# Global Kuranishi charts and a localisation formula in symplectic GW theory

partially joint work with Mohan Swaminathan

Throughout, $(X, \omega)$ will be a closed symplectic manifold.
We will be interested in the case where $X$ admits a Hamiltonian $T$-action for some compact Lie group $T$, in particular $T = (S^1)^k$.

Applications: * localisation formula (Atiyah-Bott)

$$\int_M \alpha = \sum_{P \subset M^T} \int_P \frac{\alpha}{e(N_P)}$$

* Frobenius structure on $H_T^*(X; \mathbb{Q})$
* mirror symmetry (Givental)
* quantum Kirwan maps

Def: An equivariant global Kuranishi chart $\mathcal{K}$ for a compact $T$-space $\overline{\mathcal{M}}$ consists

- a compact Lie group $G$
- a $G$-manifold $\mathcal{T}$ with finite isotropy (thickening)
- a $G$-vector bundle $\mathcal{E} \to \mathcal{T}$ (obstruction bundle)
- a $G$-equivariant section $s : \mathcal{T} \to \mathcal{E}$ with (obstruction section)

  $$\overline{\mathcal{M}} \cong s^{-1}(0)/G$$

- a $T$-action on $\mathcal{T}$ and $\mathcal{E}$ commuting with the $G$-action

  so that

  a) all maps are equivariant,

  b) it induces the $T$-action on $\overline{\mathcal{M}}$.

$\overline{\mathcal{M}}$ has virtual dimension

$$\operatorname{vdim}(\overline{\mathcal{M}}) := \dim(\mathcal{T}) - \dim(G) - \operatorname{rank}(\mathcal{E}).$$

Its equivariant virtual fundamental class

$$[\overline{\mathcal{M}}]^{vir}_T : \check{H}^{vdim+*}_T(\overline{\mathcal{M}};\mathbb{Q}) \longrightarrow H^*_T(pt;\mathbb{Q})$$

is the composition

$$\check{H}^{vdim+*}_T(\overline{\mathcal{M}}) \xrightarrow{s^* Th_T(\mathcal{E}/G)\cdot} H^{\dim(\mathcal{T}/G)+*}_{T,c}(\mathcal{T}/G) \xrightarrow{\int^T_{\mathcal{T}/G}} H^*_T(pt)$$

Rem: We 'recover' the ordinary virtual fundamental class via the commutative diagram

$$\begin{tikzcd}
\check{H}^{vdim+*}_T(\overline{\mathcal{M}}) \arrow[r, "{[\overline{\mathcal{M}}]^{vir}_T}"] \arrow[d] & H^*_T \arrow[d] \\
\check{H}^{vdim+*}(\overline{\mathcal{M}}) \arrow[r, "{[\overline{\mathcal{M}}]^{vir}}"] & \mathbb{Q}
\end{tikzcd}$$

Thm (H.): a) If $(X, \omega, \mu)$ is a Hamiltonian $T$-manifold and $J$ is $T$-invariant, then $\overline{\mathcal{M}}_{g,n}(X, A; J)$ admits an equivariant global Kuranishi chart.

b) The equivariant virtual fundamental class is independent of auxiliary choices.

Rem: * This generalises work of Abouzaid-McLean-Smith and H.-Swaminathan to the equivariant setting.

* There are constructions of equivariant Kuranishi charts by Fukaya and of equivariant GW invariants by Chen-Tian.

Def: We define the equivariant GW class of $(X, \omega)$ to be

$$GW^{X,\omega}_{g,n,A} := (ev \times st)_* [\overline{\mathcal{M}}_{g,n}(X, A; J)]^{vir}_T$$

being a map $H^*_T(X^n \times \overline{\mathcal{M}}_{g,n}) \longrightarrow H^*_T$.

Thm (H.): Suppose $T = (S^1)^k$ and $(X, \omega)$ is a Hamiltonian $T$-manifold such that

i) $X^T$ is finite

ii) the union of the 1-dimensional orbits is a disjoint union of symplectic planes.

Then

$$GW^{X,\omega,T}_{g,n,A}(\alpha_1 \times \dots \times \alpha_n) = \sum_{\Gamma \in G(X,A)} \frac{1}{|Aut(\Gamma)|} \langle j_\Gamma^* ev^* \alpha \cdot h(\Gamma), [\bar{\mathcal{M}}_\Gamma] \rangle$$

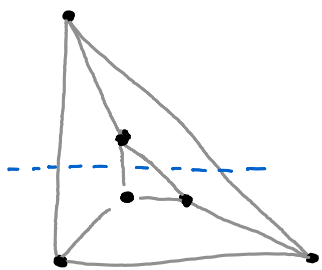
where

- $\Gamma$ is a decorated graph
- $\bar{\mathcal{M}}_\Gamma$ is a product of moduli spaces of stable curves
- $h(\Gamma) \in H^*(\bar{\mathcal{M}}_\Gamma) \otimes_{\mathbb{Q}} Frac(H_T^*)$ is the weight of $\Gamma$

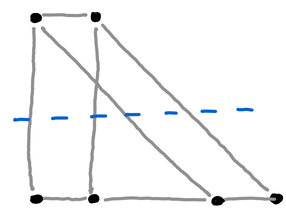
Rem: In the algebraic setting, the same formula was found for the GW invariants by Kontsevich, Graber-Pandharipande and Liu.

Cor: If $X$ is a symplectic toric manifold, its symplectic and algebraic GW invariants agree.

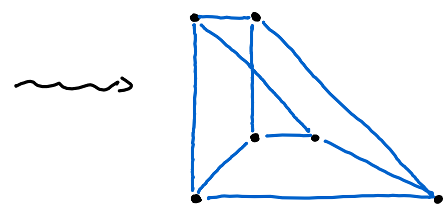
Ex: There are examples, constructed by Tolman, of 6-dimensional $T^2$-manifolds that satisfy the assumptions but are not Kähler.



$M \subset \mathbb{C}P^3 \times \mathbb{C}P^2$



$\mathbb{C}P^2 \times \mathbb{C}P^1$



Proof of Thm 1: The thickening $\mathcal{T}$ is again of the form

$$\mathcal{T} = \{(u, C, x_1, \dots, x_n, z, \eta)\}/_\sim \quad , \text{ where}$$

- $u : C \longrightarrow X$ smooth and stable, $x_i \in C$
- $z : C \to \mathbb{P}^N$ regular, holomorphic, automorphism-free with

$$z^* \mathcal{O}(1) \cong (\omega_C \otimes u^* \mathcal{O}_X(1)^{\otimes 3})^{\otimes p} \qquad p \gg 1$$

- $\eta$ is a perturbation satisfying

$$\bar{\partial}_J u + \langle \eta \rangle \cdot dz = 0$$

We require that $\mathcal{O}_X(1) \longrightarrow X$ admits a lift of the $T$-action.

Such bundles exist by work of Mundet i Riera. Then

$$t \cdot (u, C, x_1, \dots, x_n, z, \eta) = (t \cdot u, C, x_1, \dots, x_n, z, t \cdot \eta)$$

$\square$

Rem: Only this step requires the action to be Hamiltonian.

If $\mathcal{K}$ is an equivariant global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,A;J)$ and

$$\overline{\mathcal{M}}_{g,n}(X,A;J)^T = \bigsqcup_{\Gamma} \overline{\mathcal{M}}(\Gamma)$$

then each $\overline{\mathcal{M}}(\Gamma)$ admits a Kuranishi chart $\mathcal{K}_\Gamma$ with

* $\mathcal{T}_\Gamma \hookrightarrow \mathcal{T}$ with normal bundle $N_\Gamma$
* $\mathcal{E}|_{\mathcal{T}_\Gamma} = \mathcal{E}_\Gamma \oplus \mathcal{E}_\Gamma^m$

Virtual Atiyah-Bott localisation formula:

$$[\overline{\mathcal{M}}_{g,n}(X,A;J)]^{vir}_T = \sum_\Gamma j_{\Gamma *}\left(\frac{e_T(\mathcal{E}^m_\Gamma)}{e_T(N_\Gamma)} \cap [\overline{\mathcal{M}}(\Gamma)]^{vir}_T\right)$$

as maps $\check{H}^*_T(\overline{\mathcal{M}}_{g,n}(X,A;J)) \otimes \mathrm{Frac}(H^*_T) \longrightarrow \mathrm{Frac}(H^*_T)$.

Proof of Theorem 2:

Key point: If $[u, C, x_1, \dots, x_n] \in \overline{\mathcal{M}}^T$, then

- $\mathrm{im}(u)$ contained in 0- and 1-dimensional orbits of $X$
- if $(C', x'_*)$ is a stable component of $(C, x_*)$: $u(C') = p \in X^T$

1) We have a bijection

$$\{\text{components of } \overline{\mathcal{M}}^T\} \longleftrightarrow G(X, A)$$

$$\{[u, C, x_*]\} \longmapsto \begin{cases} V(\Gamma) = \pi_0(u^{-1}(X^T)) \\ \text{with } V(\Gamma) \longrightarrow X^T \\ \qquad\quad v \longmapsto u(C_v) \end{cases}$$

2) * $\overline{\mathcal{M}}(\Gamma) \cong \prod_{v \in V^s(\Gamma)} \overline{\mathcal{M}}_{g_v, n_v} =: \overline{\mathcal{M}}_\Gamma$

* $\dfrac{e_T(\mathcal{E}^m_\Gamma)}{e_T(N_\Gamma)} = h(\Gamma)$

For this, we use the (pointwise) exact sequence

$$0 \longrightarrow \ker(D\bar{\partial}_J(u)) \longrightarrow T_y\mathcal{T} \xrightarrow{d^v s(y)} \mathcal{E}_y \longrightarrow \operatorname{coker}(D\bar{\partial}_J(u)) \longrightarrow 0$$

for any $y \in \mathcal{T}^T$. Taking the fixed and the moving part, we get

* $K_\Gamma$ is regular ($s_\Gamma \pitchfork 0$ of $\mathcal{E}_\Gamma$)

* $0 \to \ker(D\bar{\partial}_J)^m \longrightarrow N_\Gamma \longrightarrow \mathcal{E}_\Gamma^m \longrightarrow \operatorname{coker}(D\bar{\partial}_J)^m \longrightarrow 0$

form vector bundles!

$$\Longrightarrow \quad \frac{e_T(\mathcal{E}_\Gamma^m)}{e_T(N_\Gamma)} = \frac{e_T(\operatorname{coker}(D\bar{\partial}_J)^m)}{e_T(\ker(D\bar{\partial}_J)^m)} = h(\Gamma)$$

$\square$