Chern-Euler intersection theory and Gromov-Witten invariants

Seminar on Real and Complex Geometry

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joint project with
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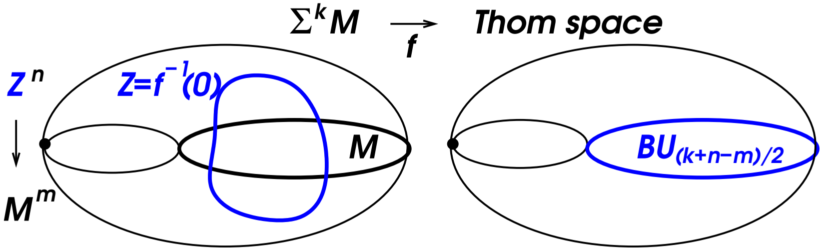
Complex cobordisms

$$U^*(M) := \lim_{k\to\infty} \Pi \left(\Sigma^k M, T(\text{universal } \mathbb{C}^{(k+*)/2} \text{ bundle})\right)$$

Pontryagin – Thom construction:

M — compact stably almost complex m-fold

$\Rightarrow \; U^{m-n}(M) = U_n(M)$



Mishchenko's logarithm

$$z = u + [\mathbb{C}P^1]\frac{u^2}{2} + [\mathbb{C}P^2]\frac{u^3}{3} + [\mathbb{C}P^3]\frac{u^4}{4} + \cdots$$

$$z \in H^*(\mathbb{C}P^\infty; \mathbb{Q}\otimes U^*(pt)) \xrightarrow{\text{Chern-Dold}^{-1}} U^*(\mathbb{C}P^\infty)\otimes\mathbb{Q} \ni u$$

Cohomological (z) and cobordism-valued (u)
universal 1st Chern classes
Poincaré-dual to hyperplane $\mathbb{C}P^{N-1} \subset \mathbb{C}P^N$

$$U^*(pt)\otimes\mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, \ldots], \ \deg[\mathbb{C}P^k] = -2k$$

Hirzebruch-Riemann-Roch for $\pi : M \to pt$

$$ChD(\pi_* A) = \int_M ChD(A)\, Td(T_M)$$

$$Td(-) = e^{\sum_{k>0} s_k ch_k(-)}, \ \ \mathbb{Q}[s_1, s_2, \dots] = U^*(pt) \otimes \mathbb{Q}$$

$$Td\left(\mathcal{O}(1)_{\mathbb{C}P^\infty}\right) = \frac{z}{u(z)} = e^{\sum_{k>0} s_k z^k/k!}$$

Specializations $U^*(-) \to \mathcal{H}^*(-)$

U^* is universal among abstract cohomology theories $\mathcal{H}^*$ where complex vector bundles are oriented

Cohomology theory: $\pi_*^H(C) := \int_M C$

$\pi_*^H(\mathbb{C}P^{k-1}) = 0$ for $k > 1 \Rightarrow z = u$

Complex K-theory:

$\pi_*^K(V) := \sum (-1)^k \dim \check{H}^k(M; V)$

$\pi_*^K(\mathcal{O}_{\mathbb{C}P^{k-1}}) = 1 \in K^0(pt)$

$\Rightarrow z = \sum_{k>0} u^k/k = -\ln(1-u)$

$\Rightarrow \; u = 1 - e^{-z}, \; Td = \frac{z}{1-e^{-z}}$

$\Rightarrow ch(\pi_*^K(V)) = \int_M ch(V) \; td(T_M)$

Specializations (contd.)

Hirzebruch χ_y-theory:
$\pi_*^y(M) = \chi_y(M) := \sum y^p(-1)^q h^{p,q}(M)$

$y=-1$: *Euler characteritic of a stably almost complex manifold is bordism-invariant because the Euler class = top Chern class is stable.*

Chern-Euler theory: $\pi_*(f: Z \to M) := \chi(Z)$
$\pi_*(\mathbb{C}P^{k-1}) = k \Rightarrow z = \sum_{k>0} u^k = u/(1-u)$
$\Rightarrow u = z/(1+z)$, $Td = 1+z \Rightarrow$
$Td(-) = 1 + c_1(-) + c_2(-) + \cdots = c(-)$

$$\mathcal{H}^*(-) = H^{even}(-) \oplus H^{odd}(-)$$

Intersection numbers $\langle f, f' \rangle = \chi(f(Z) \pitchfork f'(Z'))$

Gromov-Witten theory

X — compact (almost) Kähler manifold
$\mathcal{M} := \overline{\mathcal{M}}_{g,n}(X,d)$ — moduli space of holomorphic maps $\phi : (\Sigma, \sigma_1, \dots, \sigma_n) \to X$,
Σ — genus-g compact complex curves,
n — number of marked points σ_i,
$d = \phi_*[\Sigma] \in H_2(X;\mathbb{Z})$ — degree of ϕ,
$\overline{\mathcal{M}}_{\dots}(\cdots)$ — compactification by *stable* maps of *nodal* curves (in the spirit of Deligne - Mumford).

GW-invariants of X $:=$ *intresection numbers on* $\mathcal{M}$.

GW-invariants

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“Primary” GW-invariants:

$$\langle Z_1, \dots Z_n \rangle_{g,n,d} := \int_{[\mathcal{M}]^{vir}} \prod_i \mathrm{ev}_i^*(Z_i)$$

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Gravitational descendants:

$$\langle Z_1 \psi_1^{d_1}, \dots Z_n \psi_n^{d_n} \rangle_{g,n,d} := \int_{[\mathcal{M}]^{vir}} \prod_i \mathrm{ev}_i^*(Z_i) \psi_1^{d_1} \cdots \psi_n^{d_n}$$

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“Quantum cobordisms” (Tom Coates, 2003)

$$\int_{[\mathcal{M}]^{vir}} \prod_i \mathrm{ev}_i^*(Z_i) \psi_i^{d_i} e^{\sum_k s_k \, \mathrm{ch}_k(T_{\mathcal{M}}^{vir})}$$

Chern-Euler GW-invariants

Fake version:

$$\chi^{fake}([\mathcal{M}] \pitchfork_{i=1}^{n} \mathrm{ev}_i^*(Z_i)) := \int_{[\mathcal{M}]^{vir}} \prod_i \mathrm{ev}_i^* \, Ch(Z_i) \, c(T_{\mathcal{M}}^{vir})$$

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True version:

$$\chi([\mathcal{M}] \pitchfork_{i=1}^{n} \mathrm{ev}_i^*(Z_i)) = \int_{[I\mathcal{M}]} \prod_{i=1}^{n} \mathrm{ev}_i^* \, Ch(Z_i) \; c(T_{I\mathcal{M}})$$

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Permutation-equivariant version:

$$\chi([\mathcal{M}/S_n] \pitchfork_i \mathrm{ev}_i^*(Z)) := \int_{[I(\mathcal{M}/S_n)]} \prod_i \mathrm{ev}_i^* \, Ch(Z) \, c(T_{I\mathcal{M}})$$

Motivation

2D Yang-Mills and Hurwitz spaces

In $SU(N)$-2DYM theory, coefficients of the $1/N$-expansion of partition function are orbifold Euler characteristics of Hurwitz spaces (Cordes, Moore, Ramgoolam, 1994)

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Ekedahl-Lando-Shapiro-Vainshtein's formula:

$$h(g,\vec{k}) = \frac{m!}{|Aut(\vec{k})|} \prod_i \frac{k_i^{k_i}}{k_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{c(\text{Hodge}^*)}{\prod_i (1 - k_1 \psi_1)}$$

$\vec{k} = (k_1, \dots, k_n)$ – ramification over $\infty \in \mathbb{C}P^1$,
$m = 2g - 2 + n + \sum k_i$ – simple ramifications

Adelic product formula

$$\overline{\mathcal{D}}_X^{perm}(\mathbf{t}_1, \mathbf{t}_2, \dots) = e^{\sum_{r,\zeta\neq\eta^{-1}} \partial_{t_{\zeta,r}} \partial_{t_{\eta,r}}} \bigotimes_{m=1}^{\infty} \overline{\mathcal{D}}_{X/\mathbb{Z}_m}^{fake}\Big|_{t_{\zeta,r}=\mathbf{t}_r}$$

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Open invariants: $\chi([\mathcal{M}] \pitchfork_{i=1}^{n} \mathrm{ev}_i^*(Z_i))$,
$\mathcal{M}$ = uncompactified $\mathcal{M}_{g,n}(X,d)$.

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$$\overline{\mathcal{D}}_X(\mathbf{t}_1, \mathbf{t}_2, \dots) = e^{\frac{1}{2}\sum_r \left(\sum_\zeta \partial_{t_{\zeta,r}}\right)^2} \bigotimes_{m=1}^{\infty} \mathcal{D}_{X/\mathbb{Z}_m}^{fake}|_{t_{\zeta,r}=\mathbf{t}_r}$$

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Works in "no-descendants" Chern-Euler theory

Euler characteristics of $\mathcal{M}_{0,n}$

$$G(t) := \sum_{n\geq 2} \chi(\mathcal{M}_{0,1+n}) \frac{t^n}{n!} = (1+t)\ln(1+t) - t$$

$$= \frac{t^2}{1\cdot 2} - \frac{t^3}{2\cdot 3} + \frac{t^4}{3\cdot 4} - \frac{t^5}{4\cdot 5} + \cdots$$

Indeed, $\mathcal{M}_{0,n+1}$ is fibered over $\mathcal{M}_{0,n}$ with the fiber $\mathbb{C}P^1 - n$ pts, $\Longrightarrow$

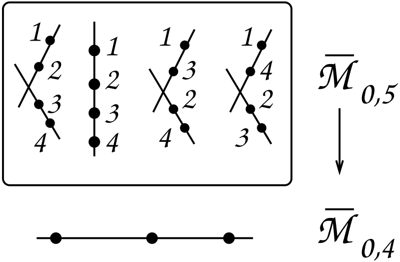
$$\chi(\mathcal{M}_{0,1+n}) = (2-n)\ \chi(\mathcal{M}_{0,n}) = (-1)^n (n-2)!$$

since $\mathcal{M}_{0,3} = pt$.

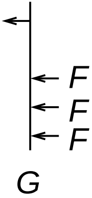
Euler charactersitics of $\overline{\mathcal{M}}_{0,1+n}$

$$F(t) := t + \sum_{n\geq 2} \chi(\overline{\mathcal{M}}_{0,1+n}) \frac{t^n}{n!}$$

$$= t + \frac{t^2}{2!} + 2\frac{t^3}{3!} + 7\frac{t^4}{4!} + 34\frac{t^5}{5!} + \cdots$$



$F = t +$



$$F = t + G(F) = t - F + (1 + F)\ln(1 + F)$$

$\chi^{orb}(\overline{\mathcal{M}}_{0,1+n}/S_n)$

$$E^{orb}(t) := t + \sum_{n\geq 2} \chi^{orb}(\overline{\mathcal{M}}_{0,1+n}/S_n) t^n = F(t)$$

$$\chi^{orb}(\mathcal{M}) := \int_{\mathcal{M}} c_{top}(T^{vir}_{\mathcal{M}}) =: \chi^{fake}(\mathcal{M}) \in \mathbb{Q}$$

Lemma (Burnside-Cauchy-Kawasaki-Lefschetz):

$$\chi(\text{orbifold}) = \chi^{fake}(\text{inertia orbifold})$$

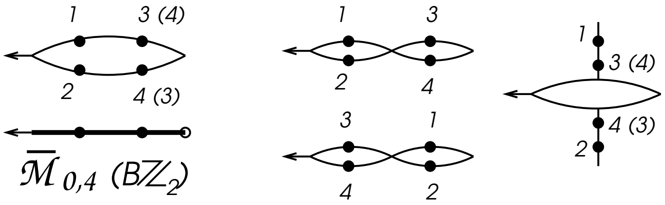
Example: $\chi(\mathcal{M}/G) = \frac{1}{|G|} \sum_{g\in G} \chi(\mathcal{M}^g)$

The *inertia orbifold* (of a global quotient):

$$I(\mathcal{M}/G) := \frac{1}{G} \bigsqcup_{g\in G} \mathcal{M}^g$$

$\chi(\overline{\mathcal{M}}_{0,1+n}/S_n)$, a false try

$$E(t) := t + \sum_{n\geq 2} \chi(\overline{\mathcal{M}}_{0,1+n}/S_n)t^n$$
$$= t + t^2 + 2t^3 + 4t^4 + 8t^5 + ??t^6 + \cdots$$

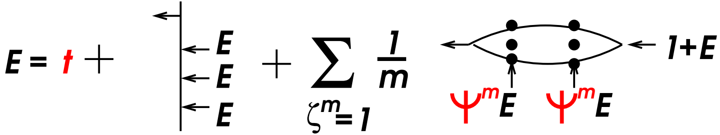


$\chi(\overline{\mathcal{M}}_{0,1+n}/S_n)$, a better way

$$E(t) = t - E + (1+E)\sum_{m\geq 1}\frac{\phi(m)}{m}\ln(1+\Psi^m E)$$

$$= t - E + (1+E)\prod_{p}\frac{1-\Psi^p/p}{1-\Psi^p}\ln(1+E)$$

$$= t + 1t^2 + 2t^3 + 4t^4 + 8t^5 + 17t^6 + 36t^7 + 79t^8 + \cdots$$



Adams operations: $\Psi^m(t) = t^m$, $\Psi^k\Psi^l = \Psi^{kl}$.

Fixed point localization

$$\int_{\mathcal{M}} \omega = \int_{i:\mathcal{M}^{\mathbf{T}} \subset \mathcal{M}} \frac{i^* \omega}{Euler_{\mathbf{T}}(N_{\mathcal{M}^{\mathbf{T}}})}$$

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Example: $X = \mathbb{C}P^n$.

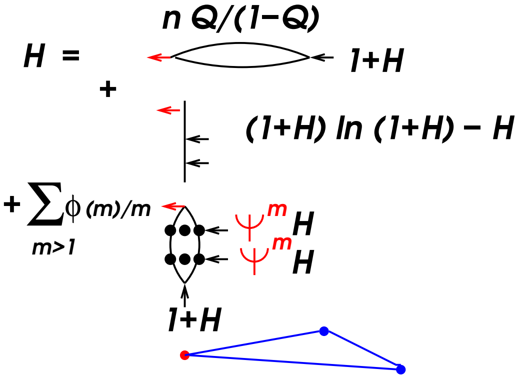
$$F(Q) := \sum_{d>0} Q^d \chi(\overline{\mathcal{M}}_{0,1}(\mathbb{C}P^n, d)) =: (n+1)H(Q)$$

$\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{C}P^n, d) \to \mathbb{C}P^n$

H — curves with a fixed value at the marked point

$n + 1 = \chi(\mathbb{C}P^n)$.

Recursion

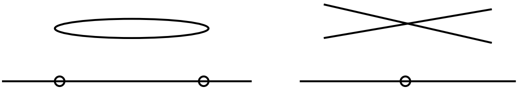


$$H = -H + n\frac{Q}{1-Q}(1+H) + (1+H)\sum_{m=1}^{\infty}\frac{\phi(m)}{m}\Psi^m \ln(1+H)$$

Sanity check

$$H_{\mathbb{C}P^1} = Q + 3Q^2 + 9Q^3 + 30Q^4 + 102Q^5 + 371Q^6 + \cdots$$
$$H_{\mathbb{C}P^2} = 2Q + 9Q^2 + 44Q^3 + 240Q^4 + 1388Q^5 + \cdots$$

$$\overline{\mathcal{M}}_{0,1}(\mathbb{C}P^1, 2) \xrightarrow{\mathbb{C}P^1} \overline{\mathcal{M}}_{0,0}(\mathbb{C}P^1, 2) \cong \frac{\mathbb{C}P^1 \times \mathbb{C}P^1}{S_2} \cong \mathbb{C}P^2$$



A surprizing by-product:

$$\sum_{m=1}^{\infty} \frac{\phi(m)}{m} \ln(1 + a_1 t^m + a_2 t^{2m} + a_3 t^{3m} + \cdots)$$

has integer coefficiend if $a_1, a_2, a_3, ... \in \mathbb{Z}$.

Equivalently: If symmetric polynomials of α_i are integers, then $\sum_{d|n} \phi(d) \sum_i \alpha_i^{n/d}$ is divisible by n.

Equivalently: If symmetric polynomians of α_i are integers, then for each prime p and $k \geq 1$

$$\sum \alpha_i^{p^k} \equiv \sum \alpha_i^{p^{k-1}} \mod p^k$$

Euler's theorem:

$$a^{p^k} - a^{p^{k-1}} = a^{p^{k-1}}(a^{\phi(p^k)} - 1) \equiv 0 \mod p^k \text{ if } a \in \mathbb{Z}$$

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"Arnold's conjecture" (2004)
proved by Zarelua (2004, 2006)
but known to Schur (1937) and Jänichen (1921)

Proof by M. Mazur and B. Petrenko (2010)

$$\text{(i)}\ \operatorname{ch}(A) \equiv \operatorname{ch}(B) \mod p^k (k \geq 1)$$

$$\Longrightarrow \operatorname{ch}(A^p) \equiv \operatorname{ch}(B^p) \mod p^{k+1}$$

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$$\sigma_i(\alpha_1, \ldots, \alpha_n)^p = \sigma_i(\alpha_1^p, \ldots, \alpha_n^p) + p\, \Sigma_{i,p}(\sigma)$$

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$$(\sigma_i + \mathcal{O}(p^k))^p = \sigma_i(\beta_1^p, \dots, \beta_n^p) + p\, \Sigma_{i,p}(\sigma + \mathcal{O}(p^k))$$