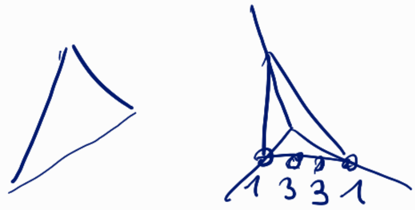
Enumerative geometry of quantum periods

(j. Ruddat, Tarlow, Zhou)

$\longrightarrow$ $\mathbb{P}^2/\mathbb{Z}_3$



§1 Mirror symmetry

$\mathbb{P}$ sm. proj. Fano surface $\leftrightarrow$ f Laurent polynomial (up to mutation)

(LG potential $f : (\mathbb{C}^*)^2 \to \mathbb{C}$)

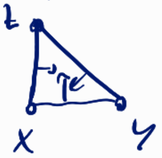
over $\mathbb{C}[NE(\mathbb{P})]$

$\ni$

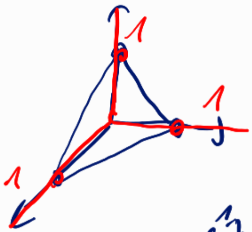
z^β

$\mathbb{P}$ toric

$\mathbb{P}^2$

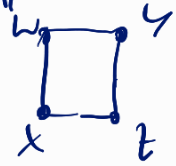


$\leftrightarrow$

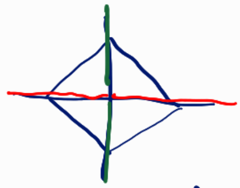


$f = x + y + \frac{q^{-1} z}{xy}$

$\mathbb{P}^1 \times \mathbb{P}^1$



$\{xy = zw\} \subset \mathbb{P}^3$

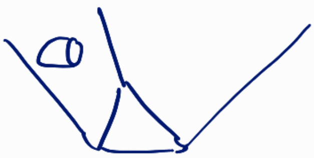


$f = x + y + \frac{z_1}{x} + \frac{z_2}{y}$

$\}$

$K_{\mathbb{P}}$ "local" CY $\leftrightarrow$ $\{uv = p(x,y)\} \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$

$p = 1 - f$



$L \subset K_{\mathbb{P}}$ AV A-brane $\leftrightarrow$ B-brane on $\{u = 0\}$
$\cong S^1 \times \mathbb{R}^2$ SL submfd

$= \mathbb{R}^2$

moduli space: $\Sigma = \{p = 0\} \subset (\mathbb{C}^*)^2$

"mirror curve"

mirror maps

closed Q^β sympl. $\leftrightarrow$ z^β cplx

$z^\beta(Q, q) = \ldots$

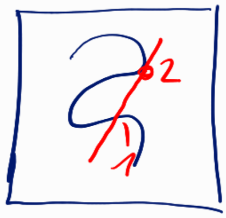
open U area of L $\leftrightarrow$ x local coords on mirror curve

$x = U \cdot M(Q, q)$

$q = 1 \to$ old

natural q-refinement by "quantum torus" $YX = qXY$

Def.: genus g, 2-marked log GW, $E \subset -K_{\mathbb{P}}$ smooth $(\mathbb{P}^2, E)$



$$R_{g,(a,b)}(\mathbb{P}(\log E),\beta) = \int_{[\;]^{\text{virt}}} (-1)^g \lambda_g \, ev^* pt$$

↑
fixed

$$\left.\begin{aligned} R_{0,(2,1)}(\mathbb{P}^2(\log E), L) &= 1 \\ R_{0,(1,2)}(\;\cdot\quad - \;) &= 4 \end{aligned}\right\} \cdot \frac{2^2}{1^2} \quad \text{(Cadman-Chen)}$$

Thm: $$M(Q,q) = 1 + \sum_{\beta\in NE(\mathbb{P})} \sum_{g\geq 0} \frac{(-1)^{\beta\cdot E}}{\beta\cdot E-1} R_{g,(1,\beta\cdot E-1)}(\mathbb{P}(\log E),\beta)\, \hbar^{2g} Q^{\beta}$$

$$\left(\frac{(\vartheta_1)_\infty}{y}\right) \text{ s.th.}$$

$q=1 \Rightarrow g=0$ $\qquad\qquad q = e^{i\hbar}$

$g=0$: = open GW (LLW, CLLT, ...)

$g>0$: discrepancy & open

Upshot: $\mathbb{P} \overset{\text{"HS"}}{\rightsquigarrow} M(Q,q)$ gen. fct. for 2-marked log GW

§2 Quantum period & mirror maps

Def.: quantum A-period

$$a(t,q) = \mathrm{Const}_{x,y}(\log \overset{1-f}{\overset{\parallel}{p}}) = \boxed{-\sum_{n>0} \tfrac{1}{n}\, \mathrm{Const}_{x,y}(f^n)}$$

$\int \log p \frac{dx}{x}\frac{dy}{y}$ [CKYZ]

Expl: $\mathbb{P}^2$ $\quad f = x + y + \frac{q^{1/2}t}{xy}$

$$a(t,q) = -(q^{1/2}+q^{-1/2})\,t + \dots$$

$$f^3:\; x\cdot \frac{q^{1/2}t}{xy}\cdot y,\; \doteq yx,\; yx \doteq$$

or [ACDKV] rep. of quantum torus by $V = \mathbb{C}[NE(\mathbb{P})][q^{\pm 1/2}][x^{\pm 1}] \ni \Psi$

$$x\cdot\Psi(x) = x\Psi(x)$$
$$y\cdot\Psi(x) = \Psi(qx)$$

↗ exact WKB methods

$\Psi \in V$ def. by $p\cdot \Psi(x) = 0$

$$a(t,q) = \mathrm{Const}_x\left(\log \frac{\Psi(qx)}{\Psi(x)}\right)$$

$\int \log y \frac{dx}{x}$ [AV]

Thm.: $a(z,q) = \underline{a}(z,q)$

Pf idea: $Y \cdot \Psi = Y(X)\,\Psi(X)$

chain rule & substitution

for Const $= \int \frac{dx}{x}$

Def.: $Q^\beta = z^{\beta} \cdot e^{-(\beta \cdot E)\, a(-z,q)} \overset{\text{Lagr. inv.}}{\rightsquigarrow} z^\beta(Q,q)$

$M(Q,q) = e^{\boxed{a(-z(Q,q),q)}}$

Expl.: $\mathbb{P}^2$

$$M(Q,q) = 1 - (q^{1/2} + q^{-1/2})Q + (q^2 + q + 1 + q^{-1} + q^{-2})Q^2 + \dots$$

$q=1$: $\quad 2 \qquad 5 \qquad 32 \qquad 286$

$2 = \frac{1}{2}\underbrace{R_{0,(1,2)}(\mathbb{P}^2,-1)}_{=4}$

§3 Theta function

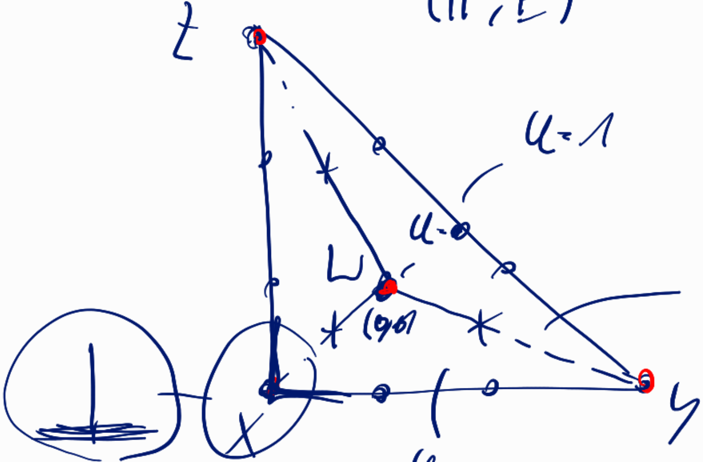
$\mathbb{P} \not\cong \mathbb{P}_\Delta$ toric proj. Fano

$\Delta \leftrightarrow -K_X$

+ central subdiv. + φ PL fct.

+ affine sing. → straight. bdry

$\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$ $\quad (\mathbb{P}^2, E)$



DCT $\longleftrightarrow$

+ scattering

+ broken lines

$\rightsquigarrow \mathcal{X} = \{XYZ = t^3(W + f_3)\} \subset \mathbb{P}(1,1,1,3)$

$\Delta_\varphi = \{(m,h) \mid m \in \Delta,\ h = \varphi(m)\}$

$\boxed{(W_1)|_0 = x + y + \frac{t}{xy}}$

$\boxed{W_1|_\varphi = \text{counts } R_{g,(1,\cdot)}}$

$t \neq 0$: $\mathbb{P}^2$

$t = 0$: $\bigcup_3 \mathbb{P}(1,1,3)$

$(\mathbb{P}^2, A)$