Galois realizations of Special linear groups

(jt. with Stefan Reiter)

IGP: Let G finite gp., does there exist a Galoisextension $L/\mathbb{Q}$ such that $G \simeq \mathrm{Gal}(L/\mathbb{Q})$?

IGP unsolved, answer known for solvable groups (Shafarevich), sporadic simple groups except M_{23} (Thompson, Matzat, ...), certain classes of finite gps of Lie-Type (Matzat, Malle, Völklein, D.-Reiter, ...)

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RIGP: Let G finite gp, does there ex. Gal. ext. $L/\mathbb{Q}(t)$: $\cdot\; G \simeq \mathrm{Gal}(L/\mathbb{Q}(t)) \simeq \mathrm{Gal}(\tilde{L}/\bar{\mathbb{Q}}(t))$

$$\begin{tikzcd} & \tilde{L} \text{ compositum} \arrow[dl, dash, "G"'] \arrow[dr, dash] & \\ \bar{\mathbb{Q}}(t) \arrow[dr, dash] & & L \arrow[dl, dash, "G"] \\ & \mathbb{Q}(t) & \end{tikzcd}$$

In the Situation of RIGP: ②

("G occurs regularly as Galois gp over $\mathbb{Q}(t)$")

THM 1: (D.-Reiter) Let $q = \ell^k$ $(k \geq 1)$ with ℓ odd prime number and let $n > 8\varphi(q-1) + 11$, then $Sl_n(\mathbb{F}_q)$ occurs regularly as Galois group over $\mathbb{Q}(t)$.

Rem: Similar result for $Gl_n(\mathbb{F}_q)$ known since 1992 for n even (Völklein) and n odd (D.-Reiter 1999).

Main ingredients of the proof of THM 1:

- Convolution of sheaves
- Relation to Laumon's theory of ℓ-adic Fourier transform

Let X connected top. sp., connected variety, ...

<u>Local system</u> $\mathcal{L}$ on X $\hat{=}$ <u>monodromy representation</u>

$$\rho_{\mathcal{L}} : \pi_1(X, *) \to GL(V)$$

$$V = \mathcal{L}_* \simeq K^n$$

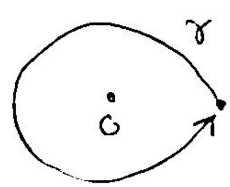
$(K = \overline{\mathbb{Q}}_\ell, \mathbb{F}_q, \overline{\mathbb{F}}_q, \mathbb{Q}_\ell, \ldots$ field$)$

<u>Rank of $\mathcal{L}$</u>: $rk(\mathcal{L}) := \dim V$

(X scheme, same formalism with (ℓ-adic) étale LS)

<u>Ex 2:</u> 1) <u>Kummer-sheaves:</u> Let $X = \mathbb{G}_m(\mathbb{C}) = \mathbb{C} \setminus \{0\} = \mathbb{C}^\times$

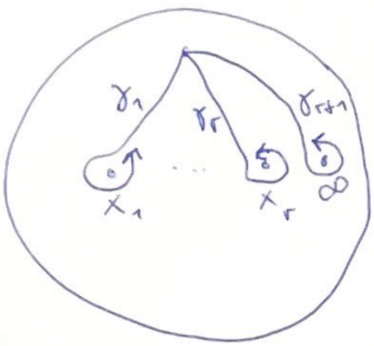
and let $\lambda \in K^\times$.



$\pi_1(\mathbb{C}^\times) = \langle \gamma \rangle$ free abelian gp

$$\left\{ \begin{array}{c} \pi_1(\mathbb{C}^\times) \dashrightarrow GL_1(K) \\ \gamma \longmapsto \lambda \end{array} \right\} \hat{=} \left\{ \begin{array}{c} \text{local system} \\ \mathcal{L}_\lambda \text{ on } \mathbb{C}^\times \end{array} \right\}$$

2) $X = \mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \dots, x_r, x_{r+1} = \infty\}$
$= \mathbb{C} \setminus \{x_1, \dots, x_r\}$



product relation

$$\pi_1(X) = \langle \gamma_1, \dots, \gamma_{r+1} \mid \gamma_1 \cdots \gamma_{r+1} = 1 \rangle$$

$\mathcal{L}$ local system of rank n on X $\hat{=}$ $\rho_{\mathcal{L}} : \pi_1(X) \to \mathrm{Gl}_n(K)$

$\hat{=}$ Monodromy-tuple

$(T_1, \dots, T_{r+1}) \in \mathrm{Gl}_n(K)^{r+1}$

$T_i = \rho_{\mathcal{L}}(\gamma_i)$, s.t.

$T_1 \cdots T_{r+1} = 1$

Vanishing cycles:

$$\phi_{x_i}(\mathcal{L}) = \phi_{x_i} := V / V^{T_i} \circlearrowleft T_i, \quad V = K^n;$$

$\leftrightarrow$ Jordan form of T_i, i.e. local monodromy.

Relation to RIGP?

Let $\mathcal{L}$ étale local System on $X = \mathrm{Spec}(\mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]} \setminus \underline{x})$

$= \mathrm{Spec}\left(\mathbb{Z}[\tfrac{1}{N}][t][\frac{1}{(t-x_1)\cdots(t-x_n)}]\right)$

$$\begin{tikzcd}
 & & & \langle \mathrm{Frob}_p \rangle \arrow[d, phantom, "\shortparallel"] & \\
1 \arrow[r] & \pi_1^t(\mathbb{A}^1_{\overline{\mathbb{F}}_p} \setminus \underline{x}) \arrow[r] \arrow[dd, phantom, "\parallel"] & \pi(\mathbb{A}^1_{\mathbb{F}_p} \setminus \underline{x}) \arrow[r] \arrow[d] & G_{\mathbb{F}_p} \arrow[l, bend right] \arrow[r] & 1 \\
 & & & \rho_{\mathcal{L}}(\mathrm{Frob}_p) \arrow[d, phantom, "\cap"] & \\
 & & \pi_1(\mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]} \setminus \underline{x}) \arrow[r, "\rho_{\mathcal{L}}"] & GL(V) \arrow[d, phantom, "\ni"] & \\
 & & & T_i = \rho_{\mathcal{L}}(\gamma_i) & \\
1 \arrow[r] & \pi_1(\mathbb{C} \setminus \underline{x}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_1(\mathbb{A}^1_{\mathbb{Q}} \setminus \underline{x}) \arrow[r] \arrow[uu, two heads] & G_{\mathbb{Q}} \arrow[l, bend right] \arrow[r] \arrow[d, phantom, "\shortparallel"] & 1 \\
 & \langle \gamma_1, \dots, \gamma_{r+1} \mid \gamma_1 \cdots \gamma_{r+1} = 1 \rangle & & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \\
 & & G_{\mathbb{Q}(t)} = \mathrm{Gal}(\overline{\mathbb{Q}(t)}/\mathbb{Q}(t)) \arrow[uu, two heads] & &
\end{tikzcd}$$

<u>Conclusion:</u> • $\mathrm{im}(\rho_{\mathcal{L}})$ is quotient of $G_{\mathbb{Q}(t)}$

• Frobenius elements "come" from upper short exact sequence

Strategy of proof of THM 1:

- •) Construct $\mathcal{L}$ as above with

$$\mathrm{im}(\rho_{\mathcal{L}}) = Sl_n(\mathbb{F}_q) = \langle T_1, \ldots, T_{r+1} \rangle$$

by Tensorproducts and convolutions:

$$(*) \quad \mathcal{L} = \mathcal{N}_2 \otimes \left(\mathcal{L}_{-1} * \left(\mathcal{N}_1 \otimes \underbrace{(\mathcal{L}_{-1} * \mathcal{F})}_{=: MC_{-1}(\mathcal{F})} \right) \right)$$

$MC_{-1}(\mathcal{F}) := \mathcal{L}_{-1} * \mathcal{F}$ <u>middle convolution</u>
of $\mathcal{F}$ with Kummer sheaf $\mathcal{L}_{-1}$

- •) Bound $\mathrm{im}(\rho_{\mathcal{L}})$ from above by proving $\det(\rho_{\mathcal{L}}(\mathrm{Frob}_p)) = \pm 1$
(Cebotarev's density theorem), using Fourier-Transform

Explanation of $\mathcal{L} = \mathcal{N}_2 \otimes MC_{-1}(\mathcal{N}_1 \otimes (MC_{-1}(F)))$:

(1) F étale local $\mathbb{F}_q$-system on $\mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]} \setminus \underline{x}$ with finite monodromy and monodromy-tuple $(T_1, \dots, T_{r+1})$ s.t.

$$T_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \dots, \quad T_{r-3} = \begin{pmatrix} \lambda_{r-3} & \\ & \lambda_{r-3}^{-1} \end{pmatrix},$$

$$T_{r-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_{r-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_r = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$T_{r+1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Rank of $\mathcal{L}$ depends on r and $\mathcal{N}_1, \mathcal{N}_2$ see below)

Here $\lambda_1 = \zeta_{q-1} \in \mathbb{F}_q^\times$ multiplicativ generator and $\lambda_1, \dots, \lambda_{2\varphi(q-1)}$ run twice through the primitive powers of λ_1. Explicit construction of F as étale-cover

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{A}^1 \setminus \underline{x} \end{array}$$

by abelian covers + pushforward.

(2) $$\mathcal{L}_{-1} * F = \mathrm{MC}_{-1}(F)$$
$$= R^1 \overline{pr}_2 \left(j_*(pr_1^* F \otimes d^* \mathcal{L}_{-1}) \right)$$

$$\begin{tikzcd}
& \mathbb{P}^1 \times \mathbb{A}^1 \arrow[dr, "\overline{pr}_2"] & \\
& \mathbb{A}^1 \times \mathbb{A}^1 \arrow[u, hook, "j"] \arrow[r, "pr_2"'] \arrow[dl, "pr_1"] \arrow[d, "d"] & \mathbb{A}^1 \\
\mathbb{A}^1 & \mathbb{A}^1 \rightsquigarrow \mathcal{L}_{-1} &
\end{tikzcd}$$

$\mathbb{A}^1$
$\wr$
$\tilde{F}$ = ext. of F to $\mathbb{A}^1$

Properties of $\mathrm{MC}_{-1}(F)$, F irred. on $\mathbb{A}^1 \setminus \underline{x}$ with monodromy tuple $(T_1, \dots, \underbrace{T_{r+1}}_{=-\mathbb{1}}) \in GL(V)^{r+1}$

<u>Irreducibility preserved</u>) $\mathrm{MC}_{-1}(F)$ again irred.

<u>Rank</u>: ii) $\mathrm{rk}(\mathrm{MC}_{-1}(F)) = \mathrm{rk}(F)(r-1) - \sum_{i=1}^{r} \dim(V^{T_i})$

(depends only on local monodromy)

iii) <u>Effect of MC_{-1} on vanishing cycles:</u>

$$\phi_{x_i}(\mathrm{MC}_{-1}(F)) = \phi_{x_i}(F) \circlearrowright_{-T_i}$$

3) $\mathcal{N}_1, \mathcal{N}_2$ suitable Rank-1-local systems on $\mathbb{A}^1 \setminus \underline{x}$ with monodromy tuple $(\pm 1, \ldots, \pm 1)$.

<u>Conclusion:</u> If $\mathcal{L} = \mathcal{N}_2 \otimes MC_{-1}(\mathcal{N}_1 \otimes MC_{-1}(\tilde{F}))$ with F as in ①, MC_{-1} as in ②, $\mathcal{N}_1, \mathcal{N}_2$ as in ③ $\Rightarrow$

- $\mathcal{L}$ irred (by i))
- $n = rk(\mathcal{L}) = \begin{cases} n_1 = 4r-9 \\ n_2 = 4r-11 \\ n_3 = 4r-10 \\ n_4 = 4r-12 \end{cases}$

  (suitable choice of $\mathcal{N}_1, \mathcal{N}_2$ using ii))

- Using iii) above $\rightsquigarrow$ Jordan forms of $T_1, \ldots, T_r$ where $(T_1, \ldots, T_{r+1}) \in GL_n(\mathbb{F}_q)$ is the monodromy tuple of $\mathcal{L}$ (case n_1):

$$T_i \sim \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \;\; 1 \end{pmatrix} \qquad i = 1, \ldots, r-3$$

$$T_{r-2} \sim \begin{pmatrix} \begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix} & \\ 0 & & & \begin{smallmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{smallmatrix} \end{pmatrix} \qquad T_r = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$T_{r-1} \sim \begin{pmatrix} i & & & & 0 \\ & -i & & & \\ & & \begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix} & & \\ & & & \ddots & \\ 0 & & & & \begin{smallmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{smallmatrix} \end{pmatrix} \qquad T_{r+1} = -\mathbb{1}$$

+ Group-Theory $\Rightarrow$ $\langle T_1, \ldots, T_r \rangle = SL_n(\mathbb{F}_q)$

So that $\mathrm{im}(\rho_\lambda) \supseteq \langle T_1, \ldots, T_{r+1} \rangle = Sl_n(\mathbb{F}_q)$.

Problem: Upper bound for $\mathrm{im}(\rho_\lambda)$:

This follows from Cebotarev's density theorem and

THM 2: (D.) Let F irred. on $\mathbb{A}^1_{\mathbb{F}_p} \setminus \underline{x}$, nontrivial, s.t.

i) $T_{r+1} = -\mathbb{1}$

ii) selfdual local monodromy

iii) $\forall y \in |\mathbb{A}^1_{\mathbb{F}_p}| \ \exists m \in \mathbb{Z}$:

$$\det(\mathrm{Frob}_y, (j_* F)_{\bar{y}}) = \pm q^m$$

$\Rightarrow$ i) – iii) hold for $MC_{-1}(F)$.

Idea of Proof for THM 2:

Fourier-convolution-theorem

$$MC_{-1}(F) = F * \mathcal{L}_{-1} = FT^{-1}\left(j_*\left(FT(F) \otimes FT(\mathcal{L}_{-1})\right)\right)$$

with FT étale Fourier transform

⌜ Fix additive char. $\psi: \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^\times \rightsquigarrow$ local system $\mathcal{L}_\psi$

$$\begin{array}{ccccc} & & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{pr_2} & \mathbb{A}^1 \\ & {}^{pr_1}\swarrow & & \searrow{}^{mult} & \\ & \mathbb{A}^1 & & \mathbb{A}^1 & \\ F & & & & \mathcal{L}_\psi \end{array}$$

$$FT(F) = R\,pr_{2*}(pr_1^* F \otimes m^* \mathcal{L}_\psi)$$ ⌟

<u>local monodromy of F:</u>

$$\phi_{x_i}(F) = \bigoplus_{j=1}^{k} \mathcal{J}_{u_j} \otimes \mathcal{L}_{\chi_j} \otimes F_j$$

$\langle T_i \rangle \rtimes \langle Frob_p \rangle$

local Kummer sheaf, T_i acts via $\chi_j \in \mathbb{F}_q^\times$

constant, $rk(F_j) = 1$, T_i acts trivially

- T_i acts via $\begin{pmatrix} 1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & 1 \end{pmatrix} \Big\} u_j$

- $Frob_p$ acts trivially on $\mathcal{J}_{u_j}^{T_i}$

$$T_i \sim \begin{pmatrix} \boxed{\begin{matrix} x_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & x_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} x_k & 1 & 0 \\ & \ddots & 1 \\ 0 & & x_k \end{matrix}} \end{pmatrix}$$

Laumons theory of local Fourier transform:

$$FT^{loc}(\mathcal{L}_{\chi_i}) = \mathcal{L}_{\chi_i} \otimes G(\chi, \psi)$$

constant sheaf of rk one on which Frob_p act as $-$Gauß-sum

$$FT^{loc}(J_{n_j}) = J_{n_j}$$

Fourier-convolution thm $\Longrightarrow$

$$\phi_{x_i}(MC_{-1}(F)) = \bigoplus_{j=1}^{k} J_{n_j} \otimes \mathcal{L}_{-\chi_i} \otimes F_j \otimes J(\chi_j, -\mathbb{1})$$

Jacobi-sum constant sheaf

$$J(\chi_j, -\mathbb{1}) = G(-\bar{\chi}_j, \psi) \otimes G(\chi_j, \psi) \otimes G(-\mathbb{1}, \psi).$$

together with Laumons theory of local ε-Constant $\Rightarrow$ THM 2. $\square$

Jt. project with Meirav Amram and Evgenii Shustin:

- consider convolution on $\mathbb{A}^2$
- role of FT is taken by Radon-transformation
- hope for new applications in Galois theory.