Real plane sextic curves with smooth real part

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Topology of algebraic varieties

Fix a deformation family $\mathcal{X}$ of complex varieties.
Consider a generic member $X \in \mathcal{X}$ "well understood."
Try to study "topological" ($\approx$ deformation invariant) properties of specific members $X \in \bar{\mathcal{X}}$, with extra structures.
Ultimate goal: classification up to appropriate deformation.

E.g., $\bar{\mathcal{X}} =$ degree d curves in $\mathbb{P}^2$, with $\mathcal{X} =$ smooth curves.

singular curves/ equisingular deformation	smooth real curves/ equivariant deformation
$\downarrow$	$\downarrow$

singular real curves/equisingular equivariant deformation

Real $=$ with an anti-holomorphic involution $\sigma \colon X \to X$.
Equisingular $\approx$ stratum $\mu =$ const. No question for $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$.

→ Topology of algebraic varieties

Degree	Singular complex	Smooth real
$d = 2$	Ancient Greeks (?)	Ancient Greeks (?)
$d = 3$	Newton (?)	Newton (?)
$d = 4$	Klein (?)	Klein (?)
$d = 5$	D–, 1987	Kharlamov, 1981 (used nodal real)
$d = 6$ **!!**	Akyol, 2012 (*)	Gudkov (topological) Nikulin, 1979; Kharlamov
$d = 7$	??	?? Viro (topological)
$d = 8$	??	?? Korchagin, Orevkov, Shustin, . . .

*Artal, Carmona, Cogolludo, Tokunaga, . . . ; Eyral, Oka; . . .

*Persson, Urabe, Yang, Shimada, D–, . . . ($K3$-theoretic)

***A–D–E**-singularities; non-simple: easier D–, 1989

→ **Topology of algebraic varieties**

Singular complex *vs.* smooth real curves: totally different, but:
compatible difficulty,
often similar tools.

Singular **and** real curves are less studied:

- Gudkov *et al.*, 1966: irreducible quartics.
- Gudkov *et al.*: rational quintics (sets of singularities) ??
- Kharlamov, 1981: one-nodal quintics (to do nonsingular).
- Jaramillo–Puentes, 2021: nodal rational quintics.
- Itenberg, 1991: one-nodal sextics (**problem discovered**).
- Josi, 2017: nodal rational sextics (**same problem**).

The magic of $K3$-surfaces

The degree $\deg = 6$ seems a natural threshold for curves in $\mathbb{P}^2$. (Similar, $\deg = 4$ is a threshold for surfaces in $\mathbb{P}^3$.)

- $\deg < 6$: everything is classical and straightforward.

- $\deg > 6$: nothing is known; there are no tools.

- $\deg = 6$: classical means do not work well, but using the "magic of $K3$-surfaces" any "reasonable" geometric question can be reduced to an arithmetical problem.

Remarkable fact: these arithmetical problems are very difficult, but they turn out feasible as long as they are $K3$-related.

Remark 1 *If $C \subset \mathbb{P}^2$ is a sextic with at worst $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$-singularities, the (minimal resolution of singularities of) the double covering $X \to \mathbb{P}^2$ ramified over C is a $K3$-surface.*

→ The magic of $K3$-surfaces

- The global Torelli theorem [Pjateckiĭ-Šapiro–Šafarevič, 1971].
- Surjectivity of the period map [Kulikov, 1977].
- Projective models of $K3$-surfaces [Saint-Donat, 1974].
- The Riemann–Roch theorem (for $K3$):
  every (-2)-class is a (unique) algebraic curve!
- Discriminant forms and embeddings of lattices [Nikulin, 1979]
  extended in [Miranda–Morrison, 1985, 1986, 2009].
- Groups generated by reflections [Vinberg, 1972].

See also [Beauville, Dolgachev, Huybrechts, ...].

$\rightarrow$ **The magic of $K3$-surfaces**

In a nutshell: we record the geometric data in the homology

$$H_2(X) \cong \mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$$

and call this the *homological type*.
State simple necessary conditions: they turn out sufficient.
Hence, we have

$$\{\text{homological types}\}/\cong \quad = \quad \{\text{geometric objects}\}/\sim$$

It remains to classify the homological types.

Remark 2 *Extracting geometric invariants back from the homological type may still be difficult.*

Even more difficult is writing down the defining equations.

→ The magic of $K3$-surfaces

Singular complex sextics:

polarization $h \in H_2(X) \supset \mathbf{S}$ exceptional divisors

Algebraic geometry is in the *lattice type* $\tilde{\mathbf{S}}_h := (\mathbf{S} \oplus \mathbb{Z}h)^{\sim}$:
this is the "typical" Néron–Severi lattice.

Conditions: $\mathbf{S}$ is a root lattice, $h^2 = 2$, and $h \perp \mathbf{S}$;
besides, there are no "bad" vectors $e \in \tilde{\mathbf{S}}_h$ s.t.

$$e^2 = -2,\ e \cdot h = 0,\ e \notin \mathbf{S} \qquad \text{or} \qquad e^2 = 0,\ e \cdot h = 1.$$

Smooth real sextics:

polarization $h \in H_2(X) \circlearrowleft \theta := \sigma_*$ real structure

Conditions ($\mathbf{L}_\pm$ are the ± 1-eigenlattices of θ):

$$h^2 = 2, \quad \theta^2 = \mathrm{id}, \quad h \in \mathbf{L}_-, \quad \sigma_+(\mathbf{L}_-) = 2.$$

(By Nikulin, everything is in the 2-elementary lattice $\mathbf{L}_-$.)

→ The magic of $K3$-surfaces

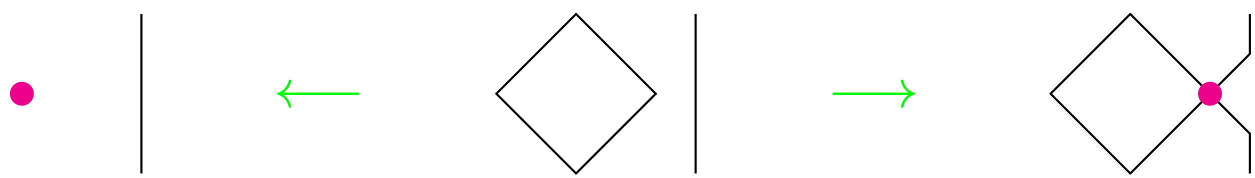
Singular real sextics

We combine the two, recording h, $\mathbf{S}$, and θ.

The extra condition is that $\mathbf{S}$ must be θ-invariant and,
moreover, $\mathbf{S}$ must have a θ-skew-invariant Weyl chamber (??)

The result is called the *real homological type*.

However, this does not determine the deformation class!



Here, the two nodes are indistinguishable homologically!

Remark 3 *Everything* **is** *in the homology; we need extra data*:
a distinguished fundamental polyhedron of a certain group.

Difficult to compute, often infinite; hence, try to avoid.

In the smooth case, that would be the one containing h.

Real sextics with smooth real part

The problem is due to the real singular points: 
(First observed by Josi in the nodal case.)

Theorem 4 *Two real simple sextics without real singular points are in the same equisingular equivariant deformation class if and only if their real homological types are isomorphic.*

Hence, we have a complete list of such sextics.
(The arithmetical problem is doable due to its $K3$-origin.)

Observations: (to be discussed later)

- lack of "obvious" topological invariants;
- extremal congruences for singular curves/surfaces;
- "dominance" of type I real schemes.

→ Real sextics with smooth real part

Proof of Theorem 4

Since we want to compare several surfaces, $H_2(X)$ will not do.
The classical approach:

1. Consider *abstract homological types* $(h \in \mathbf{L} \supset \mathbf{S}, \theta)$.

2. Work with *marked* surfaces $(X, \varphi\colon H_2(X) \xrightarrow{\sim} \mathbf{L})$.

3. Use (1) to define the *period space* Ω of **marked** surfaces.

4. This space is **disconnected**:
   it has *walls* $H_v \approx v^\perp \cap \Omega$, where $v \in \mathbf{L}$, $v^2 = -2$, $v \cdot h = 0$.

5. (!!) Prove that (wall crossing) = (change of marking).
   It is this part that fails for real singular points.

$\to$ Real sextics with smooth real part

In the problem at hand, periods are

$$\omega := (\omega_+, \omega_-) \in \Omega := \mathcal{C}(\tilde{\mathbf{S}}_h{}^\perp \cap \mathbf{L}_+) \times \mathcal{C}(\tilde{\mathbf{S}}_h{}^\perp \cap \mathbf{L}_-),$$

where

$$\mathcal{C}(\mathbf{N}) := \left\{ x \in \mathbf{N} \otimes \mathbb{R} \,\middle|\, x^2 > 0 \right\} \Big/ \mathbb{R}^\times$$

is (the projectivization of) the positive cone.

A wall H_v has codimension 1 iff $v \in \mathbf{L}_+$ or $v \in \mathbf{L}_{-h} := h^\perp \subset \mathbf{L}_-$.
It does intersect the positive cone iff $\mathbf{S} + \mathbb{Z}v$ is negative definite.
(Otherwise, x, h, ω_+, ω_- would be four positive squares.)
We need an autoisometry $r\colon \mathbf{L} \to \mathbf{L}$ “reflecting” against H_v
and preserving the structure, i.e., h, $\mathbf{S}$, *and* θ.

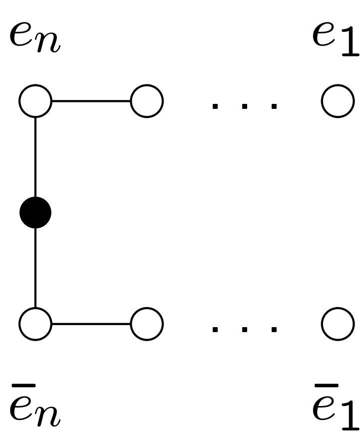
For example, in the smooth case $\mathbf{S} = 0$, ref_v would do.
If $\circ$ is a real node and $v = \bullet$ is like $\circ\!\!-\!\!\!-\!\!\bullet$, there is no hope!

→ Real sextics with smooth real part

Thus, (Dynkin basis for $\mathbf{S}$) $\cup \{v\}$ must be a "Dynkin diagram" with dotted edges (intersection -1) allowed.

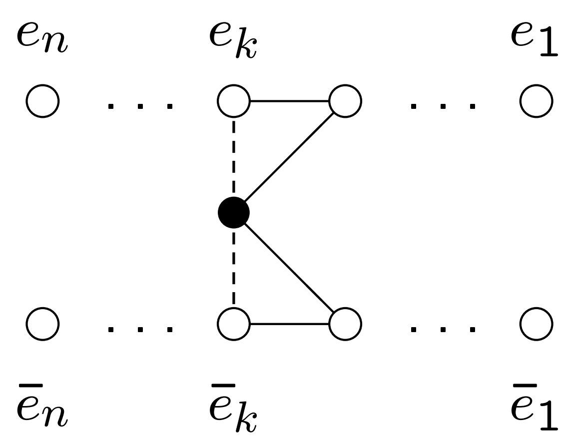
Crucial: the $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$-components of $\mathbf{S}$ split into pairs interchanged by θ (no real singular points).

Easy to classify: a symmetric graph without induced $\tilde{\mathbf{A}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$.



$$\Sigma \simeq \mathbf{A}_n$$

$$\bar{\Sigma} = \mp\theta(\Sigma)$$



Both are $\mathbf{A}_{2n+1}$: $e_{n+1} := v - (e_k + \ldots + e_n + \bar{e}_n + \ldots + \bar{e}_k)$.
"Reflection" preserving $\{e_i, \mp\bar{e}_i\}$: either $-\theta$ or

$$e_i \leftrightarrow e_{i+n+1},\ \ i = 1, \ldots, n, \qquad e_{n+1} \mapsto -(e_1 + \ldots + e_{2n+1}).$$

Acts trivially on the discriminant ⇒ product of reflections. □

Topological invariants

The classification is topological **by definition**:
distinct deformation families differ homologically.
One can try to compute more "visible" invariants:

- irreducible components
- singularities w.r.t. components
- splitting conics $\approx \tilde{\mathrm{S}}_h/(\mathbf{S} \oplus \mathbb{Z}h)$

(the three items above) $\leftarrow$ lattice type

- components real/conj?
- splitting conics real/conj/...?

(the two items above) $\leftarrow$ RLT

- types I/II of the components
- ovals w.r.t. the components
- intersection of halves: number, where

No fundamental polyhedron $\Rightarrow$ not enough data but
we use heuristic approach + adjacencies
Still, not enough to distinguish everything!
Interesting phenomena: *e.g.*, differ by degenerations.
Some can be explained by the "geometric construction."

Extremal congruences

Locally irreducible simple singularities: $\mathbf{A}_{2k}$, $\mathbf{E}_6$, $\mathbf{E}_8$.
In other words, no 2-torsion in the discriminant.
The *additive Kronecker–Jacobi symbol*

$$J(\mathbf{S}) := \frac{1}{2} - \frac{1}{2}\left(\frac{2}{|\det \mathbf{S}|}\right) = \begin{cases} 0 \bmod 2, & \text{if } \det \mathbf{S} = \pm 1 \bmod 8, \\ 1 \bmod 2, & \text{if } \det \mathbf{S} = \pm 3 \bmod 8. \end{cases}$$

Example 5 $J(\mathbf{S}_1 \oplus \mathbf{S}_2) = J(\mathbf{S}_1) + J(\mathbf{S}_2) \bmod 2$ *and*

$$J(\mathbf{A}_2) = J(\mathbf{A}_4) = J(\mathbf{E}_6) = 1, \quad J(\mathbf{A}_6) = J(\mathbf{A}_8) = J(\mathbf{E}_8) = 0.$$

Theorem 6 X*: a real* $(M-d)$*-surface,* $H_1(X) = 0$ (?)*,*
locally simple singularities $2\,\mathbf{S}$*, none real.*

- $d = 0 \Rightarrow \chi(X_{\mathbb{R}}) = \sigma(X) + 8J(\mathbf{S}) \bmod 16$;
- $d = 1 \Rightarrow \chi(X_{\mathbb{R}}) = \sigma(X) + 8J(\mathbf{S}) \pm 2 \bmod 16$;
- $d = 2$ & $\chi(X_{\mathbb{R}}) = \sigma(X) + 8J(\mathbf{S}) + 8 \bmod 16 \Rightarrow X$ *is type* I.

→ Extremal congruences

Example 7 *Type I real M-schemes of sextics with n ovals*:

$$\begin{array}{lll} n=9: & 2\mathbf{A}_2: & 2\langle 6\rangle,\ 6\langle 2\rangle \\ n=7: & 2\mathbf{A}_4: & 1\langle 5\rangle,\ 5\langle 1\rangle \\ & 4\mathbf{A}_2: & 3\langle 3\rangle \\ n=5: & 6\mathbf{A}_2,\ 2\mathbf{E}_6: & 0\langle 4\rangle \\ & 2\mathbf{A}_4\oplus 2\mathbf{A}_2: & 2\langle 2\rangle \end{array}$$

Real $(M-1)$-schemes:

$$\begin{array}{lll} n=8: & 2\mathbf{A}_2: & 1\langle 6\rangle,\ 2\langle 5\rangle,\ 5\langle 2\rangle,\ 6\langle 1\rangle \\ n=7: & 2\mathbf{A}_4: & 0\langle 5\rangle,\ 1\langle 4\rangle,\ 4\langle 1\rangle,\ 5\langle 0\rangle = 6 \\ & 4\mathbf{A}_2: & 2\langle 3\rangle,\ 3\langle 2\rangle \\ n=4: & 6\mathbf{A}_2,\ 2\mathbf{E}_6: & 0\langle 3\rangle,\ 3\langle 0\rangle = 4 \\ & 2\mathbf{A}_4\oplus 2\mathbf{A}_2: & 1\langle 2\rangle,\ 2\langle 1\rangle \end{array}$$

Agrees with "modifications of real schemes of type I."

$\to$ Extremal congruences

Remark 8 *Need to distinguish between two "types I":*

- *the curve itself, or upon smoothing* (type I scheme)*;*
- *the curve upon the normalization* (type I component)*.*

The two coincide if all singularities are locally irreducible.

Example 9 $2\mathbf{A}_2$ *does not appear in* $3\langle 3\rangle$ *of type II.*
$4\mathbf{A}_2$ *does not appear in* $0\langle 4\rangle$ *or* $4\langle 0\rangle = 5$ *of type II.*
$2\mathbf{A}_4$ *does not appear in* $2\langle 2\rangle$ *of type II.*
They do appear in all other schemes with the same n*.*

Conjecture 10 *A similar statement should hold for*

$$\mathbf{A}_3,\ \mathbf{A}_7,\ \mathbf{A}_{11},\ldots,\ \mathbf{D}_5,\ \mathbf{D}_7,\ldots,$$

(*no rigorous proof or statement yet*)*, but not for*

$$\mathbf{A}_1,\ \mathbf{A}_5,\ldots,\ \mathbf{D}_4,\ \mathbf{D}_6,\ldots,\ \mathbf{E}_7.$$

In other words, no hope if $\operatorname{discr}\mathbf{S}$ *has* $\mathbb{Z}/2$*-summands.*

Modifications of real schemes of type I

A phenomenon specific to real plane sextics (K3):

every empty oval can be contracted (and then erased)

Can also be proved in the presence of other singularities.
Induces a transformation of the sets of deformation classes.
(May depend on the oval chosen.)

An observation (still needs understanding):

$$\begin{array}{ccc} \textbf{type I} \text{ real scheme} & \longrightarrow & \text{(any) one oval erased} \\ \downarrow & & \downarrow \\ \text{deformation classes} & = & \text{deformation classes} \end{array}$$

First observed for $\langle\langle 1\rangle\rangle \to \langle 1\rangle$.
For "very" singular curves $\Leftarrow$ the "geometric construction."

The geometric construction

"Very" singular sextics have *stable involutions* ($\Leftarrow K3$),
i.e., involutions $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ *persistent throughout the family*

$$\begin{array}{ccccccl} & C & \subset & \mathbb{P}^2 & \supset & O \cup L & \\ & & & \downarrow & & & \\ \text{trigonal curve} & \bar{C} & \subset & \Sigma_2 & \supset & E \cup \bar{L} & \text{except. + ord. section} \end{array}$$

The trigonal curve $\bar{C} \subset \Sigma_2$ is *extremal* $\approx$ rigid $\approx \mu(\bar{C}) = 8$.
Exceptional section: $y = \infty$; ordinary section $\bar{L}$: a "parabola."

Also provides **examples** in the non-extremal case:
involution may exist, but it is no longer stable.

$\rightarrow$ The geometric construction

This picture shows all "algebraic geometry", such as

- components of C and the action of σ,

- splitting conics and the action of σ, *etc.*,

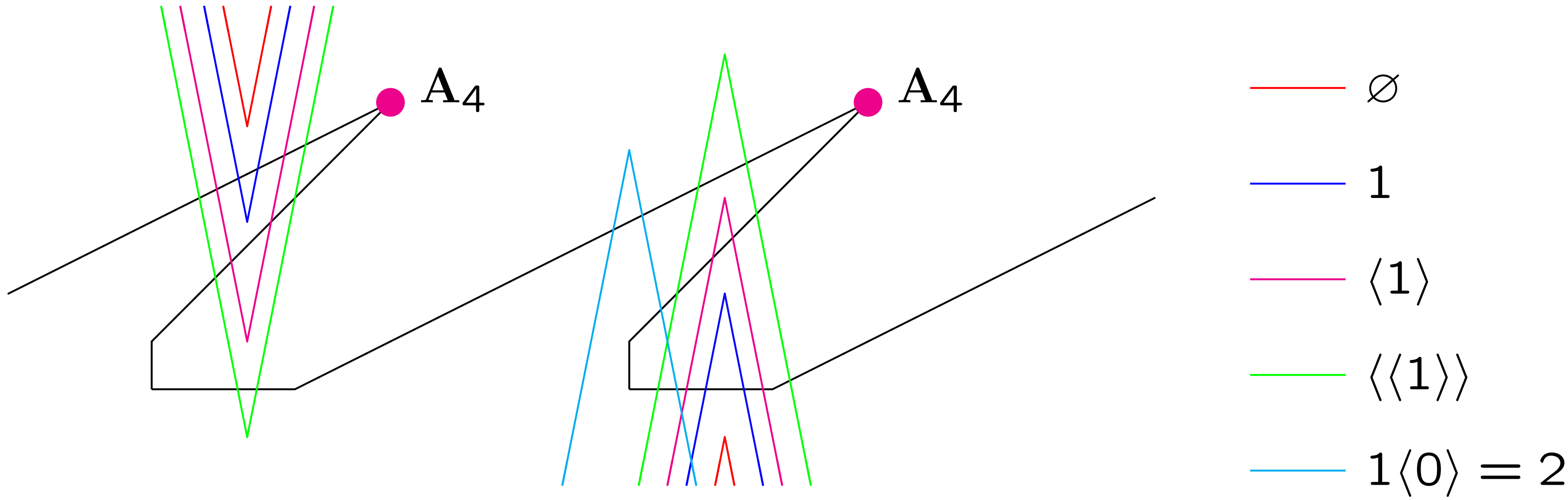
and a bit of topology (not detectable otherwise), *e.g.*,

- $\bar{C}$ may have several real forms (*e.g.*, if $\operatorname{Sing}\bar{C} = \mathbf{A}_7 \oplus \mathbf{A}_1$),

- "upwards" *vs.* "downwards" parabolas.

→ The geometric construction

Example 11 *The set of singularities* $4\mathbf{A}_4$*, with* Sing $\bar{C} = 2\mathbf{A}_4$.

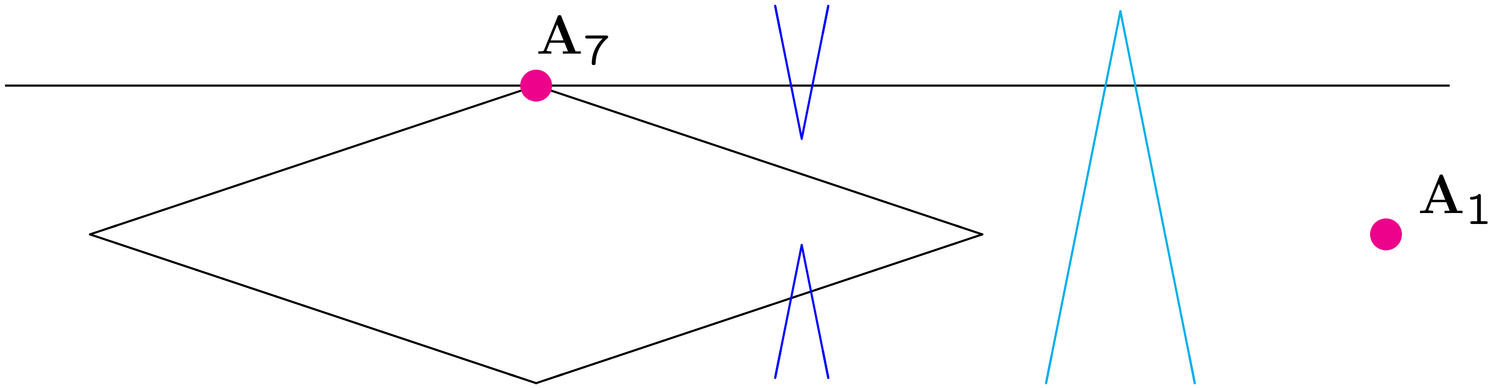


Can **always** *create an oval inside a nest* $\langle 1 \rangle \to \langle\langle 1 \rangle\rangle$:
once we see two branches, we also see the third one!
May **not** *be able to create an extra oval in* $1\langle 0 \rangle = 2$.

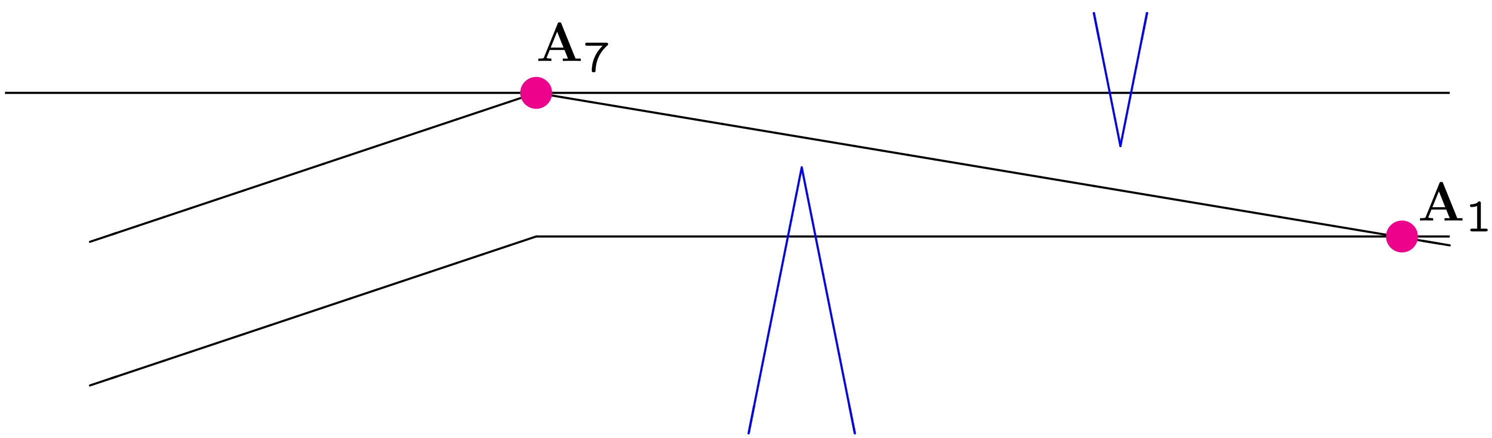
Remark 12 *Typically, the "upwards"/"downwards" parabolas differ by the action of* σ *on splitting conics & their pull-backs.*
The splitting conics are also clearly seen in these pictures:
sections passing in a certain way through singular points.

→ The geometric construction

Example 13 Sing $C = 2\mathbf{A}_7 \oplus 2\mathbf{A}_1$, *with* Sing $\bar{C} = \mathbf{A}_7 \oplus \mathbf{A}_1$.
An "ellipse" and a parabola:



May not be able to convert 1 *to* $\langle 1 \rangle$.
There is another real form, a "hyperbola" and a parabola;
the distinction upstairs is not clear.



Thank you!