Real Thom polynomials

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joint with András Szenes (1. part) and Ákos Matszangosz (2. part)

Thom polynomials: universal polynomials (characteristic classes) used to solve enumerative problems and find obstructions to the existence of not too singular maps.

1. Example: $f:\mathbb{CP}^2\to\mathbb{CP}^2$ $\deg f=2$ # of cusps? $(x,y)\mapsto(x^3+xy,y)$

Theory: ! $\eta:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ singularity:
equiv. class of hol. map germs : $\mathcal{A}$: reparametrization in source & target

$\mathcal{K}$: contact ($\mathcal{K}\subset\mathrm{Diff}(n+p)$ preserving graphs)

Thm: $\exists$ universal poly $tp(\eta)\in\mathbb{Z}[c_1,\dots,c_n]$ s.t. if

$f:N^n\to P^p$ "nice" (sg. transversal) then

$[\eta(f)]=tp(\eta)(c_f)$

where $c_f=\dfrac{c(f^*(TP))}{c(TN)}$

1.Ex cont'd: $tp(\underbrace{A_2(2,2)}_{\text{cusp}})=c_1^2+c_2$ $\quad c_f=\dfrac{(1+2x)^3}{(1+x)^3}=1+3x+0\Rightarrow\boxed{3\text{ cusps}}$ (c_1 c_2)

Rm: f is a Net of Conics!

Real case: Borel–Haefliger: $tp(\eta^{\mathbb{R}})=tp(\eta^{\mathbb{C}})(c_i\mapsto w_i)\in\mathbb{Z}_2[w_1,\dots,w_i,\dots]$

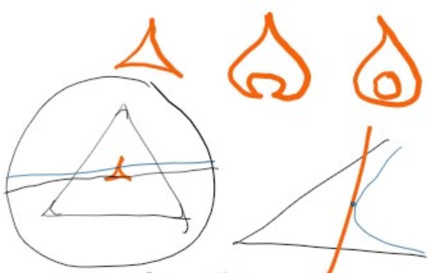
e.g. $tp(A_2^{\mathbb{R}}(0))=w_1^2+w_2$

Bonus: $f:N^n\to P^p$ real smooth & generic $\Rightarrow$

$[\eta(f)]=tp(\eta)(w_f)$

Example: $f:\mathbb{RP}^2\to\mathbb{R}^2$ $\quad w(\mathbb{RP}^2)=(1+x)^3=1+x+x^2$

$w_f=\dfrac{1}{1+x+x^2}=1+x\Rightarrow$ # of cusps $\equiv 1$



Tp doesn't tell us which odd numbers can occur!

Real Thom polynomials with integer coefficients

$tp^{\mathbb{Z}}(\eta)\in\mathbb{Z}[p_1,\dots,p_i]$

Examples: $\ell=p-n$

1. $tp^{\mathbb{Z}}(\Sigma^{2i}(2d))=\underbrace{\begin{vmatrix} p_i & p_{i+1} & \cdots \\ p_{i-1} & p_i & \\ & & \ddots \\ & p_{i-1} & p_i \end{vmatrix}}_{d+i}$ Ronga

2. $tp^{\mathbb{Z}}(A_2(2d-1))=p_d$ Szűcs

with sign

Application: $M^4\to\mathbb{R}^5$ $\quad tp^{\mathbb{Z}}(A_2(1))=p_1\ \overset{\text{Rokhlin}}{\Rightarrow}$ # of cusps $=3\sigma(M)$

This is a signed sum, but $\#\geq 3|\sigma|$ lower bound

New results:

A_4 $tp^{\mathbb{Z}}(A_4(2d-1))=p_d^2+3\sum_{i=1}^{d}4^{i-1}p_{d-i}\,p_{d+i}$ $\quad$ e.g. $tp^{\mathbb{Z}}(A_4(1))=p_1^2+3p_2$

Notice similarity w. Ronga formula:

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$tp(A_2^{\mathbb{C}}(d\dot{+})) = c_d^2 + \sum_{i=1}^{d} 2^{i-1} c_{d-i} c_{d+i}$

A_6 $tp^2(A_6(1)) = p_1^3 + 11p_1p_2 + 24p_3$

$tp^2(A_6(3)) = p_2^3 + 11p_1p_2p_3 + 24p_1^2p_4 + 312p_1p_5 + 864p_6 + 84p_3^2 + \overset{?}{k}(p_3^2 - p_2p_4)$

Parity of k: Bérczi-Szenes $\Rightarrow tp(A_6^{\mathbb{C}}(3))$ can be calc.
but TL;DR (2 pages) $\Rightarrow k \equiv 0$

Method: Restriction ='s of Rimányi

Why no more result? (More details: F-Rimányi: Tp's with integer coeff's)

1. Not all real singularity has tp!
   Reason: $\eta \subset M$ real alg $\Rightarrow [\eta \subset M]_2 \in H^*(M, \mathbb{Z}_2)$ exists but
   $[\eta \subset M] \in H^*(M, \mathbb{Z})$ not always!
   orientation & ∂ condition. (+ trivial cond. $\deg p_i = 4i$)
2. Restriction = method requires symmetries. Not enough in real case. (V_a family has $O(2)$ sym.)

In fact I don't know if $tp^2(A_4(2d-1))$ exists!

Conjecture: All "components" of ∂A_4 are non-coorientable
if f avoids "exotic" singularities then tp-formula is correct.

All we know is in this formula:

$$[A_{2d}(2, 2l+1)] = \prod_{i=1}^{l}\prod_{j=1}^{d}(\beta_i^2 - (j\alpha)^2).$$

Euler characteristics of singularity loci (joint with Ákos Matszangosz)

$\exists ?\ f: \mathbb{RP}^{20} \to \mathbb{R}^{27}$ Morin map? ($\dim \ker df \leq 1$)

Facts: $[\Sigma^2(f)]_{\mathbb{Z}_2} = 0$ & $[\Sigma^2(f)]_{\mathbb{Z}}$ doesn't exist.

Theorem:

For a generic map $f: \mathbb{RP}^{20} \to \mathbb{R}^{27}$ the Σ^2 locus is a smooth surface of odd Euler characteristic. In particular it is nonempty and unorientable.

How to get such result? Enhance the Thom polynomial!

① $\mathbb{C}$-world: c^{SM} class. (Chern-Schwartz-McPherson class)

Def: $i: X \hookrightarrow M$ smooth $\mathbb{C}$-alg $c^{SM}(X \subset M) := i_! c(TX) \in H^*(M)$

(*) Rem: $c^{SM}(X \subset M) = \boxed{[X]} +$ higher terms $+ \chi(X) \cdot [\text{point}]$ (Poincaré-Hopf)

Thm (McPherson) $\exists$ extension for X subvariety $c^{SM}(X \subset M) \in H^*(M)$ with nice functorial properties. (in part. (*) holds!)

For $X \subset \mathbb{CP}^n$ $c^{SM}(X \subset \mathbb{CP}^n) \overset{\text{Ohmoto-Aluffi}}{\longleftrightarrow} \{\chi(X \cap \mathbb{CP}^i) : i = 0, \dots, n\}$

(similarly for $Gr_k\mathbb{C}^n$!)

A close cousin: $s^{SM}(X \subset M) := \dfrac{c^{SM}(X \subset M)}{c(TM)}$ Segre-SM: better pull-back property

Thm (Ohmoto Toru) $\exists$ s^{SM}-Thom polynomial: $f: N^n \to P^p$ "nice" $\to$

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Thm (Ohmoto Toru) $\exists$ s^{SM}-Thom polynomial: $f: N^n \to P^p$ "nice" $\Rightarrow$

$$s^{SM}(\eta(f)) = s^{SM}tp(\eta)(c_f)$$

Parusinski-Pragacz: $s^{SM}tp(\Sigma^i)$

Rimányi: s^{SM}-$tp(\eta)$ can be calc. (for low codim & degree)

$\Rightarrow$ for f "nice" $\chi(\eta(f))$ can be calc.

② The Real world: Singular Stiefel-Whitney classes. $w(X \subset M)$

real analog, for smooth X $w(X \subset M) = i_! w(TX)$ $\quad s^{SW}(X \subset M) = \dfrac{w(X \subset M)}{w(M)}$

Stiefel $\rightsquigarrow$ Sullivan $\rightsquigarrow$ Deligne $\rightsquigarrow$ McPherson

simplicial real alg c^{SM}

③ s^{SW}-tp can be defined

Thm: $f: N^n \to P^p$ generic $\Rightarrow$ $s^{SW}(\eta(f)) = s^{SW}$-$tp(\eta)(w_f)$ (generic condition very technical) help from Jörg Schürmann

④ Thm: $s^{SW}tp(\eta^{\mathbb{R}}) = c^{SM}tp(\eta^{\mathbb{C}})(c_i \mapsto w_i)$

Proof: Use [Brasselet-Schürmann-Yokura]

$\Rightarrow$ $\chi(\Sigma^2(f))$ can be calc. in our example $f: \mathbb{RP}^{20} \to \mathbb{R}^{27}$

s^{SW}-$tp(\eta)$ is not known for any other singularity!

Conjecture: s^{SW}-$tp(A_2(\ell+1)) = s^{SW}$-$tp(\Sigma^2(\ell))$ (lots of computer evidence)

$\Downarrow$ geometry? 1. step: [Csépai-Szűcs-Terpai]

$f: \mathbb{RP}^{20} \to \mathbb{R}^{28}$ generic $\Rightarrow$ $\chi(A_2(f)) \equiv 1$

Thank you