A log/tropical take on Hurwitz Numbers

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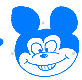
Plan:

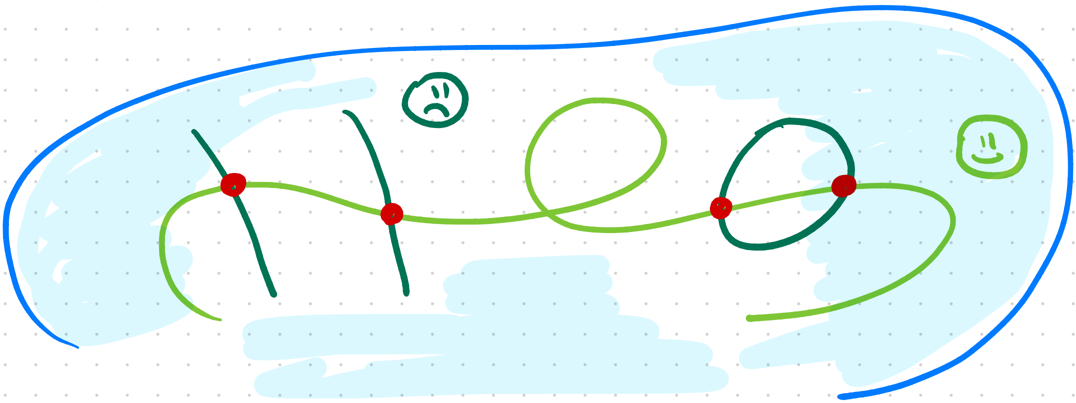
1. Double Hurwitz numbers
2. Tropical Hurwitz numbers
3. Tautological ring of $\overline{M}_{g,n}$
4. Hurwitz numbers as intersection numbers in $\log CH(\overline{M}_{g,n})$
5. New enumerative problems arising from this perspective

Based on joint work with Paul Johnson, Hannah Markwig, Dhruv Ranganathan and Johannes Schmitt.

Enumerative geometry & moduli spaces

"How many 's satisfy conditions  and  ?"

becomes: $\mathcal{M}$ 

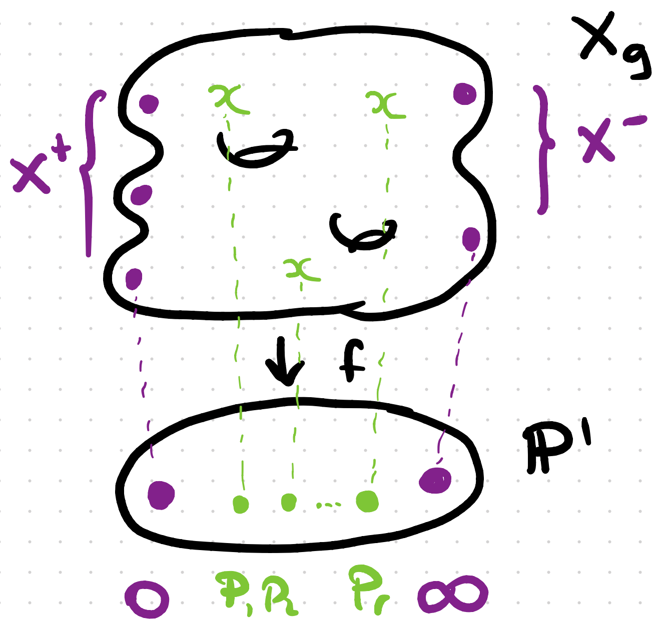


Q: do Hurwitz numbers arise from intersection theory on the moduli space of curves?

Double Hurwitz #'s

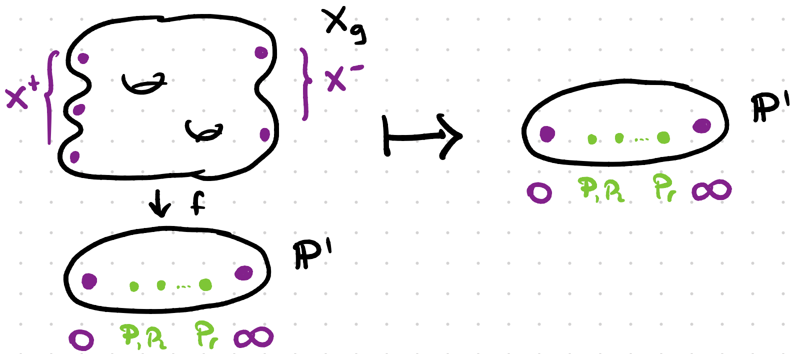
$\boxed{H_g^r(\vec{x})}$

- $\vec{x} \in \mathbb{Z}^n \cap \{\sum x_i = 0\} = H$
- $\mathrm{div}(f) = \sum x_i Q_i$
- r fixed simple branch points.



DHN's as a tautological degree

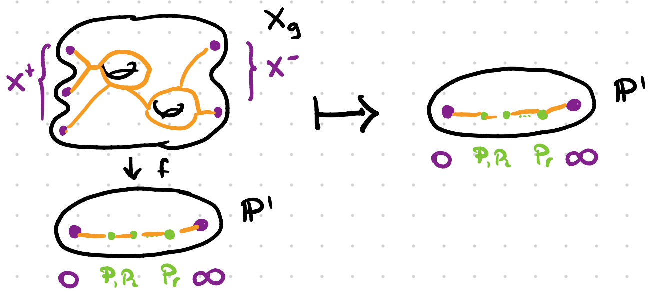
$$br : DR_g(\vec{x}) \to \mathrm{Div}(\mathbb{P}^1)$$



$$H_g^r(\vec{x}) := \deg(br)$$

Hannah-lyfication:

$$\mathrm{br}^{\mathrm{trop}} : DR_g^{\mathrm{trop}}(\vec{x}) \longrightarrow \mathrm{Div}^{\mathrm{trop}}(\mathbb{P}^1)$$



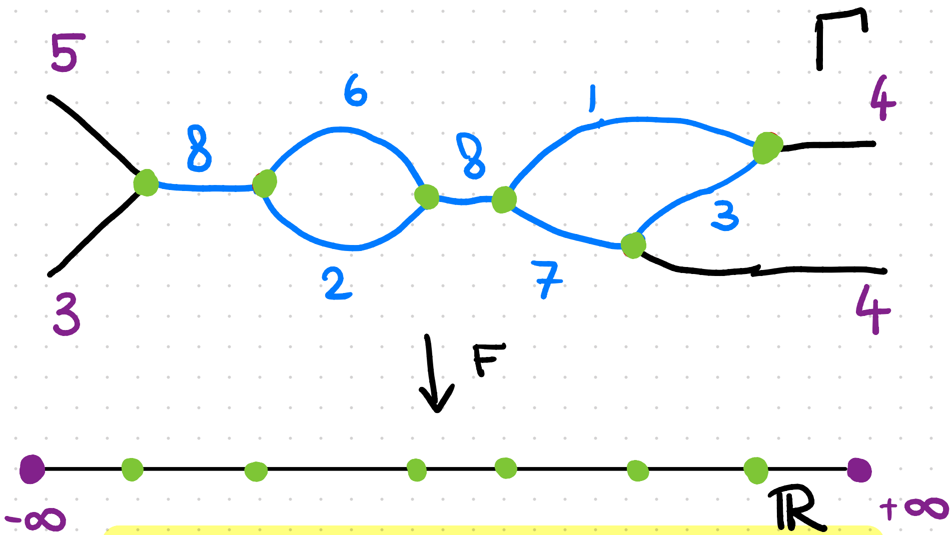
$$H_g^{r,\mathrm{trop}}(\vec{x}) := \deg^{\mathrm{trop}}(\mathrm{br})$$

natural lattice geom way to define this

CORRESPONDENCE THM (CJM)

$$H_g^{r,\mathrm{trop}}(\vec{x}) = H_g^{r}(\vec{x})$$

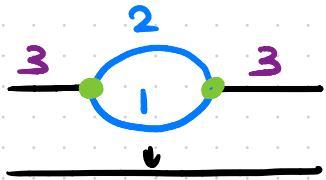
Tropical Hurwitz numbers



$$g=2\ ,\ \vec{x}=(5,3,-4,-4)$$

$$H_g^{trop}(\vec{x}) = \sum_{(\Gamma,f)\in M.G.} \frac{\prod_{c.e.} \omega_e}{|Aut(\Gamma,f)|}$$

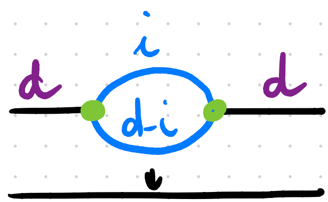
$$H_1^2(3,-3) =$$



$$\rightsquigarrow \frac{2 \cdot 1}{1}$$

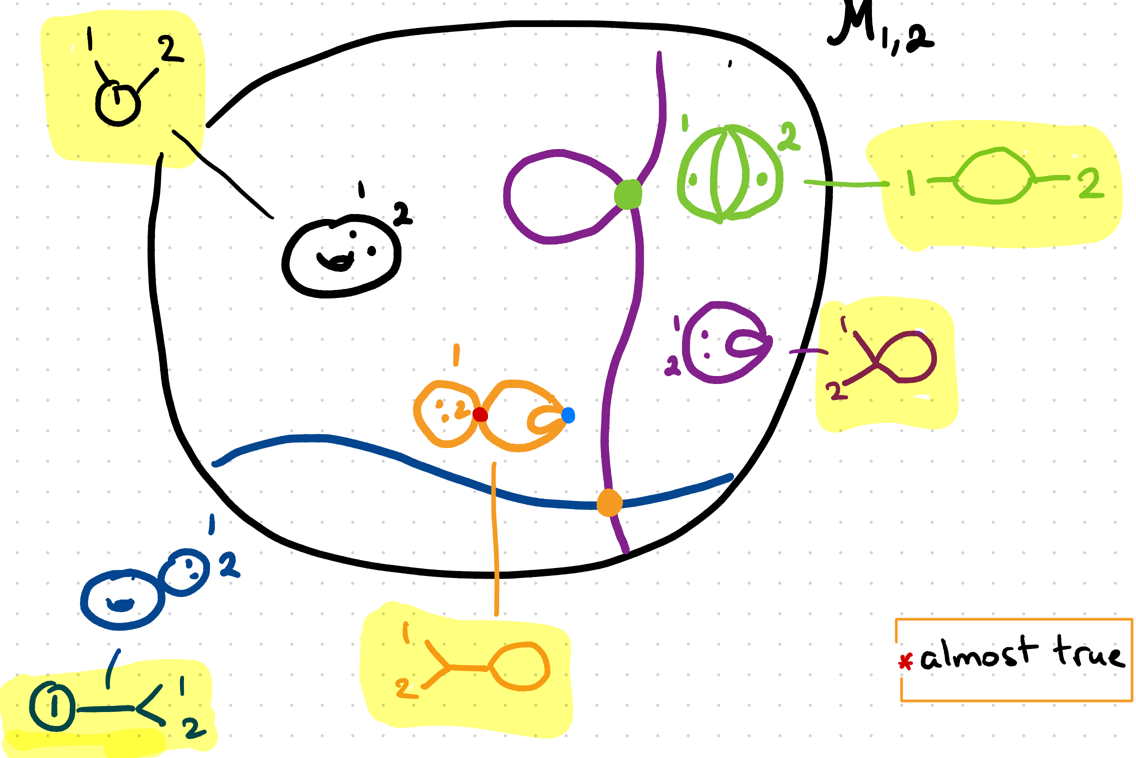
$$H_1^2(d,-d) = \frac{1}{2} \sum_{i=1}^{d-1} i \cdot (d-i)$$

$$= \frac{d^3 - d}{12}$$



The moduli space of curves $\boxed{\overline{M}_{g,n}}$

- stratified:



- Tautological ring*: part of Chow generated by strata classes.

$$\boxed{\text{!! COMBINATORIAL !!}}$$

Geometric tautological classes

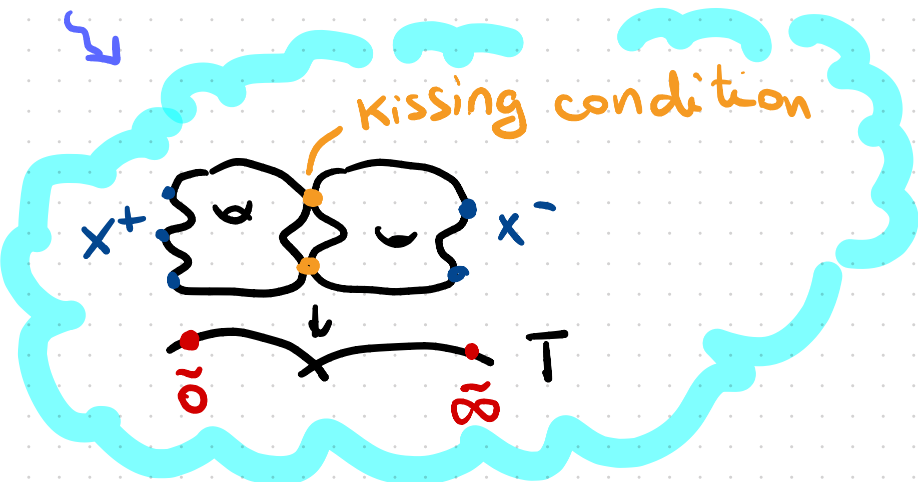
- **DR-cycles:**

$$DR(\vec{x})\big|_{\mathcal{M}_{g,n}} = \left\{ C, \vec{P} \,\middle|\, \sum x_i P_i \equiv 0 \right\}$$

$\Updownarrow$

curves w/ rat'l map w/ given 0's and poles

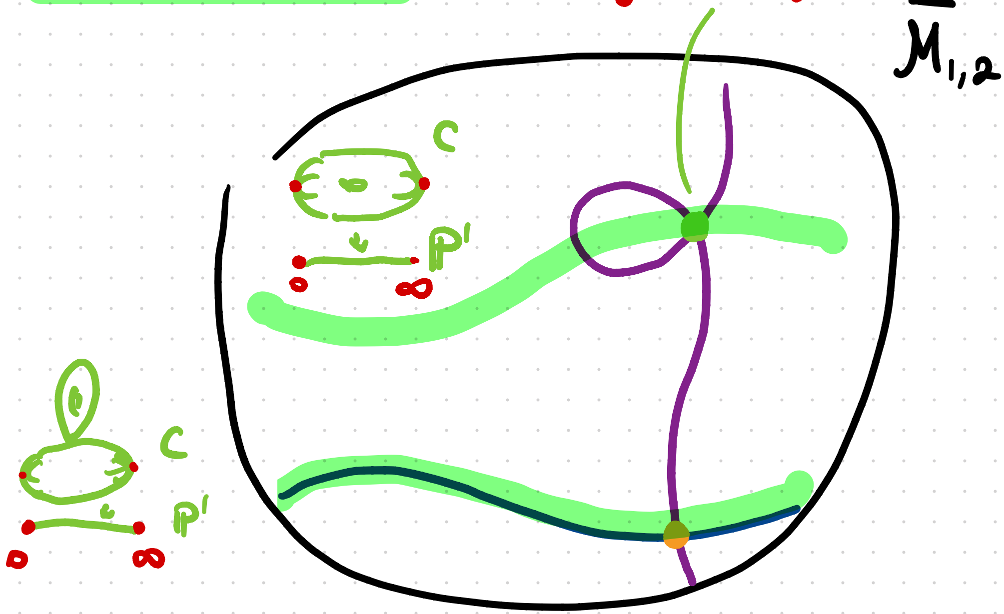
compact. by RSM



don't always play well with stratification

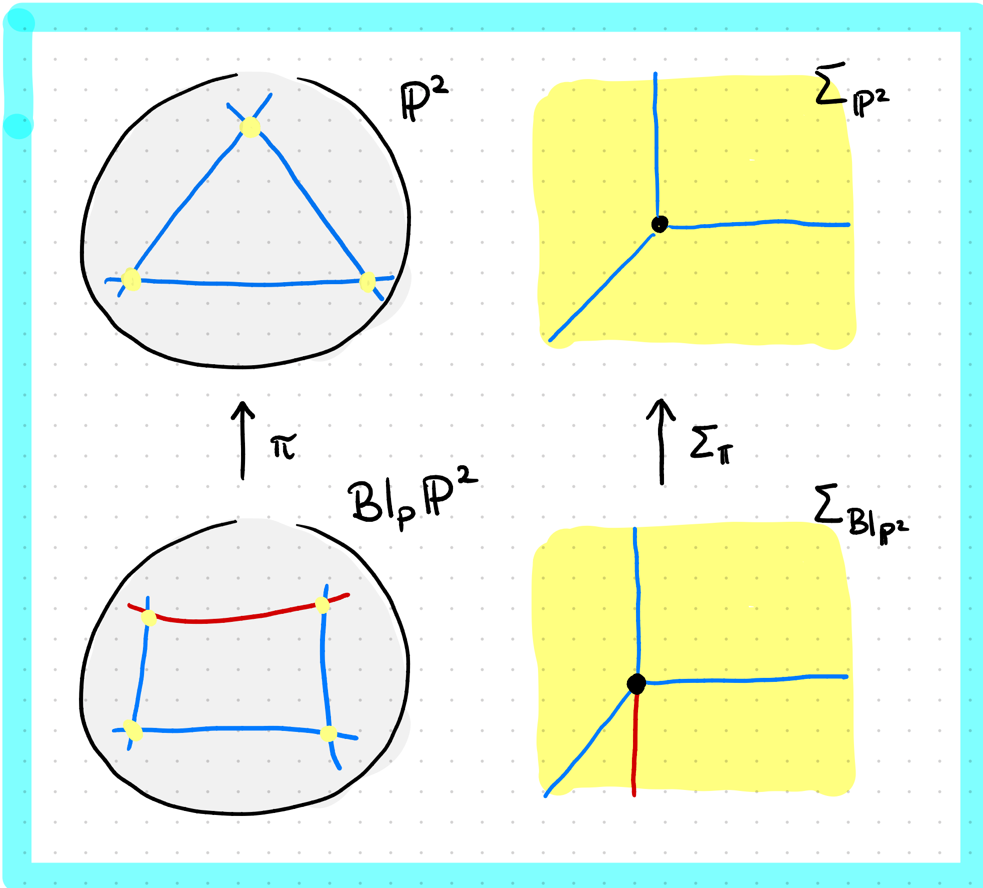
1-dim'l

Eg: $\boxed{DR'(-d,d)}$



Log = tool to fix this!

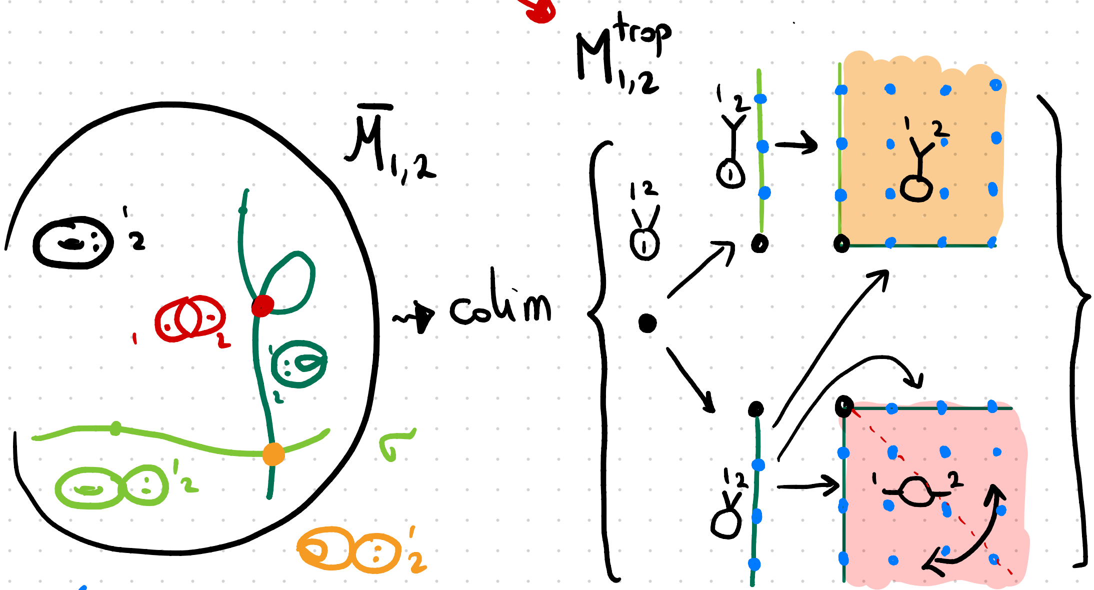
Toric geometry



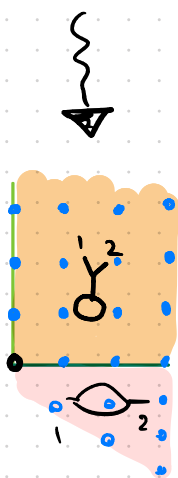
1. Stratification
2. Orbit-cone correspondence
3. Subdivisions $\rightsquigarrow$ blow-ups
4. $A_T^*(X_\Sigma) = PP(\Sigma) \longrightarrow A^*(X_\Sigma)$

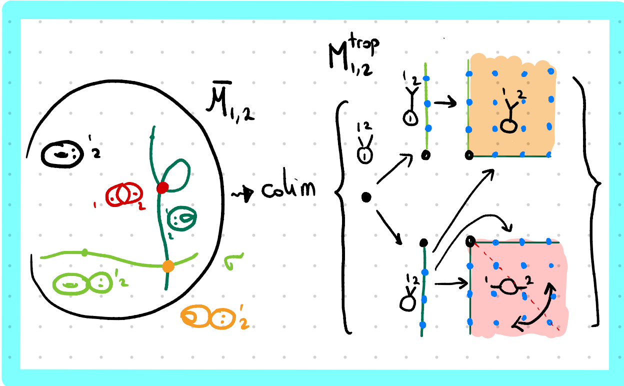
Artin fan of log space

<u>Log geometry</u>

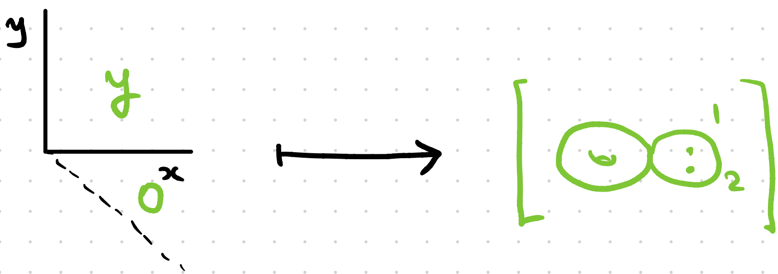


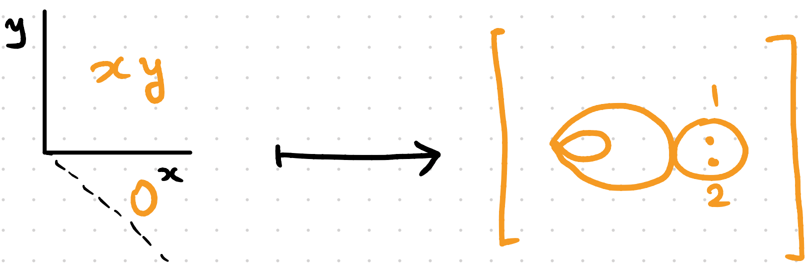
stalks of char. sheaf of monoids gives one cone for each stratum

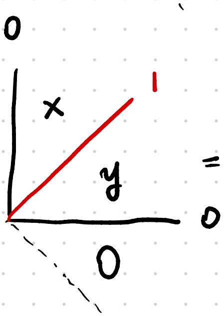




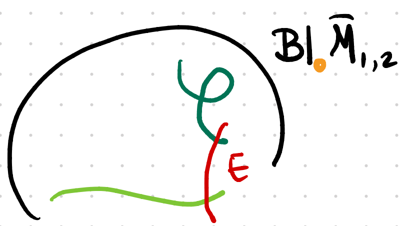
$$P.P.(M^{trop}_{g,n}) \longrightarrow \log CH^*(\bar{M}_{g,n})$$







$= \min(x,y) = E$



logarithmic DR-cycle

$$\boxed{\mathrm{DR}_g^{\log}(\vec{x}) \in \log\mathrm{CH}(\bar{\mathcal{M}}_{g,n})}$$

$$\begin{tikzcd}
\mathrm{Pic}_{g,n} \arrow[r] \arrow[d] & \bar{\mathrm{Pic}}_{g,n} \arrow[d] & \tilde{\mathrm{Pic}}_{\vec{x}} \arrow[l] \arrow[d] \\
\mathcal{M}_{g,n} \arrow[r, hook] \arrow[u, bend left, "\mathcal{O}"] \arrow[u, bend right, "\sigma_{\vec{x}}"'] & \bar{\mathcal{M}}_{g,n} \arrow[u, bend left, "\mathcal{O}"] \arrow[u, bend right, dashed, "\sigma_{x}"'] & \tilde{\mathcal{M}}_{\vec{x}} \arrow[l] \arrow[u, bend left, "\mathcal{O}"] \arrow[u, bend right, "\sigma_{x}"']
\end{tikzcd}$$

$$\mathcal{O}(C, p_1, \dots, p_n) = \mathcal{O}_C$$

$$\sigma_{\vec{x}}(C, p_1 \dots p_n) = \mathcal{O}_C\left(\sum x_i p_i\right)$$

BACK TO HURWITZ # PROBLEM:

$DR_g(\vec{x}) \subseteq \overline{M}_{g,n}$

but does anything in $\overline{M}_{g,n}$ cut down the branch condition?

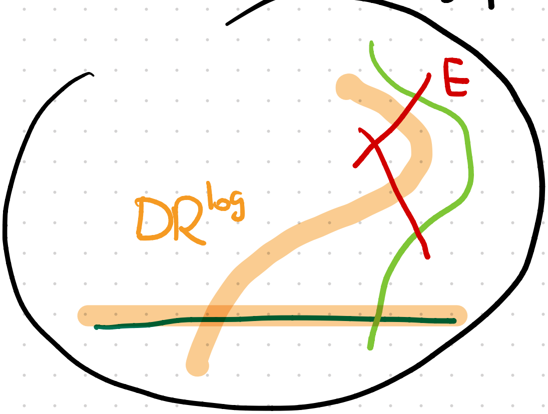
Thm (–, Markwig, Ranganathan)

There is $b_{\vec{x}} \in \log CH(\overline{M}_{g,n})$ s.t.

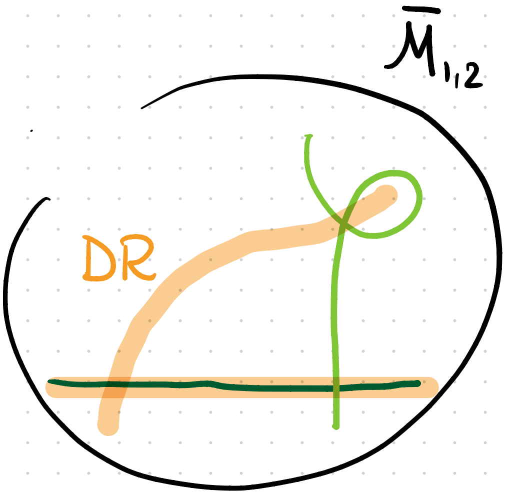
$$H_g(\vec{x}) = \deg\left(DR_g^{\log}(\vec{x}) \cdot b_{\vec{x}}\right)$$

Further: $b_{\vec{x}}$ AND $\widetilde{M}_{\vec{x}}$ are explicitly determined by $DR_g^{trop}(\vec{x})$

$\sim 2pt$ $= 2\varphi_p \cdot DR^{\log}$



$\downarrow$

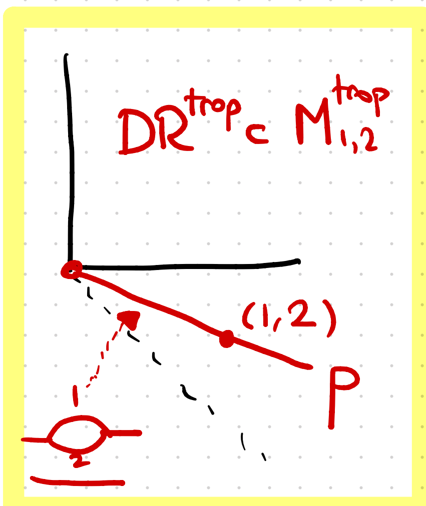


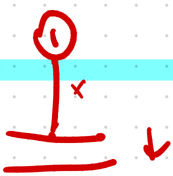
$$H_1(3,-3) = 2$$

$$= \prod \frac{\omega(e)}{|\mathrm{Aut}(\Gamma)|}$$

$$= \deg DR^{\log} \cdot 2 \cdot \varphi_p$$

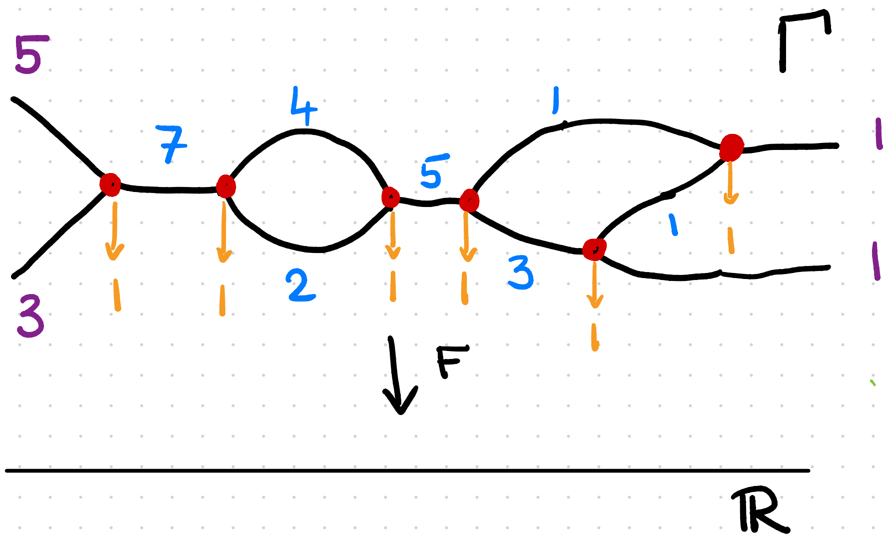
TROPICAL SIDE





New Enumerative Problems

① Leaking.

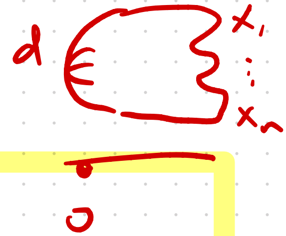


$$\boxed{k=1\ ,\ g=2\ ,\ x=(5,3,-1\ -1)}$$

related to moduli spaces of meromorphic k-differentials on curves.

$$H_g^k(\vec{x}) := \log DR_g^k(\vec{x})\cdot br_{\vec{x}}$$

Example : one pont, $g=0$



Hurwitz:

$$H_0(\vec{x}) = (n-2)!\, d^{n-3}$$

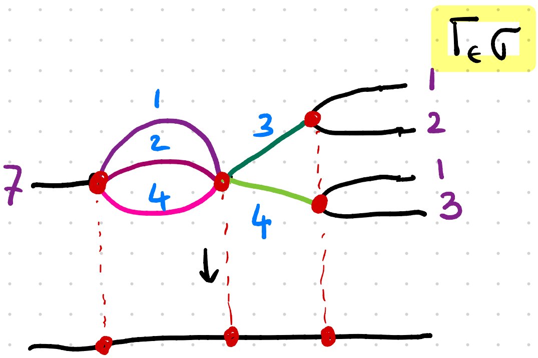
k-leaky

$$H_0^{k}(\vec{x}) = (n-2)! \prod_{i=1}^{n-3} \left(d - \frac{ik}{2} \right)$$

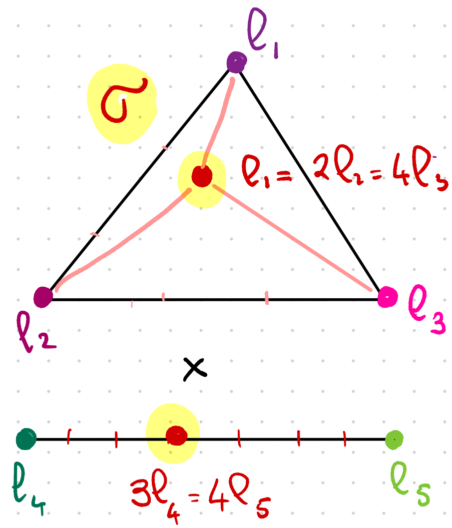
② Partial br p.polys

$$b_{i,\vec{x}} := br^{trop\,*}\left(\prod_{i=1}^{i} t_i\right)$$

imposes a non-maximal degeneration on the curves, leaving room for pairing with other tautological classes.



$$br_{2,\vec{x}}\big|_\sigma = 4^2\cdot 3\cdot 2\cdot P_\sigma$$



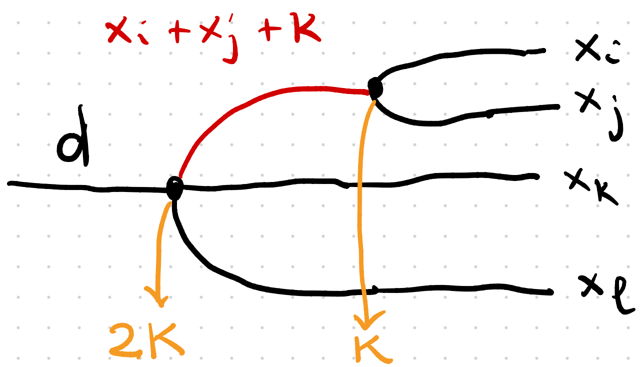
$$P_\sigma = \min(\ell_1, 2\ell_2, 4\ell_3)\cdot\min(3\ell_4, 4\ell_5)$$

ψ^I - Hurwitz #'s

joint w/
Hannah Markwig
Johannes Schmitt

$$H_g^k(\vec{x}, \psi^I) := \psi^I \cdot br_{i,\vec{x}} \cdot DR_g^{\log,k}(\vec{x})$$

$$H_0^k(d, -x_2, \dots, -x_5, \psi_1) = 3d - 3k$$



Γ_{ij}

K-LEAK: $d - \sum x_i = 3k$ i.e. $\sum x_i = d - 3k$

$$\sum_{\Gamma_{ij}} (x_i + x_j + k) = 3 \sum x_i + 6k = 3d - 3k$$

Example : one point, g=0

Hurwitz:

$$H_0(\vec{x}) = (n-2)!\, d^{n-3}$$

K-leaky

$$H_0^K(\vec{x}) = (n-2)! \prod_{i=1}^{n-3} \left(d - i\frac{K}{2}\right)$$

K-leaky w/ ψ_1

$$H_0^K(\vec{x}, \psi_1^e) = \frac{(n-2)!}{(e+1)!} \prod_{i=e+1}^{n-3} \left(d - i\frac{K}{2}\right)$$

Thanks!