A log/tropical take on Hurwitz Numbers

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Plan:

1. Double Hurwitz numbers
2. Tropical Hurwitz numbers
3. Tautological ring of $\bar{M}_{g,n}$
4. Hurwitz numbers as intersection numbers in $\log CH(\bar{M}_{g,n})$
5. New enumerative problems arising from this perspective

Based on joint work with Paul Johnson, Hannah Markwig, Dhruv Ranganathan and Johannes Schmitt.
Enumerative geometry & moduli spaces

"How many 😊's satisfy conditions 😊 and 😞?"

becomes:

Q: do Hurwitz numbers arise from intersection theory on the moduli space of curves?
Double Hurwitz #'s

- \( \tilde{x} \in \mathbb{Z}^n \setminus \{ \sum x_i = 0 \} = \mathcal{H} \)
- \( \text{div}(f) = \sum x_i Q_i \)
- Fixed simple branch points.

\[ H^r_g(\tilde{x}) \]
DHN's as a tautological degree

\[ \text{br}: \text{DR}_g(\mathbb{P}^1) \to \text{Div}(\mathbb{P}^1) \]

\[ H_g^r(\mathbb{P}^1) := \text{deg} (\text{br}) \]
Hannah-lyfication:

\[ \text{br} : \text{DR}_g(\tilde{x}) \rightarrow \text{Div}(\mathbb{P}^1) \]

\[ H_{g,r,(\tilde{x})}^{\text{trop}} \] := \text{deg} (\text{br})

CORRESPONDENCE THM (CJM)

\[ H_{g,r,(\tilde{x})}^{\text{trop}} = H_{g}^{r}(\tilde{x}) \]
Tropical Hurwitz numbers

\[ g = 2, \overrightarrow{x} = (5, 3, -4, -4) \]

\[ H^\text{trop}_{g}(\overrightarrow{x}) = \sum_{(\Gamma, f) \in \text{M.G.}} \frac{\prod_{c.e.} \omega_e}{|\text{Aut}(\Gamma, f)|} \]
$H_{i}^{2} (3, -3) = \frac{2}{1}$

$H_{i}^{2} (d, -d) = \frac{1}{2} \sum_{i=1}^{d-1} i \cdot (d-i) = \frac{d^3 - d}{12}$
The moduli space of curves $\bar{M}_{g,n}$

- stratified:

- Tautological ring*: part of Chow generated by strata classes.

!! COMBINATORIAL !!
Geometric tautological classes

- DR-cycles:
  \[ \text{DR}(\bar{x}) | M_{g,n} = \{ C, \hat{p} | \sum x_i \hat{p}_i \equiv 0 \} \]

- Compact by RSM

- Kissing condition

- Curves w/ rat'le
  - Map w/ given O's and poles
don’t always play well with stratification

\[ \log = \text{tool to fix this!} \]
1. Stratification
2. Orbit-cone correspondence
3. Subdivisions $\rightarrow$ blow-ups
4. $A^*_\tau(X_\Sigma) = PP(\Sigma) \rightarrow A^*(X_\Sigma)$
Artin fan
of log space

Log geometry

\[
\overline{\mathcal{M}}_{1,2} \rightarrow \text{colim} \rightarrow \mathcal{M}^{\text{trop}}_{1,2}
\]

stalls of char. sheaf of monoids gives one cone for each stratum
\[ P.P. \left( M_{g,n}^{\text{trop}} \right) \rightarrow \log CH^* (\overline{M}_{g,n}) \]
logarithmic DR-cycle

$$\text{DR}^\log_g (\hat{\tau}) \subseteq \text{log CH}(\bar{M}_{g,n})$$

$$\text{Pic}_{g,n} \rightarrow \bar{\text{Pic}}_{g,n} \rightarrow \tilde{\text{Pic}}_x$$

$$\mathcal{M}_{g,n} \leftarrow \bar{\mathcal{M}}_{g,n} \leftarrow \tilde{\mathcal{M}}_x$$

$$\mathcal{O}(C, P_1, \ldots, P_n) = \Theta_c$$

$$\sigma^x (C, P_1, \ldots, P_n) = \Theta_c (\Sigma x : P_i)$$
BACK TO HURWITZ # PROBLEM:

\[ \text{DR}_g(x) \subseteq \mathcal{M}_{g,n} \]
but does anything in \( \mathcal{M}_{g,n} \) cut down the branch condition?

\[ \text{Thm (-, Markwig, Ranganathan)} \]

There is \( b_x \in \log CH(\mathcal{M}_{g,n}) \) s.t.

\[ H_g(x) = \deg \left( \text{DR}_g^{\log}(x) \cdot b_x \right) \]

Further: \( b_x \) AND \( \tilde{M}_x \) are explicitly determined by \( \text{DR}_g^{\text{trop}}(x) \)
\[
\tilde{M} \sim 2 \mathbb{P} \cdot \text{DR}^{\log}
\]

\[
H_1(3, -3) = 2
\]

\[
= \prod_{\frac{1}{2}} \frac{\omega(E)}{|\text{Aut}(\Gamma)|}
\]

\[
= \deg \text{DR}^{\log} \cdot 2 \psi_p
\]

Tropical Side

\[
\text{DR}^{\text{trop}} \subset M_{1,2}^{\text{trop}}
\]

\[
(1, 2)
\]

\[
P
\]
New Enumerative Problems

1. Leaking.

\[ \Gamma \]

\[ \mathfrak{R} \]

\[ k=1, \ g=2, \ x=(5,3,-1,-1) \]

related to moduli spaces of meromorphic \[ k \]-differentials on curves.

\[ H^k_{g}(x) := \log DR^k_{g}(x) \cdot br_{x} \]
Example: one part, \( q = 0 \)

**Hermitez:**

\[
H_0(z^2) = (n-2)! \, d^{n-3}
\]

**k-keaky**

\[
H_0^k(z^2) = (n-2)! \prod_{i=1}^{n-3} \left( d - i \frac{k}{2} \right)
\]
2 Partial br p-polys

\[ b_{i,x} := \text{br}^* \left( \prod_{i=1}^{\text{trop}} t_i \right) \]

Imposes a non-maximal degeneration on the curves, leaving room for pairing with other tautological classes.

\[ \text{br}_{2,x} |_{\Gamma} = 4^2 \cdot 3 \cdot 2 \cdot \pi_0 \]

\[ \pi_0 = \min (\ell_1, 2\ell_2, 4\ell_3), \min (3\ell_4, 4\ell_5) \]
\[ H^k_g(\tilde{x}, \psi^i) := \psi^i \cdot b_{r_i, \tilde{x}} \cdot DR_{g}^{log, K}(\tilde{x}) \]

\[ H^k_0(d, -x_2, \ldots, -x_5, \psi_i) = 3d - 3K \]

**K-Leak:**
\[ d - \sum x_i = 3K \quad \text{i.e.} \quad \sum x_i = d - 3K \]

\[ \sum_{\Gamma_{ij}} (x_i + x_j + K) = 3 \sum x_i + 6K = 3d - 3K \]
Example: one part, $g=0$

Hurwitz:

$$H_0(\tilde{x}) = (n-2)! \, d^{n-3}$$

$k$-leaky

$$H_0^k(\tilde{x}) = (n-2)! \prod_{i=1}^{n-3} \left( d - i \frac{k}{2} \right)$$

$k$-leaky w/ $\psi_i$

$$H_0^k(\tilde{x}, \psi_i^e) = (n-2)! \frac{\prod_{i=e+1}^{n-3} (d - i \frac{k}{2})}{(e+1)!}$$
Thanks!