(Refined) Verlinde and Segre formula for Hilbert schemes of points

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Hilbert scheme of points:

S smooth projective surface, $S^{[n]}$ Hilbert scheme of points parametrizes 0-dimensional subschemes of length n on S
general points of $S^{[n]}$ is set of n distinct points on S
when points come together get non reduced scheme structure
e.g. in $(\mathbf{A}^2)^{[2]}$ limit of $\{(0,a),(0,-a)\} = Z(x, y^2 - a^2)$ for $a \to 0$ is
$Z(x, y^2) = \{(0,0)\}$ with structure sheaf $\mathbb{C}[x,y]/(x,y^2)$
$S^{[n]}$ is smooth projective of dimension $2n$

Related to symmetric power $S^{(n)} = S^n/\mathcal{S}_n$
$S^{(n)}$ parametrizes effective 0-cycles of degree n

$$n_1x_1 + \ldots + n_rx_r, \quad x_i \in S, \quad \sum n_i = n$$

Hilbert-Chow morphism

$$\omega : S^{[n]} \to S^{(n)}, Z \mapsto supp(Z)$$

is a resolution of singularities

Examples:

- $S^{[0]} = \{\emptyset\}$ is a point
- $S^{[1]} = S$
- if $Z \in S^{[2]}$ then either $Z = \{x_1, x_2\} \subset S$
  or $Z = (x, t) \in S$ with $x \in S$, t tangent direction to S at x
  $\Longrightarrow S^{[2]}$ is blowup of S^2 along the diagonal divided by exchanging factors
  equivalently $S^{[2]}$ is blowup of $S^{(2)}$ along diagonal

Why care about it?

1. via $Z \mapsto I_Z$ Hilbert scheme of points is moduli space of rank 1 sheaves
   Simple example of moduli spaces of sheaves
   model case for all one wants to study about them
2. Building block of moduli spaces, used to study them
3. Important example of higher dimensional varieties
   e.g. if S is K3 surface, then $S^{[n]}$ is hyperkähler
4. Enumerative applications, counting point configurations, curves and other things

(Refined) curve counting: Let L sufficiently ample line bundle on S of arithmetic genus g, let $B \subset |L|$ general δ-dimensional sub-linear system

$$\mathcal{C}^{[n]} = \left\{(Z, C) \in S^{[n]} \times B \,\middle|\, Z \subset C\right\}$$

relative Hilbert scheme. Write

$$\sum_{n \geq 0} e(\mathcal{C}^{[n]}) q^n = \sum_{i=0}^{\delta} n_i \frac{q^i}{(1-q)^{2(i+1-g)}}$$

Then n_δ is the number of δ-nodal curves in B

Refinement: $\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)$. Write

$$\sum_{n \geq 0} \chi_{-y}(\mathcal{C}^{[n]}) q^n = \sum_{i=0}^{\delta} N_i(y) \frac{q^i}{\big((1-q)(1-yq)\big)^{i+1-g}}$$

Then $N_\delta(y) \in \mathbb{Z}[y]$ is refined count of δ-nodal curves in B. Related to real and refined tropical curve counting

A lot of work by many authors, see slides of talk by Shustin

Universal subscheme:

$$Z_n(S) = \{(x,[Z]) \mid x \in Z\} \subset S \times S^{[n]}$$

$p : Z_n(S) \to S^{[n]}, \quad q : Z_n(S) \to S$ projections
Fibre $p^{-1}([Z]) = Z$

Tautological sheaves:
V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$
$V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_S^{[n]}([Z]) = H^0(\mathcal{O}_Z) = \mathcal{O}_Z$ (as vector space)
These tautological bundles are useful for many applications of Hilbert schemes

Determinant bundles: $\det(V^{[n]}) \in \mathrm{Pic}(S^{[n]})$ generate $\mathrm{Pic}(S^{[n]})$

$Z_n(S) = \{(x,[Z]) \mid x \in Z\} \subset S \times S^{[n]}$
$p : Z_n(S) \to S^{[n]}, \quad q : Z_n(S) \to S$ projections

Tautological sheaves: V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$
$V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_S^{[n]}([Z]) = H^0(\mathcal{O}_Z)$
Extends to map of Grothendieck groups $K^0(S) \to K^0(S^{[n]})$ by
$(V - W)^{[n]} = V^{[n]} - W^{[n]}$

Line bundles on $S^{[n]}$: $\det(V^{[n]}) \in \mathrm{Pic}(S^{[n]})$, these generate $\mathrm{Pic}(S^{[n]})$
Want formulas for

$$\chi(S^{[n]}, \det(V^{[n]})) \quad \textbf{Verlinde formula}$$

$$\int_{S^{[n]}} c_{2n}(V^{[n]}) = \int_{S^{[n]}} s_{2n}(-V^{[n]}) \quad \textbf{Segre formula}$$

Verlinde formula: Via the correspondence $Z \mapsto \mathcal{I}_Z$ have $S^{[n]} = M_S^H(1,0,n)$ (moduli sp. of rk 1 stable sheaves E with $\det(E) = 0$, $c_2(E) = n$)
Verlinde formula is rk 1 case of surface analogue of the celebrated Verlinde formula for curves.

Segre formula has enumerative meaning, counts configurations of points in special positions
e.g. $S \subset \mathbb{P}^{3n-2}$ surface. H hyperplane bundle on $\mathbb{P}^{3n-2}$

$$\int_{S^{[n]}} s_{2n}(H^{[n]})$$

is number of $n-2$ planes in $\mathbb{P}^{3n-2}$ which are n-secant to S
More generally many enumerate questions can be reduced to computing intersection numbers of Chern classes of tautological bundles and possibly of the tangent bundle of $S^{[n]}$ (also e.g. true for (refined) curve counting)

Aim: For $V \in K^0(S)$ want formula for generating functions

$$I^{Verlinde}_{S,V}(x) = \sum_{n \geq 0} x^n \chi(S^{[n]}, \det(V^{[n]})), \quad \text{Verlinde formula}$$

$$I^{Chern}_{S,V}(x) = \sum_{n \geq 0} x^n \int_{S^{[n]}} c_{2n}(V^{[n]}), \quad \text{Segre formula}$$

Why care about the generating functions? The numbers are interesting, contain much information on geometry of $S^{[n]}$
For series of related numbers should study generating functions
They bring hidden relations between the numbers to the surface

Note: The $S^{[n]}$ for different n are closely related. E.g. have rational maps

$$S \times S^{[n]} \to S^{[n+1]}; (x, Z) \mapsto \{x\} \cup Z$$

$\Longrightarrow$ gives inductive structure
can expect nice generating functions for invariants

Recall universal subscheme $Z_n(S) \subset S \times S^{[n]}$
can show: blowup of $S \times S^{[n]}$ along $Z_n(S)$ is

$$S^{[n,n+1]} = \{(Z, W) \in S^{[n]} \times S^{[n+1]} \mid Z \subset W\}$$

This allows to compute intersection numbers on $S^{[n]}$ recursively:
From $S^{[n]}$ pullback to $S^{[n-1,n]}$, pushforward to $S \times S^{[n-1]}$, etc until arriving at S^n.

Tautological sheaves: V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$, $V^{[n]}([Z]) = H^0(V|_Z)$
Ellingsrud-G-Lehn (2001): Let $P((d_i)_{i=1}^{rn}, (e_j)_{j=1}^{2n})$ polynomial. Write

$$P(S^{[n]}, V) := P\big((c_i(V^{[n]}))_i, (c_j(S^{[n]}))_j\big) \in H^*(S^{[n]}, \mathbb{Q})$$

Note that $(S_1 \sqcup S_2)^{[n]} = \coprod_{n_1+n_2=n} S_1^{[n_1]} \times S_2^{[n_2]}$
Let $P_n((d_i)_{i=1}^{rn}, (e_j)_{j=1}^{2n})$, $n \geq 0$ polynomials. Assume, when $n = n_1 + n_2$

$$P_n((S_1 \sqcup S_2)^{[n]}, V)|_{S_1^{[n_1]} \times S_2^{[n_2]}} = \pi_1^*(P_{n_1}(S_1^{[n_1]}, V|_{S_1}))\pi_2^*(P_{n_2}(S_2^{[n_2]}, V|_{S_2})),$$

$$\Longrightarrow \sum_{n \geq 0} \int_{S^{[n]}} P(S^{[n]}, V) x^n = A_0^{c_2(V)} A_1^{c_1(V)^2} A_2^{c_1(V)K_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}$$

for universal $A_0, \ldots, A_4 \in \mathbb{Q}[[x]]$ depending only on P_1 and $r = \mathrm{rk}(V)$

$$I^{Verlinde}_{S,V}(x) = \sum_{n\geq 0} x^n \chi(S^{[n]}, \det(V^{[n]})), \quad \text{Verlinde formula}$$

$$I^{Chern}_{S,V}(x) = \sum_{n\geq 0} x^n \int_{S^{[n]}} c_{2n}(V^{[n]}), \quad \text{Segre formula}$$

Universality (Ellingsrud-G-Lehn) implies

$$I^{Verlinde}_{S,V}(x) = A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3^{c_1(V)K_S - \frac{1}{2}K_S^2} A_4^{K_S^2},$$
$$I^{Chern}_{S,V}(x) = B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3^{c_1(V)K_S - \frac{1}{2}K_S^2} B_4^{K_S^2}$$

$A_1, \ldots, A_4, B_0, \ldots, B_4 \in \mathbb{Q}[[x]]$ universal power series (depending only on $k = \mathrm{rk}(V)$)

$$I_{S,V}^{Verlinde}(x) = A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3^{c_1(V)K_S - \frac{1}{2}K_S^2} A_4^{K_S^2}$$

Verlinde Series $I_{S,V}^{Verlinde}(x)$ (EGL (2001)): With the change of variables $x = -t(1-t)^{r^2-1}$ ($r = \mathrm{rk}(V)$) have

$$A_1(x) = (1-t), \quad A_2(x) = \frac{(1-t)^{r^2}}{1-r^2 t}.$$

and $A_3(x) = A_4(x) = 1$ for $|r| \leq 1$

Segre Series:

Lehn conjecture (1999): formula for $I_{S,-L}^{Chern}(x)$ for $L \in \mathrm{Pic}(S)$

Proven by Marian-Oprea-Pandharipande, Voisin (2019)

MOP consider $I_{S,V}^{Chern}(x)$ for general $V \in K^0(S)$

$$I_{S,V}^{Chern}(x) = B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3^{c_1(V)K_S - \frac{1}{2}K_S^2} B_4^{K_S^2}$$

Theorem (MOP (2022))

Put $k = \mathrm{rk}(V)$, $r = k - 1$, with change of variable $x = -y(1-ry)^{r-1}$. Then

$$B_0(x) = \frac{(1-y)^{r+1}}{1-ry}, \quad B_1(x) = \frac{1-ry}{(1-y)^r}, \quad B_2(x) = \frac{(1-ry)^{2r}}{(1-y)(1-r^2y)}$$

Furthermore MOP determine $B_3(x)$, $B_4(x)$ as algebraic functions for $|k| \leq 2$.

(1) formulas are complicated, even when $K_S = 0$: multiplying out $A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)}$ or $B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)}$ and undoing the change of variables gives something very complicated

(2) A_1, A_2; B_0, B_1, B_2 are easier to study: can compute on K3 surface, then $S^{[n]}$ is hyperkähler and there are powerful tools

A_3, A_4, B_3, B_4 which involve K_S are much more mysterious

$$I_{S,V}^{Verlinde}(x) = A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3^{c_1(V)K_S - \frac{1}{2}K_S^2} A_4^{K_S^2},$$
$$I_{S,V}^{Chern}(x) = B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3^{c_1(V)K_S - \frac{1}{2}K_S^2} B_4^{K_S^2}$$

Mysterious relation: Verlinde series $\longleftrightarrow$ Segre series:

Conjecture (Johnson, MOP)

Put $r = k - 1$, then

$$B_3^{(k)}(-y(1-ry)^{r-1}) = A_3^{(r)}(-y(1-y)^{r^2-1})$$
$$B_4^{(k)}(-y(1-ry)^{r-1}) = A_4^{(r)}(-y(1-y)^{r^2-1})$$

Here we mean that for the Segre (B) series we take $\mathrm{rk}(V) = k$ *and for the Verlinde (A) series* $\mathrm{rk}(V) = k - 1 = r$*.*

How can this be and where could shift k to $k-1$ come from?

Theorem

The Verlinde Segre correspondence is true:
$B_3^{(r+1)}(-y(1-ry)^{r-1}) = A_3^{(r)}(-y(1-y)^{r^2-1}), \quad B_4^{(r+1)}(-y(1-ry)^{r-1}) = A_4^{(r)}(-y(1-y)^{r^2-1})$

Therefore it is enough to determine A_3, A_4

Theorem

With $x = -y(1-y)^{r^2-1}$ *we have*

$$A_3^{(r)}(x) = \frac{1}{(1-y)^{\frac{r}{2}}} \exp\left(-\sum_{n>0} \frac{y^n}{2n} \mathrm{Coeff}_{x^0}\left(\frac{x^r - x^{-r}}{x - x^{-1}}\right)^{2n}\right)$$

Looks complicated, but is much simpler than expected
Alternative formula: let $\alpha_i(y)$, $\quad i = 1, \ldots, r-1$ branches of the inverse of $\frac{(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2}{(x^{\frac{r}{2}} - x^{-\frac{r}{2}})^2} = x^{r-1} + \ldots$
i.e. $x = \alpha_i(y) = \epsilon_{r-1}^i y^{\frac{1}{r-1}} + \ldots$ sol. of $(x^r - 2 + x^{-r})y = x - 2 + x^{-1}$.
Then

$$A_3(-y(1-y)^{r^2-1})^2 = \frac{y}{(1-y)^r \prod_{i=1}^{r-1} \alpha_i(y)}.$$

Conjectural formula for A_4: Recall

$$A_3(-y(1-y)^{r^2-1})^2 = \frac{y}{(1-y)^r \prod_{i=1}^{r-1} \alpha_i(y)}.$$

Conjecture

With $x = -y(1-y)^{r^2-1}$, we have

$$\left(A_4(x)A_3(x)^r\right)^8 =$$

$$= \frac{(1-r^2y)^3}{(1-y)^{3r^2}} \prod_{i,j=1}^{r-1} (1-\alpha_i(y)\alpha_j(y))^2 \prod_{\substack{i,j=1 \\ i\neq j}}^{r-1} (1-\alpha_i(y)^r\alpha_j(y)^r)^2$$

So complete Verlinde and Segre formula. Proven when $K_S^2 = 0$

Proposition (based on computations with Don Zagier)

This conjecture is true modulo x^{50} (until 49-th Hilbert scheme).

For $V \in K^0(S)$ of rank k define

$$I_{S,V}(x,z) := \sum_{n\geq 0}(-x)^n\chi\big(S^{[n]}, \det(\mathcal{O}_S^{[n]})^{-1}\otimes\Lambda_{-z}V^{[n]}\big) \in \mathbb{Z}[[x,z]]$$

where $\Lambda_{-z}W = \sum_{n\geq 0}(-z)^n\Lambda^n W$

$I_{S,V}(x,z)$ specializes to $I_{S,V}^{Verlinde}(x,z)$ and $I_{S,V}^{Chern}(x,z)$:

$$\begin{aligned}(-1)^{n(k-1)}\mathrm{Coeff}_{x^nz^{kn}}\big(I_{S,V}(x,z)\big) &= \chi(S^{[n]}, \det(V^{[n]})\otimes\det(\mathcal{O}_S^{[n]})^{-1}) \\ &= \chi(S^{[n]}, \det((V-\mathcal{O}_S)^{[n]})\end{aligned}$$

$$\lim_{\epsilon\to 0}\left(I_{S,V}\left(\frac{-(1+\epsilon)^k}{\epsilon^{k-2}}x, \frac{1}{1+\epsilon}\right)\right) = I_{S,V}^{Chern}(x,z)$$

Note that in $\chi(S^{[n]}, \det((V-\mathcal{O}_S)^{[n]})$ the rank drops by 1

Universality says

$$I_{S,V}(x,z) = G_0^{c_2(V)}G_1^{\chi(\det(V))}G_2^{\frac{1}{2}\chi(\mathcal{O}_S)}G_3^{c_1(V)K_S-\frac{1}{2}K_S^2}G_4^{K_S^2}$$

for $G_0, G_1, G_2, G_3, G_4 \in \mathbb{Q}[[x,z]]$ depending only on $k = \mathrm{rk}(V)$.

$$I_{S,V}(x,z) = G_0^{c_2(V)} G_1^{\chi(\det(V))} G_2^{\frac{1}{2}\chi(\mathcal{O}_S)} G_3^{c_1(V)K_S - \frac{1}{2}K_S^2} G_4^{K_S^2}$$

Theorem

Let $k = rk(V)$, $r = k-1$. With the changes of variables

$$x = \frac{u(1-u)^r}{v(1-v)^r}, \quad z = \frac{v}{(1-u)^r}, \quad y = \frac{uv}{(1-u)(1-v)},$$

we have

$$G_0 G_1(x,z) = 1 - y, \quad G_0 = \frac{(1-u-v)^{r+1}}{(1-v)^r((1-u)^r - v)},$$

$$G_2(x,z) = \frac{(1-\frac{u}{v})^2(1-v)^{r^2-1}((1-u)^r - v)}{(1-u-v)^{r^2}(1-u)^{r^2-1}(1-u-v-(r^2-1)uv)}$$

$$G_3(x,z) = A_3(-y(1-y)^{r^2-1}), \quad G_4(x,z) = A_4(-y(1-y)^{r^2-1})$$

Verlinde-Segre correspondence "explained" by the fact that $G_3(x,z)$ and $G_4(x,z)$ only depend on the variable y

Let X be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \ldots, p_e$, $d = \dim(X)$
Let E be equivariant vector bundle of rank r on X
Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C}v_k$, with action $(t_1, t_2) \cdot v_k = t_1^{n_k} t_2^{m_k} v_k$, $n_k, m_k \in \mathbb{Z}$
Then the $n_k\epsilon_1 + m_k\epsilon_2 \in \mathbb{Z}[\epsilon_1, \epsilon_2]$ are called the **weights** of $E(p_i)$
Denote $u_{1,i}, \ldots, u_{d,i}$ the weights of $T_{p_i}X$

$c_i^T(E(p_i)) = i$-th elementary symm. fctn in **weights** of $E(p_i) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$

Let $P\big((c_i(E))_i\big)$ be a polynomial in Chern classes of E

Theorem (Bott residue formula)

$$\int_{[X]} P\big((c_i(E))_i\big) = \left(\sum_{k=1}^{e} \frac{P\big((c_i^T(E(p_k)))_i\big)}{u_{1,k}\cdots u_{d,k}}\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$

Sum in brackets is a **polynomial** *in* ϵ_1, ϵ_2.

Let S be a smooth toric surface, i.e. S has action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints $p_1, \ldots, p_e$
Near each fixpoint p_i have affine T-equivariant coordinates x_i, y_i
The action of T on S lifts to an action on $S^{[n]}$
$Z \in S^{[n]}$ is T-invariant $\iff Z = Z_1 \sqcup \ldots \sqcup Z_e \quad supp(Z_i) = p_i$,
and $I_{Z_i} \in k[x_i, y_i]$ is gen. by monomials i.e.

$$I_{Z_i} = (x_i^{n_0}, y_i x_i^{n_1}, ..., y_i^r x_i^{n_r}, y_i^{r+1}) \quad (n_0, \ldots, n_r) \text{ partition}$$

$\Longrightarrow$ Fixpoints on $S^{[n]}$ are in bijection to e-tuples $(P_1, \ldots, P_e)$ of partitions, of numbers adding up to n

$$V^{[n]}(Z) = \bigoplus_{i=1}^{e} V^{[n_i]}(Z_i), \quad T_{S^{[n]}}(Z) = \bigoplus_{i=1}^{e} T_{S^{[n_i]}}(Z_i)$$

The weights of the action on $V^{[n_i]}(Z_i)$ and $T_{S^{[n_i]}}(Z_i)$ are given in terms of the combinatorics of the partition P_i

By Universality enough to prove for S toric surface and V toric vector bundle on S i.e. $T=(\mathbb{C}^*)^2$ acts on S with finitely many fixpoints, action lifts to $V \implies$ can use localization

Let $p_1, \ldots, p_e$ fixpoints of T-action on S, denote $t_1^{(i)}$, $t_2^{(i)}$ wts on $T_S(p_i)$ and $v_1^{(i)}, \ldots, v_k^{(i)}$ wts on $V(p_i)$ (each weight is of the form $n\epsilon_1 + m\epsilon_2$)

On $S^{[n]}$ fixpoints are parametrized by e-tuples of partitions of numbers adding up to n. Put

$$\Omega(x, z_1, \ldots, z_k, q, t) := \sum_{\lambda partitions} \frac{\prod_{i=1}^{k} \prod_{\square\in\lambda}(1 - q^{c(\square)} t^{r(\square)} z_i)}{\prod_{\square\in\lambda}(q^{a(\square)+1} - t^{l(\square)})(q^{a(\square)} - t^{l(\square)+1})} x^{|\lambda|}$$

Identify partition with graph, and put $c(\square)$ column, $r(\square)$ row, $a(\square)$ arm length, $c(\square)$ leg length

Put $H = \log(\Omega)$
By Riemann-Roch and localization on $S^{[n]}$ have

$$I_{S,V}(x,z) = \left(\prod_{i=1}^{e} \Omega(x, e^{v_1^{(i)}}z, \dots, e^{v_k^{(i)}}z, e^{t_1^{(i)}}, e^{t_2^{(i)}})\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$
$$= \exp\left(\sum_{i=1}^{e} H(x, e^{v_1^{(i)}}z, \dots, e^{v_k^{(i)}}z, e^{t_1^{(i)}}, e^{t_2^{(i)}})\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$

So we "only" have to compute this.

Proposition

We can expand

$$H(x, z_1, \dots, z_k, e^{\epsilon_1}, e^{\epsilon_2}) = \sum_{d_1,d_2 \geq -1} H_{d_1,d_2}(x, z_1, \dots, z_k)\epsilon_1^{d_1}\epsilon_2^{d_2}$$

(not trivial could have deep pole in ϵ_1, ϵ_2)

Trick: Rewrite previous formula for $I_{S,V}(x,z)$: Inside exponential apply localization formula on S

$$I_{S,V}(x,z) = \exp\left(\sum_{i=1}^{e} H(x, e^{v_1^{(i)}}z, \ldots, e^{v_k^{(i)}}z, e^{t_1^{(i)}}, e^{t_2^{(i)}})\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$

$$= \exp\Bigg(\Bigg(\sum_{i=1}^{e} \frac{1}{t_1^{(i)}t_2^{(i)}}\Big(H_{-1,-1}(x, e^{v_j^{(i)}}z) + (t_1^{(i)} + t_2^{(i)})H_{-1,0}(x, e^{v_j^{(i)}}z)$$

$$+ t_1^{(i)}t_2^{(i)}H_{0,0}(x, e^{v_j^{(i)}}z) + ((t_1^{(i)})^2 + (t_2^{(i)})^2)H_{-1,1}(x, e^{v_j^{(i)}}z)\Big)\Bigg)\Bigg|_{\epsilon_1=\epsilon_2=0}\Bigg)$$

$$= \exp\left(c_2(V)C_2 + c_1(V)^2C_{11} + K_Sc_1(V)D_1 + e(S)F + (K_S^2 - 2e(S))E\right)$$

Put $H_{d_1,d_2,k}(x,z) = H_{d_1,d_1}(x,z,\ldots,z)$,

Then $F(x,z)$, $E(x,z)$,$D_1(x,z)$, $C_2(x,z)$, $C_{11}(x,z)$ are given in terms of $H_{-1,-1,k}(x,z)$, $H_{-1,0,k}(x,z)$, $H_{0,0,k}(x,z)$, $H_{-1,1,k}(x,z)$

So we need to understand these power series.

Want to understand

$H_{-1,-1,k}(x,z), \quad H_{-1,0,k}(x,z), \quad H_{0,0,k}(x,z), \quad H_{-1,1,k}(x,z)$

Use two properties: **regularity** and **symmetry**

1. $f(x,z) \in \mathbb{C}[[x,z]]$ is d-regular (wrt k) if $f\left(x\epsilon^{2-k}(1+\epsilon)^k, \frac{1}{1+\epsilon}\right) \in \epsilon^d \mathbb{C}[[x,\epsilon]]$,
2. $f(x,z)$ is called symmetric if $f(x,z) = f(x^{-1}, xz)$.

Theorem

1. *$H_{d_1,d_2,k}(x,z)$ is $-d_1 - d_2$ regular for $d_1 + d_1 \leq 0$*
2. $H_{d_1,d_2,k}(x,z) + \frac{B_{d_1+1}B_{d_2+1}}{(d_1+1)!(d_2+1)!}(Li_{1-d_1-d_2}(x) + kLi_{1-d_1-d_2}(z))$ *is symmetric* ($Li_d(x) = \sum_{n>0} x^n/n^d$ *polylog).*

First part follows from the fact that $I^{Chern}_{S,V}$ is limit of $I_{S,V}$
Second part is deep input from symmetric function theory: identities of generalized MacDonald's polynomials

Symmetric and regular functions fulfill very strong constraints:

Theorem

Let $f(x,z)$ be a symmetric d-regular function (wrt k).

1. *if $d > 0$, then $f(x,z) = 0$.*
2. *if $d = 0$, there exists a unique $h(y) \in \mathbb{C}[[y]]$, such that*

$$f\left(\frac{u(1-u)^{k-1}}{(1-v)^{k-1}}, \frac{v}{(1-u)^{k-1}}\right) = h\left(\frac{uv}{(1-u)(1-v)}\right).$$

The functions $F(x,z)$, $E(x,z)$, $D_1(x,z)$, $C_2(x,z)$, $C_{11}(x,z)$ can be expressed in terms of symmetric regular functions
A symmetric regular function is determined by few of its coefficients,
Trick: assume $g(x,z)$ is 1-regular, then $f(x,z) := D_z g(x,z)$ is regular. If furthermore $f(x,z)$ is symmetric, then $g(x,0)$ determines $f(x,z)$