$$H^{\leqslant}(\lambda,\mu) = \sum_{m\geqslant 0} H_m^{\leqslant}(\lambda,\mu)\hbar^m.$$

Maps – Fully simple maps duality

Theorem (Borot, Charbonnier, Do, Garcia-Failde, 2019)

$$FSMap(\lambda) = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j\leqslant 1} m_j(\lambda)! \sum_{\mu\vdash d} H^{\leqslant}(\lambda,\mu)\Big|_{\hbar=-\hbar} Map(\mu)$$

$$Map(\lambda) = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j\leqslant 1} m_j(\lambda)! \sum_{\mu\vdash d} H^{<}(\lambda,\mu) FSMap(\mu)$$

$\lambda_i,\ i = 1,\dots,\ell(\lambda)$ — parts of the partition λ.
$m_j(\lambda)$ — is the number of occurrences of j as a part in the partition λ.

Combinatorics of the symmetric group

Let $\vec{C}^{(k)} = \sum_{j_1 \leqslant \ldots \leqslant j_k} (i_1 j_1) \ldots (i_k j_k)$ be the sum of all possible monotonic decomposition of k transpositions.

Then $H_k^{\leqslant}(\lambda, \mu) = \frac{1}{d!}[C_{1^d}] C_\lambda C_\mu \vec{C}^{(k)}$

In the regular representation $\rho_{\mathbb{C}S_d}$:

$$H_k^{\leqslant}(\lambda, \mu) = \left(\frac{1}{d!}\right)^2 \mathrm{tr}(\rho_{\mathbb{C}S_d}(C_\lambda C_\mu \vec{C}^{(k)}))$$

Jucys–Murphy elements

Definition

$X_1 = 0,\ X_2 = (12),\ X_3 = (13) + (23), \dots, X_k = (1\,k) + (2\,k) + \dots + (k-1\,k);$

Lemma

$\vec{C}^{(k)} = h_k(X_1, \dots, X_d) = \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_k \leqslant d} X_{i_1} \dots X_{i_k}$ — *complete symmetric polynomials.*

Theorem (Jucys–Merphy, 1981)

Let $a \in Z\mathbb{C}S_d$ *and* $a = P_a(X_1, \dots, X_d)$ *is a symmetric polynomial polynomial, then the eigenvalue* a_λ *of the action of* a *in the irreducible representation* ρ_λ *is equal to* $P_a(c(\lambda))$*, where* $c(\lambda)$ *is the vector of contents of the Young diagram* λ*.*

Corollary

$$\sum_k H_k^{\leqslant}(\lambda, \mu)\hbar^k p_\mu q_\lambda = \sum_\eta s_\eta(p) s_\eta(q) \prod_{(i,j)\in\eta} \frac{1}{1 - \hbar c_{\boxdot}(\eta)}$$

Fock space reformulation

Definition

Let $\mathbb{C}[[p_1, p_2, \ldots]]$ is a Fock space of formal power series in p variables and

- $|\rangle$ denotes 1 in this space.
- $|\langle$ is an operator of taking the linear term of the series.
- $J_0 = 0,\ J_a = p_{-a}, a < 0,\ J_a = p_a \partial_{p_a}, a > 0.$
- $\mathcal{D}$ is a diagonal operator in the basis of Schur functions, $\mathcal{D} : s_\lambda \mapsto \prod_{i,j\in\lambda}(1 + \hbar(j - i))s_\lambda$

$$Map_g(\mu_1, \ldots, \mu_n) = [\hbar^{2g-2+n}]\langle| \prod_{i=1}^n \frac{J_{\mu_i}}{\mu_i} e^{\sum\limits_{i=1}^{\infty} \frac{J_i t_i}{i\hbar}} \mathcal{D} e^{\frac{J_{-2}}{2\hbar}} |\rangle = [\hbar^{2g-2+n}]\langle| \prod_{i=1}^n \frac{J_{\mu_i}}{\mu_i} Z|\rangle,$$

$$\sum_k H_k^{\leqslant}(\lambda, \mu)\hbar^k = \langle| \prod \frac{J_{\mu_i}}{\mu_i} \mathcal{D}^{-1} \prod \frac{J_{\lambda_i}}{\lambda_i} |\rangle\Big|_{\hbar\to-\hbar}.$$

Theorem (BDKS,21')

Let $Z = e^{\sum\limits_{i=1}^{\infty} \frac{J_i t_i}{i\hbar}} \mathcal{D} e^{\frac{J_{-2}}{2\hbar}}$, *then* $FSMap_g(\mu_1, \ldots, \mu_n) = [\hbar^{2g-2+n}]\langle| \prod\limits_{i=1}^n \frac{J_{\mu_i}}{\mu_i} \mathcal{D}^{-1} Z|\rangle.$

Topological recursion [Chekhov, Eynard, Orantin 2007]

$$(\Sigma, \omega_{0,1}, \omega_{0,2}) \rightsquigarrow \{\omega_{g,n}\}$$

Initial data

- Σ — Riemann surface
- $\omega_{0,1}(z) = y(z)dx(z)$ — differential on Σ
- $\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ — bidifferential on $\Sigma \times \Sigma$

Differential of topological recursion $\omega_{g,n}$

$\omega_{g,n}(z_1, \ldots, z_n)$ for $2g - 2 + n > 0$ are symmetric meromorphic differentials on Σ^n with poles (in each variable) only at zeroes of $dx(z_i)$.

TR: the formula

$$\omega_{g,n+1}(z_0, z_1, \dots, z_n) = \sum_{w\colon dx|_w=0} \operatorname{Res}_w \frac{1}{2} \frac{\int_w^{\sigma(w)} \omega_{0,2}(z_0, \cdot)}{\omega_{0,1}(w) - \omega_{0,1}(\sigma(w))} \cdot$$

$$\cdot \left(\omega_{g-1,n+2}(w, \sigma(w), z_1, \dots, z_n) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{z_1,\dots,z_n\}}}' \omega_{g_1,|I_1|+1}(w, z_{I_1}) \omega_{g_2,|I_2|+1}(\sigma(w), z_{I_2}) \right)$$

- $\sum'$ means that we exclude from the sum two cases when $g_i = g$ and $I_i = \{z_1, \dots, z_n\}$, $i = 1, 2$.
- σ is an involution near the ramification point w.

TR: examples

- Hurwitz numbers (simple, double, monotone, weighted etc.);
- enumeration of maps (hypermaps, fully simple, weighted etc.);
- correlators of matrix models;
- correlators of Cohomological field theories (Gromov–Witten invariants);
- Weil–Petterson volumes, Masur–Veech volumes of moduli spaces, etc.

Enumeration $\leftrightarrow$ TR

Given a collection of quantities

$$f_{g,\mu}, \quad g \geq 0, \mu = (\mu_1, \ldots, \mu_n), \mu_i \geq 1.$$

We pack them into generating series

Correlator functions:

$$H_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} f_{g,\mu} x_1^{\mu_1} \ldots x_n^{\mu_n}.$$

Correlator differentials:

$$\omega_{g,n}(x_1, \ldots, x_n) = d_1 \ldots d_n H_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} f_{g,\mu} \prod_{i=1}^{n} \mu_i x_i^{\mu_i - 1} dx_i.$$

Enumeration $\leftrightarrow$ TR

Quantities $\to$ TR

$f_{g,\mu}$ (or $H_{g,n}$ or enumerative problem) satisfies topological recursion if there is a change of variables $x = x(z)$ such that $\omega_{g,n}$ becomes *rational* in the coordinates $z_1, \ldots, z_n$ (i.e. after substitution $x_i = x(z_i)$) for all (g, n).

TR $\to$ quantities

- Σ Riemann surface;
- z uniformizing global coordinate on Σ;
- $\omega_{g,n}(z_1, \ldots, z_n)$ global meromorphic symmetric differential on Σ^n;
- $(o \in \Sigma, x)$ a distinguished point and a local coordinate at this point;
- We are interested in the power expansion of $\omega_{g,n}$ at the point $(o, \ldots, o) \in \Sigma^n$ in the local coordinates $x_i = x(z_i)$.

TR: examples

Simple Hurwitz numbers

$$\Sigma = \mathbb{C}P^1, \quad x(z) = \log(z) - z, \quad y(z) = z$$

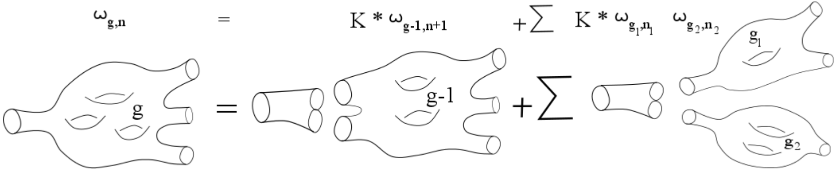
The expansion of the differentials $\omega_{g,n}$ near the points $z_1 = z_2 = \ldots = z_n = 0$ in the coordinates $X_i = \exp(x(z_i)), i = 1, \ldots, n$ give the series $H_{g,n}(X_1, \ldots, X_n)$ for simple Hurwitz numbers.

Kontsevich–Witten intersection numbers

$$\Sigma = \mathbb{C}P^1, \quad x(z) = \frac{z^2}{2}, \quad y(z) = z$$

Corresponds to the KW intersection numbers $\int\limits_{\overline{M}_{g,n}} \psi_1^{\mu_1} \ldots \psi_n^{\mu_n}$.

$$\omega_{g,n} = K * \omega_{g-1,n+1} + \sum K * \omega_{g_1,n_1} \omega_{g_2,n_2}$$



TR for maps

Example (Initial data for maps)

Let

$$V(z) = x(z) - \sum_{k=1}^{n} t_k x(z)^k; \;\; \gamma(t_1, t_2, \ldots)^{-1} = [z^{-1}]V(z); \;\; \alpha(t_1, t_2, \ldots) = [z^0]V(z)$$

Then

$$x(z) = \alpha + \gamma(z + \frac{1}{z})$$

$$y(z) = V(z)_+ x(z) - 1$$

$$(\mathbb{C}P^1, X(z) = 1/x(z), y(z), dz_1 dz_2/(z_1 - z_2)^2), \;\; \omega_1^{(0)} = ydX/X$$

x–y duality

What happens if one swaps x and y? I.e. replaces (Σ, x, y) with (Σ, y, x)?
Is it possible to express the resulting $\omega_n^{\vee,(g)}$ through the original $\omega_n^{(g)}$?

Theorem ([ABDKS'22])

$$\frac{\omega_n^{\vee,(g)}(z_{[\![n]\!]})}{\prod_{i=1}^n dx_i^\vee} = (-1)^n [\hbar^{2g}] \sum_\Gamma \frac{\hbar^{2g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}$$

$$\prod_{i=1}^n \sum_{k_i=0}^\infty \partial_{y_i}^{k_i} [u_i^{k_i}] \frac{dx_i}{dy_i}$$

$$\frac{1}{u_i} e^{u_i \mathcal{S}(\hbar u_i \partial_{x_i}) \sum_{\tilde g=0}^\infty \hbar^{2\tilde g} \frac{\omega_1^{(\tilde g)}(z_i)}{dx_i} - u_i \frac{\omega_1^{(0)}(z_i)}{dx_i}}$$

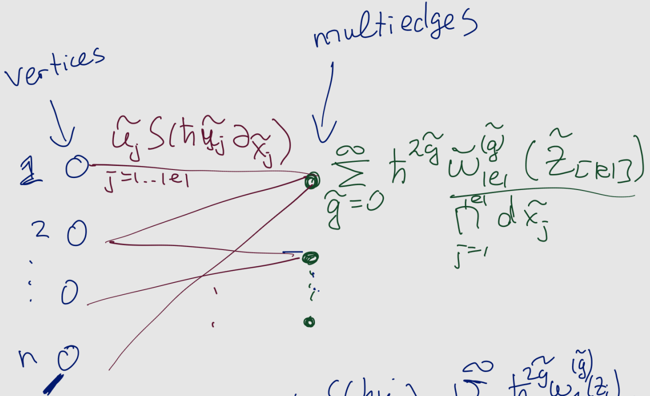
$$\prod_{e\in E(\Gamma)} \prod_{j=1}^{|e|\geq 2} \lfloor_{(\tilde u_j, \tilde x_j) \to (u_{e(j)}, x_{e(j)})} \tilde u_j \mathcal{S}(\hbar \tilde u_j \partial_{\tilde x_j})$$

$$\sum_{\tilde g=0}^\infty \hbar^{2\tilde g} \frac{\tilde\omega_{|e|}^{(\tilde g)}(\tilde z_{[\![|e|]\!]})}{\prod_{j=1}^{|e|} d\tilde x_j}$$

$$+ \delta_{(g,n),(0,1)}(-x_1).$$

- The sum is over all connected graphs Γ with n labeled vertices and multiedges $E(\Gamma)$ of index ≥ 2.
- All legs of every given multiedge e are labeled from 1 to $|e|$ in an arbitrary way.
- For a multiedge e with index $|e|$ its attachment to the vertices is controlled by the associated map $e\colon [\![|e|]\!] \to [\![n]\!]$.
- For e with $|e| = 2$
  $\tilde\omega_2^{(0)}(\tilde x_1, \tilde x_2) := \omega_2^{(0)}(\tilde x_1, \tilde x_2) - \frac{d\tilde x_1 d\tilde x_2}{(\tilde x_1 - \tilde x_2)^2}$ if $e(1) = e(2)$, and
  $\tilde\omega_2^{(0)}(\tilde x_1, \tilde x_2) := \omega_2^{(0)}(\tilde x_1, \tilde x_2)$ otherwise.
  For all $(g, n) \neq (0, 2)$
  $\tilde\omega_n^{(g)} := \omega_n^{(g)}$.
- $g(\Gamma)$ is the first Betti number of Γ.
- $|\mathrm{Aut}(\Gamma)|$ is the number of automorphisms of Γ.
- $[\hbar^{2g}]$ (respectively, $[u_i^{k_i}]$) is the operator that extracts the respective coefficient from the whole expression to the right of it, i.e. $[x^m] \sum_{i=-\infty}^\infty a_i x^i := a_m$.
- $\lfloor_{a\to b}$ is the operator of substitution $a \to b$.
- $\mathcal{S}(z) := \left(e^{z/2} - e^{-z/2}\right)/z$.

Read from the right to the left



$$\sum_{k_i=0}^{\infty} \partial_{y_i}^{k_i} [u_i^{k_i}] \frac{dx_i}{dy_i} \frac{1}{u_i} e^{u_i S(\hbar u_i \partial_{x_i}) \sum_{\tilde{g}=0}^{\infty} \frac{\hbar^{2\tilde{g}} \omega_1^{(\tilde{g})}(z_i)}{dx_i} - u_i \frac{\omega_1^{(0)}(z_i)}{dx_i}}$$

$x - y$ swap duality: basic properties

Theorem ([ABDKS, 22'])

1. The $x - y$ dual differentials are meromorphic, symmetric, and $\omega_n^{\vee,(g)} - \delta_{(g,n),(0,2)} \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ is regular on the diagonals.

2. The inverse transformation is given by the same formula with the roles of x and y swapped

3. If $\{\omega_n^{(g)}\}$ solves TR with the initial data (Σ, x, y, B), then $\{\omega_n^{\vee,(g)}\}$ solves TR with the initial data (Σ, y, x, B).

Applications of $x-y$ swap I: closed formulas for TR differentials

Consider TR for the spectral curve data (Σ, x, y, B) with

$$y(z) = z, \quad \Sigma = \mathbb{C}P^1, \quad B = \frac{dz_1 dz_2}{(z_1 - z^2)^2}$$

(E.g. KW intersection numbers, enumeration of maps, Hurwitz numbers, etc.)

$x^\vee = y = z$ has no critical points

$\Rightarrow$ the dual TR is trivial: $\omega_n^{\vee,(g)} = 0$ for $(g,n) \neq (0,2)$

$\Rightarrow$ applying $x-y$ swap we get an explicit formula for $\omega_n^{(g)}$:

$$\omega_n(z_{[\![n]\!]}) = (-1)^n \sum_{r_1,\dots,r_n \geq 0} \Big(-d_1 \tfrac{1}{dx_1}\Big)^{r_1} \dots \Big(-d_n \tfrac{1}{dx_n}\Big)^{r_n} [u_1^{r_1} \dots u_n^{r_n}]$$
$$\prod_{i=1}^n \Big(e^{u_i(\mathcal{S}(u_i\hbar\partial_{z_i})-1)x_i}\, dz_i\Big)\, (-1)^{n-1} \sum_{\sigma \in \mathrm{cycl(n)}} \prod_{i=1}^n \frac{1}{z_i - z_{\sigma(i)} + (u_i + u_{\sigma(i)})\hbar/2},$$

Similar closed formulas exist in some other cases, e.g. for $y = \log z$, *that is* $dy = \frac{dz}{z}$

Applications of $x - y$ swap II: integrability