Rational curves with (γ-)hyperelliptic singularities

Joint with R. Vidal Martins and V. Lara Lima (UFMG)

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More interesting: Inside of M_d^n, consider $M_{d,g}^n := \{\text{maps } \varphi \text{ with images of arithmetic genus } g\}$. By analogy with the $n = 2$ case, we'll call $M_{d,g}^n$ a *Severi variety*.

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Basic tool: Dimension counts for subloci $M^n_{d,\mathrm{S}} \subset M^n_{d,g}$ parameterizing maps with *cusps* of value semigroup S in their images.

Value semigroups of cusps

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Let $v : \mathbb{C}[[t]] \to \mathbb{N},\ t \mapsto 1$ be the usual t-adic valuation. Then $\mathrm{S} := \text{Im}(v \circ \phi) \subset \mathbb{N}$ is the *value semigroup* of the pair (φ, P).

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Philosophy: The topology of $M^n_{d,\mathrm{S}}$ is controlled by the value semigroup S.

A dimension-counting heuristic

$\mathrm{S} \subset \mathbb{N}$ is a *numerical* semigroup: $\#(\mathbb{N} \setminus \mathrm{S})$ is the δ-invariant of the cusp $\varphi(P)$. In particular, when $C = \varphi(\mathbb{P}^1)$ is a rational curve whose unique singularity is $\varphi(P)$, $g = \#(\mathbb{N} \setminus \mathrm{S})$ is the arithmetic genus of C.

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The g-nodal sublocus of $M^n_{d,g}$ has codimension $(n-2)g$ inside M^n_d. (To see this: each node imposes 2n-n-2=n-2 conditions, and it's easy to check that these are additive.)

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However this turns out not to be true in general!

Realizability and hyperellipticity

Every numerical semigroup $\mathrm{S} \subset \mathbb{N}$ is the value semigroup of a cusp. (Say that $\mathrm{S} = \langle e_1, \ldots, e_m \rangle$ is a minimal presentation; then $t \mapsto (t^{e_1}, \ldots, t^{e_m})$ realizes S.)

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Numerical semigroups are stratified according to their *hyperellipticity* degree: we say S is γ*-hyperelliptic* if

1) S contains exactly γ even numbers in $[1, 4\gamma]$; and
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The terminology is explained by the $\gamma = 0$ case, in which S is *hyperelliptic*, meaning that $2 \in \mathrm{S}$.

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Fact: When $g \gg \gamma$,

$\mathrm{S}_\gamma^1 := \langle 2\gamma+2, 2\gamma+4, \ldots, 2g\rangle + \langle 2g+1-2\gamma, \ldots, 2g-1, 2g+1\rangle$ and
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Goal: Compute $\mathrm{cod}(M_{d,\mathrm{S}}^n) \subset M_d^n$, with a particular focus on the cases $\mathrm{S} = \mathrm{S}_\gamma^i$, $i = 1, 2$.

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We will give a combinatorial model that conjecturally accounts for all of these conditions.

Dyck paths and semigroups

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$M^4_{d,\mathrm{S}}$ will contain maps $\varphi : \mathbb{P}^1 \to \mathbb{P}^4$ with various possible local ramification profiles near the cusp.

Typical example of rational curves with hyperelliptic cusps

A dimension-theoretically generic possibility is $t \mapsto (f_1 = t^2 + O(t^3), f_2 = t^4 + O(t^5), f_3 = t^6 + O(t^7), f_4 = t^8 + O(t^9))$, i.e. the t-adic valuation of φ in the preimage of the cusp is $(2, 4, 6, 8)$.

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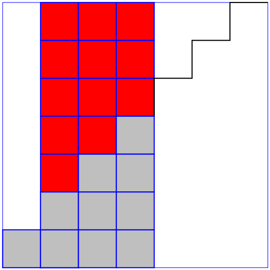

Figure: Ramification conditions r_P are in grey; the conditions contributed by f_i are in the ith column. Conditions b_P “beyond ramification” in red. We have $r_P + b_P - 1 = (n-1)g$.

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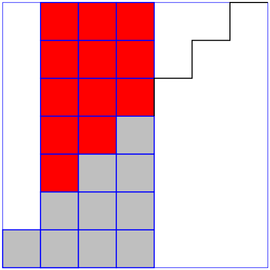

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A typical condition beyond ramification is that $\mathrm{lc}(f_j - f_1^j) = 0$, $j = 2, \ldots, n$.

Dimension count for rational curves with hyperelliptic singularities

Now set $F_j := f_j - f_1^2$. We can think of the condition $\mathrm{lc}(F_j) = 0$ imposed by F_j as encoded by the lowest red square in column j.

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Inductively, we may "walk" up column j, inducing a single new independent condition at every step. The condition encoded by the second-lowest red square is imposed by $F_j^* := F_j - [t^{2j+2}]F_j \cdot f_1^j$. To continue walking up the column, replace F_j by F_j^* and perturb by (a multiple of) a power of f_1.

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Thm (C., Lara Lima, Vidal Martins): Assume that $d \geq \max(2g-2, n)$; then $\mathrm{cod}(M^n_{d,\mathrm{S}_0} \subset M^n_d) \geq (n-1)g$, and conjecturally "$=$" holds. Moreover, M^n_{d,S_0} is the union of irreducible unirational varieties $M^n_{d,\mathrm{S}_0;\mathbf{k}}$ of fixed ramification profile $\mathbf{k}$. Here $\mathrm{S}_0 = \langle 2, 2g+1 \rangle$ is the unique hyperelliptic semigroup of genus g.

The $\gamma > 0$ case

We focus on γ-hyperelliptic cusps of maximal weight, i.e., $\mathrm{S} = \mathrm{S}_\gamma^2 = \langle 4, 4\gamma + 2, 2g - 4\gamma + 1 \rangle$.

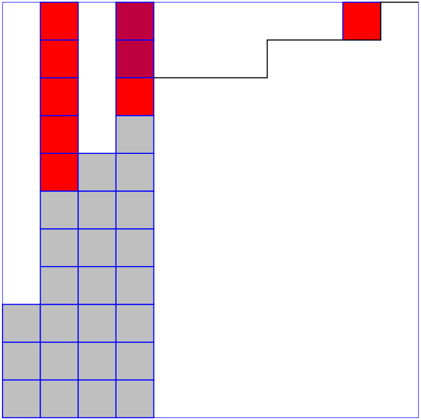

Figure: Conditions contributing to b_P and r_P for S_γ when $g = 11$, $\gamma = 2$, and $n = 4$. The dark red boxes do not contribute, i.e., they correspond to a correction to account for the fact that $2g - 4\gamma + 1 \notin \mathrm{Span}(\mathbf{k})$.

Dimension count for $M^n_{d,\mathrm{S}^2_\gamma}$ when $\gamma > 0$

Thm (C., Lara Lima, Vidal Martins): Let $\mathcal{V}_{\mathrm{S}_{g,\gamma}} := M^n_{d,g;\mathrm{S}_\gamma} \subset M^n_{d,g}$ denote the subvariety consisting of rational curves with a single singularity P that is a γ-hyperelliptic cusp with value semigroup S_γ, $\gamma > 0$. Assume as before that $n \leq 2g$, $d \geq \max(2g-2, n)$ and, moreover, that $g \geq 4\gamma + 2$. Then

$$\begin{aligned}\mathrm{cod}(\mathcal{V}_{\mathrm{S}_\gamma}, M^n_d) \geq (n-1)g &- \delta_{n\leq\gamma}(2\gamma + n + j^{**} - 4)\\ &- \delta_{\gamma+1\leq n\leq 3\gamma+1}(3\gamma + j^{**} - 3)\\ &- \delta_{n\geq 3\gamma+2; g\geq 4\gamma+j^{**}+5}(6\gamma + j^{**} - 2)\\ &+ \delta_{n\geq 3\gamma+2; g\leq 4\gamma+j^{**}+4}(g - 10\gamma - 2j^{**} - 3)\end{aligned}$$

where δ is Dirac's delta and j^{**} is either the unique nonnegative integer for which $g \in (6\gamma - 2j^{**}, 6\gamma - 2j^{**} + 2]$ or else $j^{**} = 0$.

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Upshot: The codimension of this locus is *always* at least $(n-2)g$.

Excess Severi components, and γ-hyperelliptic semigroups of minimal weight

Consider a cusp in $\mathbb{C}^\gamma$ parameterized by $t \mapsto (f_0, \ldots, f_{\gamma-1})$, where

$$f_i(t) := t^{2(\gamma+i)} + O(t^{2(\gamma+i)+1})$$

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$\mathrm{S}_1^* := \langle 2\gamma, 2\gamma+2, \ldots, 4\gamma-2 \rangle$ and $\mathrm{S}_2^* := \langle 4\gamma+5, 4\gamma+7, \ldots, 6\gamma+3 \rangle$.

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$$\mathrm{S}_1^* := \langle 2\gamma, 2\gamma+2, \ldots, 4\gamma-2 \rangle \text{ and } \mathrm{S}_2^* := \langle 4\gamma+5, 4\gamma+7, \ldots, 6\gamma+3 \rangle.$$

It is not hard to check that $g(\mathrm{S}^*) = 3\gamma + 1$.

Excess Severi components, and γ-hyperelliptic semigroups of minimal weight

Consider a cusp in $\mathbb{C}^\gamma$ parameterized by $t \mapsto (f_0, \ldots, f_{\gamma-1})$, where

$$f_i(t) := t^{2(\gamma+i)} + O(t^{2(\gamma+i)+1})$$

with *generic* higher-order coefficients.

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However, the *only* conditions imposed on rational curves in $\mathbb{P}^\gamma$ by cusps of type S^* arise from ramification, and there are $\frac{(5\gamma-3)\gamma}{2} - 1$ of these. This is less than $(\gamma-2)g(\mathrm{S}^*)$ whenever $\gamma \geq 8$.

Dimension-counting for $M^n_{d,\mathrm{S}}$ in general

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Definition. Given distinct natural numbers $k_1, \dots, k_n$, a *decomposition* of $s \in \mathbb{N}$ with respect to $k_1, \dots, k_n$ is an equation

$$s = m_1 k_1 + \ldots + m_n k_n \qquad (1)$$

with non-negative integer coefficients $m_j, j = 1, \dots, n$. Its *underlying partition* is $(k_1^{m_1}, \dots, k_n^{m_n})$. A decomposition as in (1) is *reducible* whenever some proper sub-sum of the right-hand side of (1) decomposes with respect to $k_1, \dots, k_n$; otherwise it is *irreducible*.

Dimension-counting for $M^n_{d,\mathrm{S}}$ in general

Conjecture (C., Lara Lima, Vidal Martins). Given a vector $\mathbf{k} := (k_0, \dots, k_n) \in \mathbb{N}^{n+1}_{\geq 0}$, let $M^n_{d;\mathrm{S},\mathbf{k}} \subset M^n_{d,g}$ denote the subvariety parameterizing rational curves $f : \mathbb{P}^1 \to \mathbb{P}^n$ with a unique cusp P with semigroup S and ramification profile $\mathbf{k}$. Suppose that $\mathrm{S} = \langle s_1^*, \dots, s_\ell^* \rangle$ is a presentation by minimal generators, $d = \deg(f) \geq \max(n, 2g-2)$, and set $\varphi(s) := \max\{\psi(s) - 1, 0\}$, $\rho(s) := \#\{r > s : r \notin \mathrm{S}\}$. Then

$$\mathrm{cod}(M^n_{d;\mathrm{S},\mathbf{k}}, M^n_d) = \sum_{i=1}^{n} (k_i - i) + \sum_{s \in \mathrm{S}} \varphi(s)\rho(s) - \sum_{i : s_i^* \notin \mathbf{k}} \rho(s_i^*) - D(\mathbf{k}) - 1$$

where $\psi(s)$ denote the number of irreducible decompositions for s with respect to $(k_1, \dots, k_n)$, and $D(\mathbf{k})$ is its *syzygetic defect*, which corrects for overcounting due to linear dependencies among the decompositions counted by $\psi(s)$ as s *varies*.

The syzygetic defect

Let $V = V(E)$ be the vector matroid on

$$E := \{v_{s_1,2} - v_{s_1,1}, \ldots, v_{s_1,\psi(s_1)} - v_{s_1,1}; \ \ldots \ \ldots \ ; v_{s_p,2} - v_{s_p,1}, \ldots, v_{s_p,\psi(s_p)} - v_{s_p,1}\}$$

where $s_p \in \mathrm{S}$ is the largest element strictly less than the conductor.

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Let $C_1, \ldots, C_q$ be the circuits of V and for each i, let $s(i)$ be the largest integer for which $v_{s(i),j} - v_{s(i),1} \in C_i$ for some j. Then

$$D(\mathbf{k}) := \sum_{i=1}^{q} \rho(s(i)).$$

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The syzygetic defect is 0 in the hyperelliptic case, in which all irreducible partitions are multiples of 2. It is also zero (for slightly less trivial reasons) in the $\gamma > 0$, maximal-weight case.

New families of Severi-excessive varieties

Cor. (C., Lara Lima, Vidal Martins). Let $\mathcal{V}_{(2\gamma+2,2\gamma+4,\ldots,2\gamma+2n)}$ denote the (generic stratum of a) Severi variety with underlying value semigroup S^*_γ as above.
Assume $n=\gamma+1$, $d\geq 2g-2$, that $g=3\gamma+6$ (resp., $g=3\gamma+8$) for some nonnegative integer $\gamma\geq 5$ (resp., $\gamma\geq 6$) and that the Conjecture holds; then $\mathcal{V}_{(2\gamma+2,2\gamma+4,\ldots,2\gamma+2n)}$ is of codimension

$$\frac{5}{2}\gamma^2+\frac{7}{2}\gamma+3 \text{ (resp., } \frac{5}{2}\gamma^2+\frac{7}{2}\gamma+10)$$

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Upshot: We expect unexpectedly large Severi varieties to exist in every genus $g \geq 21$ and every projective target dimension $n \geq 6$.

Additional questions

1. Is there a more *intrinsic* (i.e. coordinate-free) description of $M^n_{d,g;\mathrm{S}}$ as a degeneracy locus? This might allow for extensions of Eisenbud–Harris' dimensional transversality of Schubert varieties associated with ramification in points of linear series.

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2. What, explicitly, is the *deformation theory* of unicuspidal rational curves? E.g., we'd like a description of the space of (flat) infinitesimal deformations of maps $f : \mathbb{P}^1 \to \mathbb{P}^n$ that preserve a cusp of type S and its preimage, viewed as a subset of global sections of $f^*\mathcal{T}_{\mathbb{P}^n}$.

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3. What is the *homotopy type of the Berkovich analytification* of Severi varieties over Puiseux series?
4. What is the dimension theory of Severi varieties of unisingular rational curves whose singularities have *multiple* branches? *Realizability* and (geometric) *reconstruction* are wide open for semigroups of rank greater than one.