Rigid isotopy classification of real quintic rational plane curves

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November 16th 2017

Setting

Let $f \in \mathbb{R}[x,y,z]$ be a homogeneous polynomial of degree d. We define

$$\begin{array}{ccccc} \mathbb{R}X & = \{[x:y:z] \in \mathbb{RP}^2 \mid f(x,y,z) = 0\} & \subset & \mathbb{RP}^2 \\ \cap & & & \cap \\ \mathbb{C}X & = \{[x:y:z] \in \mathbb{CP}^2 \mid f(x,y,z) = 0\} & \subset & \mathbb{CP}^2. \end{array}$$

Let us assume f is non-singular, i.e., ∇f has no solutions in $\mathbb{C}^3 \setminus 0$.

Topology of $\mathbb{R}X$

If $\mathbb{R}X \neq \emptyset$, then $\mathbb{R}X$ is a smooth compact manifold of real dimension 1.

Therefore,

$$\mathbb{R}X \underset{\text{homeo}}{\cong} \bigsqcup_{1}^{k} S^1, \quad k \geq 0.$$

The topology of $\mathbb{R}X$ does not depend entirely on its degree. If the degree is odd, then $k \geq 1$.

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Theorem (Harnack, 1876)

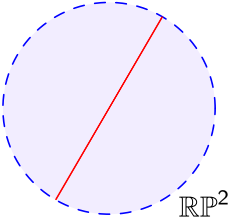
If X is a non-singular curve in $\mathbb{RP}^2$ of degree d, then

$$k \leq \frac{(d-1)(d-2)}{2} + 1.$$

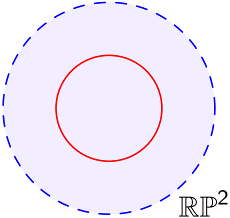
Moreover, this bound is sharp.

Topology of the pair $(\mathbb{RP}^2, \mathbb{R}X)$

There are two different homology classes of embedded circles in $\mathbb{RP}^2$:



non-contractible component, *pseudoline*;



contractible component, *oval*.

Topology of the pair $(\mathbb{RP}^2, \mathbb{R}X)$

Let X, Y be two non-singular plane curves of degree d.

$$(\mathbb{RP}^2, \mathbb{R}X) \cong (\mathbb{RP}^2, \mathbb{R}Y) \quad \text{if} \quad \exists \varphi \in \mathrm{Homeo}(\mathbb{RP}^2, \mathbb{RP}^2) \\ \varphi(\mathbb{R}X) = \mathbb{R}Y.$$

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Since $\text{Homeo}(\mathbb{RP}^2, \mathbb{RP}^2)$ is connected, this is equivalent to:

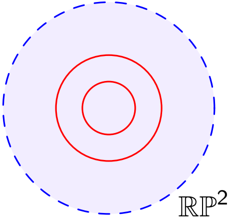
$$\exists \varphi_t \in \text{Homeo}(\mathbb{RP}^2, \mathbb{RP}^2), 0 \le t \le 1,$$

$$\varphi_0 = \text{Id}_{\mathbb{RP}^2},$$

$$\varphi_1 = \varphi.$$

Equivalently, the sets $\mathbb{R}X$, $\mathbb{R}Y$ are isotopic as subsets of $\mathbb{RP}^2$.

Two non-isotopic sets realized in degree 4:



Hilbert's 16th problem, first part

Given a degree d, classify the homeomorphism classes of pairs $(\mathbb{R}\mathbb{P}^2, \mathbb{R}X)$, where $\mathbb{R}X \subset \mathbb{R}\mathbb{P}^2$ is a non-singular curve of degree d.

Equivalently,

Given a degree d, classify all non-singular curves $\mathbb{R}X \subset \mathbb{R}\mathbb{P}^2$ of degree d up to isotopy.

In such a family φ_t, every set $\varphi_t(\mathbb{R}X) \cong \mathbb{R}X$.
However, only $\varphi_0(\mathbb{R}X) = \mathbb{R}X$ and $\varphi_1(\mathbb{R}X) = \mathbb{R}Y$ are algebraic sets.
Imposing that $\varphi_t(\mathbb{R}X)$ is the real point set of a non-singular curve of degree d gives rise to the following notion.

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Definition

Two curves X, Y are *rigidly isotopic* if there exists an isotopy φ_t, $0 \leq t \leq 1$, from $\mathbb{R}X$ to $\mathbb{R}Y$ such that $\varphi_t(\mathbb{R}X)$ is the real point set of a non-singular curve of degree d.

Let us denote by $\mathcal{C}_d$ the space of polynomials of degree d up to projective equivalence.
We call *discriminant* the subset $\mathcal{D}_d \subset \mathcal{C}_d$ of all singular curves.
Within the discriminant $\mathcal{D}_d$, a generic curve has exactly one singular point, which is a nodal singular point.

$$\begin{array}{ccc} \mathbb{R}\mathcal{D}_d & \overset{\text{codim}_{\mathbb{R}}=1}{\subset} & \mathbb{R}\mathcal{C}_d \\ \cap & & \cap \\ \mathbb{C}\mathcal{D}_d & \underset{\substack{\text{codim}_{\mathbb{C}}=1 \\ \text{codim}_{\mathbb{R}}=2}}{\subset} & \mathbb{C}\mathcal{C}_d \end{array}$$

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\end{array}$$

The connected components of $\mathbb{R}\mathcal{C}_d \smallsetminus \mathbb{R}\mathcal{D}_d$ are called *chambers*.
The classification of the chambers is equivalent to the classification of real curves up to rigid isotopy.

State of the art

The classification of non-singular curves in $\mathbb{RP}^2$ of degree 5 up to rigid isotopy was obtained by Kharlamov.
In degree 6, it was obtained by Nikulin.

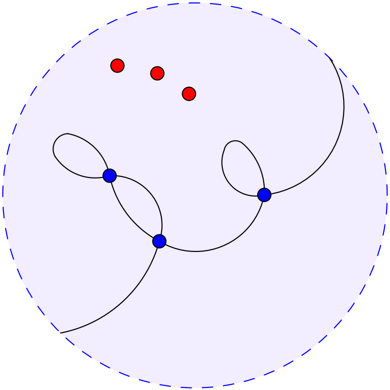
Rational curves

From now on, we assume that $X \subset \mathbb{RP}^2$ is a generic rational curve of degree 5. Thus, the curve X has a parametrization

$$\begin{array}{ccc} \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^2 \\ [u:v] & \longmapsto & [P(u,v):Q(u,v):R(u,v)], \end{array}$$

where P, Q and R are real homogeneous polynomials of degree d which do not have common zeros in $\mathbb{CP}^1$.
A generic rational curve of degree d has $\frac{(d-1)(d-2)}{2}$ nodal singular points.

Topology of $\mathbb{R}X$



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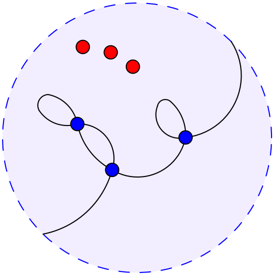
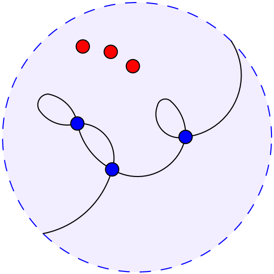


- *h hyperbolic nodes*, i.e., nodal points where two real branches cross.
- *e elliptic nodes*, i.e., where two imaginary complex conjugate branches intersect.
- *c imaginary nodes*, i.e., nodal points of $\mathbb{C}X \setminus \mathbb{R}X$.

Topology of $\mathbb{R}X$



Since the imaginary nodes come in pairs of complex conjugate points, we have that

$$e + h \in \{0, 2, 4, 6\}.$$

Topology of $(\mathbb{RP}^2, \mathbb{R}X)$

Itenberg, Mikhalkin and Rau found a classification of nodal real rational curves up to isotopy.

Main result

I obtain the rigid isotopy classification of nodal rational curves of degree 5 in $\mathbb{RP}^2$ using the fact that we can construct dessins by gluing building blocks and the correspondence

$$\left\{\mathbb{R}X \subset \mathbb{RP}^2 \text{marked nodal rational curves of degree 5}\right\} / \text{rigid isotopy}$$

$$\updownarrow$$

$$\left\{D \subset \mathbb{D}^2 \text{marked nodal dessins of degree 9}\right\} / \text{equiv. of marked dessins}$$

Rigid isotopy class of $\mathbb{R}X$

In order to study the rigid isotopy class of $\mathbb{R}X$, I use the following correspondences, from works by Orevkov, Degtyarev, Itenberg, Kharlamov, Zvonilov.

$\mathbb{R}X \subset \mathbb{RP}^2$ marked with a point of multiplicity $d-3$

$\updownarrow$ birational transformations

$C \subset \Sigma_n \longrightarrow \mathbb{CP}^1$ proper trigonal curve

$\updownarrow$ j-invariant fiberwise

$D \subset \mathbb{D}^2$ dessins

$$\left\{\mathbb{R}X \subset \mathbb{RP}^2 \text{ marked with a point of multiplicity } d-3\right\}/\text{L-isotopy}$$

$$\updownarrow \text{ birational transformations}$$

$$\left\{C \subset \Sigma_n \longrightarrow \mathbb{CP}^1 \text{ proper trigonal curve}\right\}/\text{equivariant equisingular def.}$$

$$\updownarrow\ j\text{-invariant fiberwise}$$

$$\left\{D \subset \mathbb{D}^2 \text{ dessins}\right\}/\text{elementary equivalence}$$

The j-invariant of a trigonal curve

Let $\Sigma \xrightarrow{\pi} B$ be a ruled surface endowed with an exceptional divisor E of negative self-intersection.
Let $C \subset \Sigma$ be a reduced irreducible curve such that

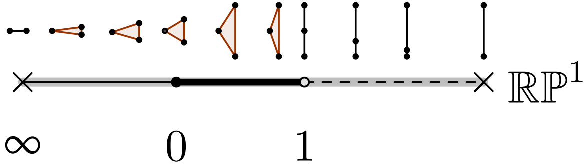
- $\pi|_C \colon C \longrightarrow B$ is a morphism of degree 3,
- $C \cap E = \varnothing$.

We define the function

$$\begin{aligned} j_C \colon \quad & B \longrightarrow \mathbb{CP}^1 \\ & b \longmapsto j\text{-inv}(\pi^{-1}(b) \cap (C \cup E)). \end{aligned}$$

Real property of the j-invariant

The value of the j-invariant determines the relative position of the points $\pi^{-1}(b) \cap C$.
The set $\pi^{-1}(b) \cap C$ has a symmetry if and only if $j_C(b) \in \mathbb{RP}^1$.
Moreover, the configuration of these points has the form



Dessin definition

To a trigonal curve C we associate the dessin

$$D_C := j^{-1}(\mathbb{RP}^1) \subset B,$$

along with the decorations of the $\mathbb{RP}^1$ in the target $\mathbb{CP}^1$.

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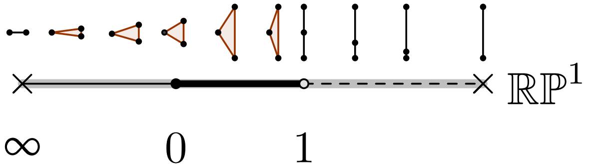
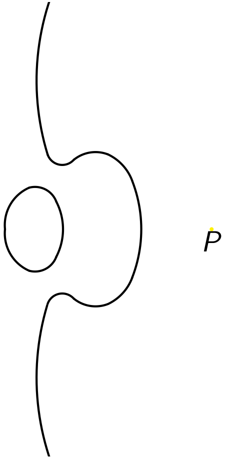


Example of a dessin



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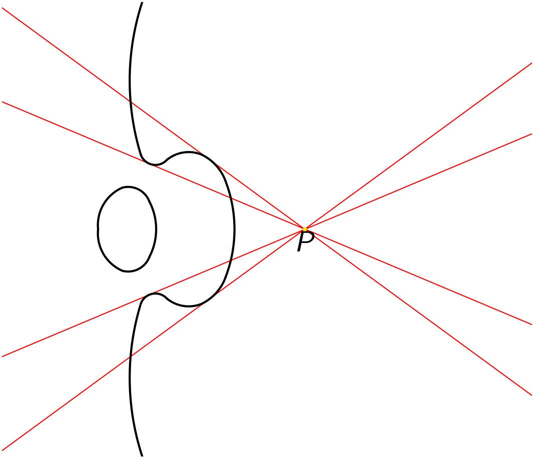

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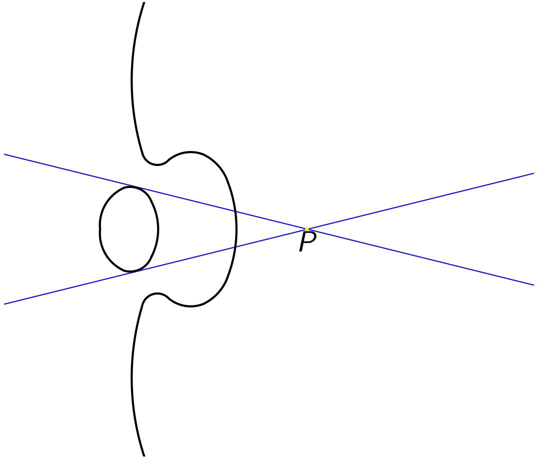

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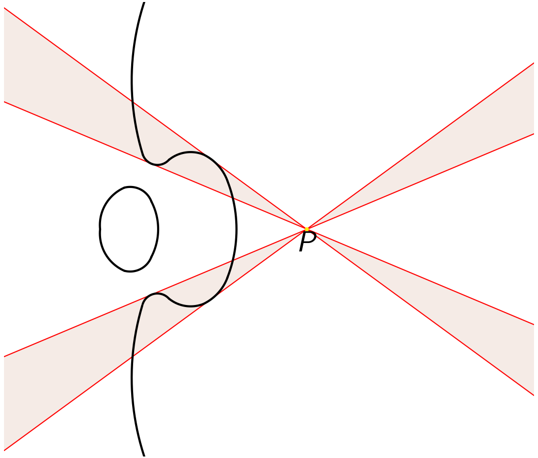

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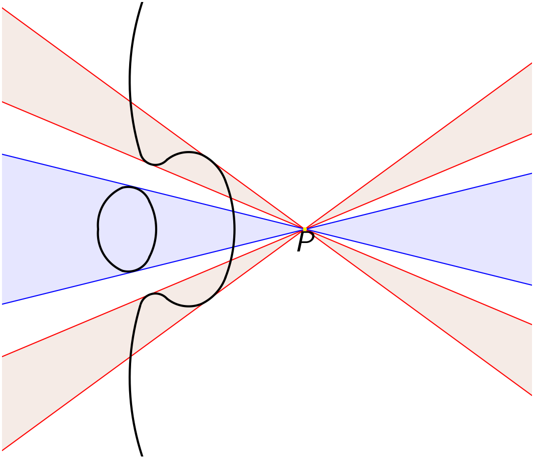

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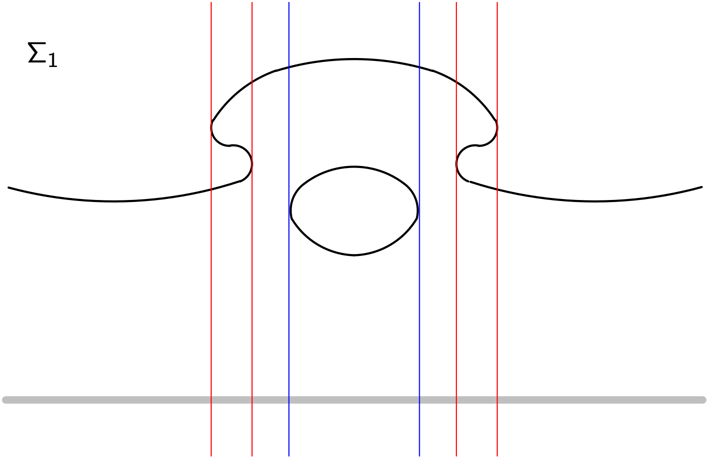

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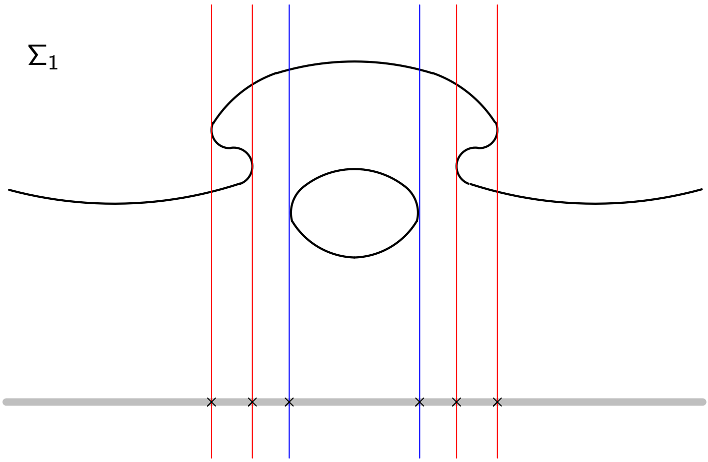

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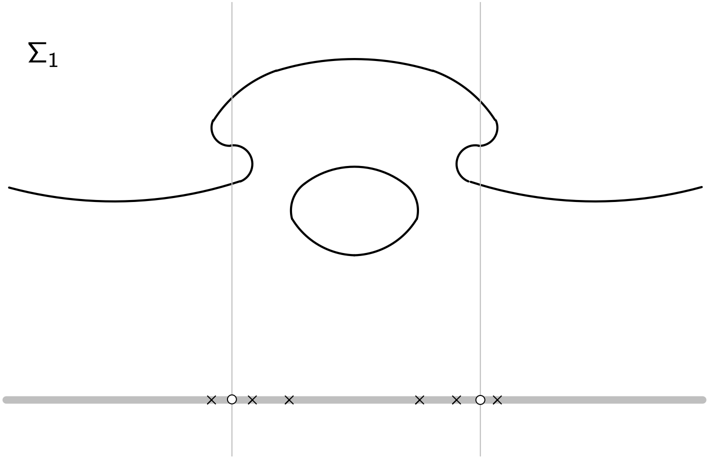

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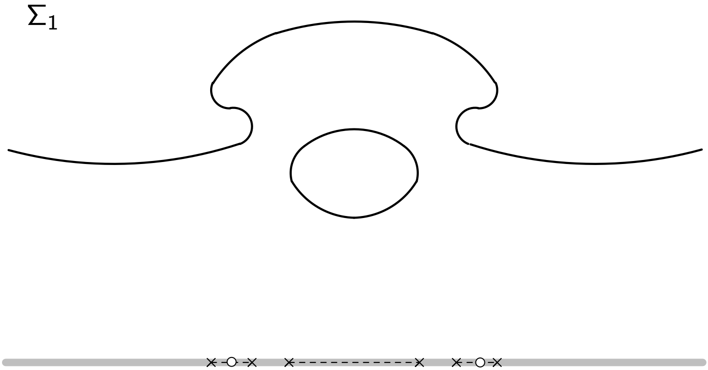

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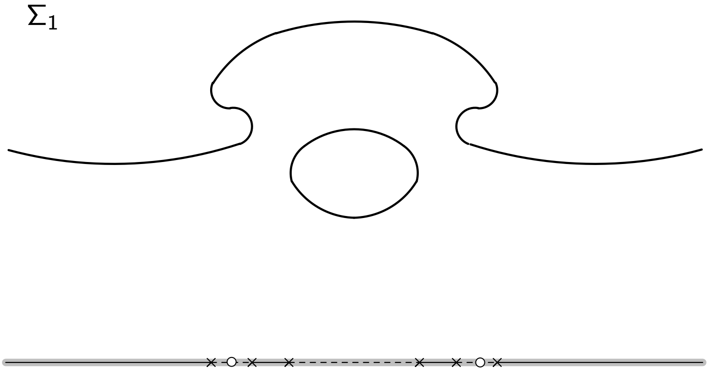
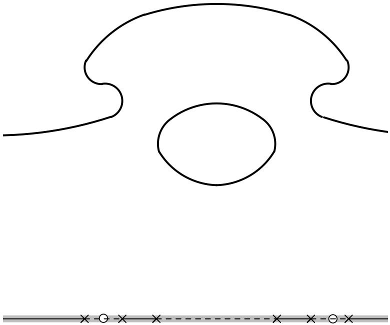
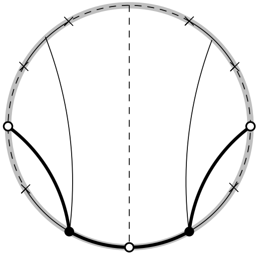
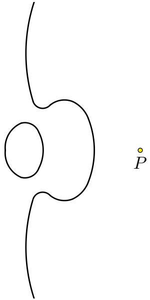
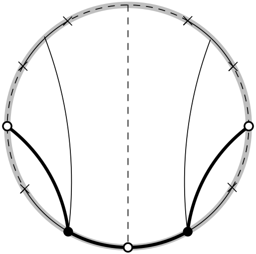






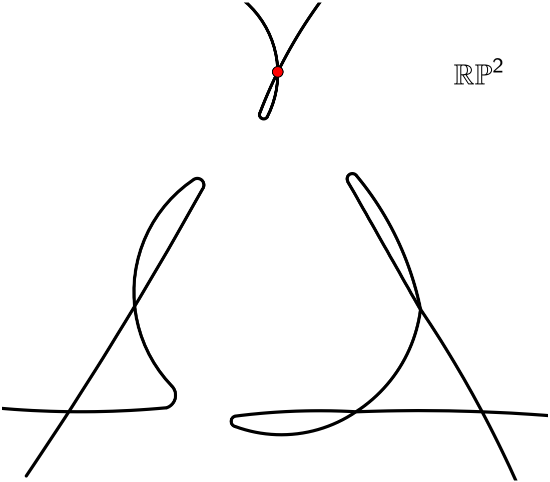
I_2





I_2

Dessin associated to a nodal real rational curve



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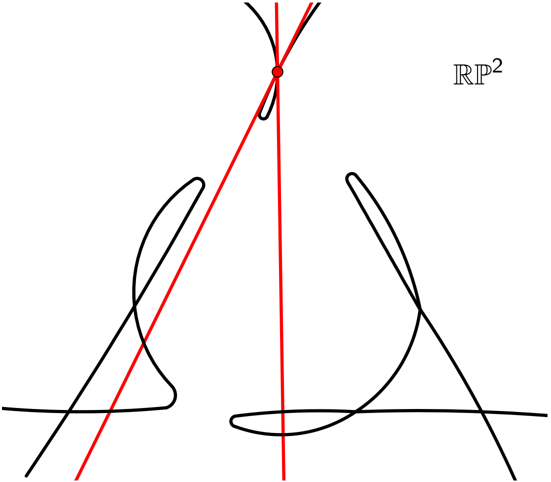
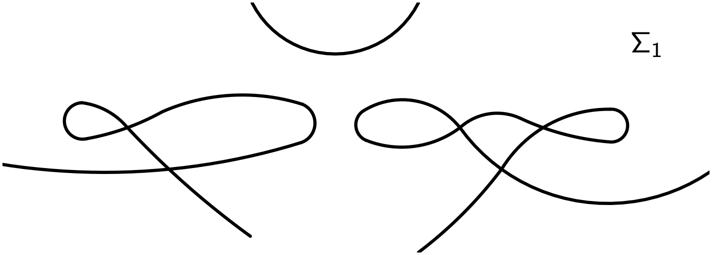


Trigonal curve in Σ_1



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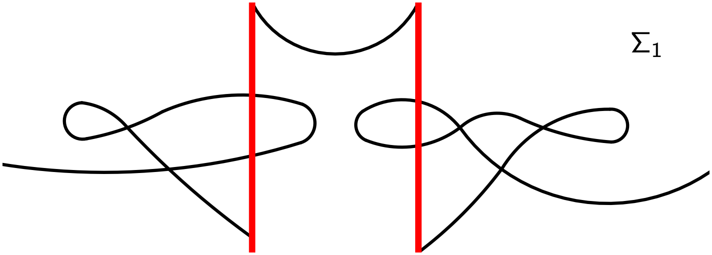
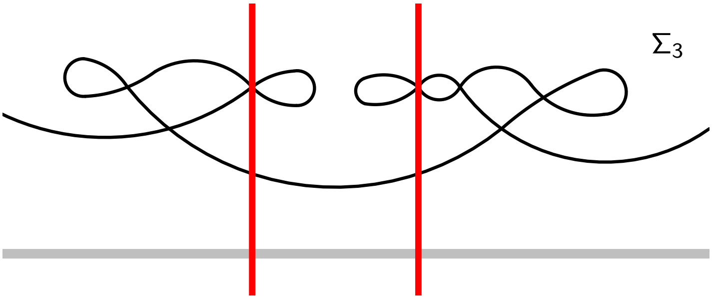


Marked proper trigonal curve in Σ_3



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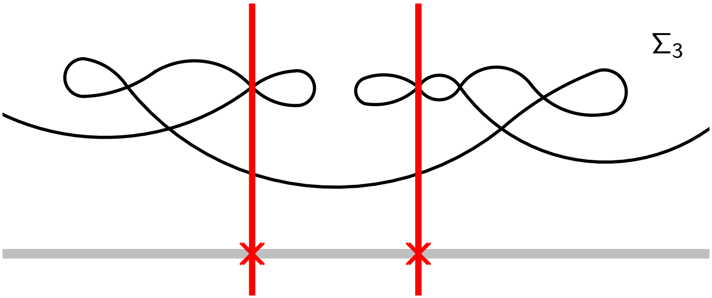

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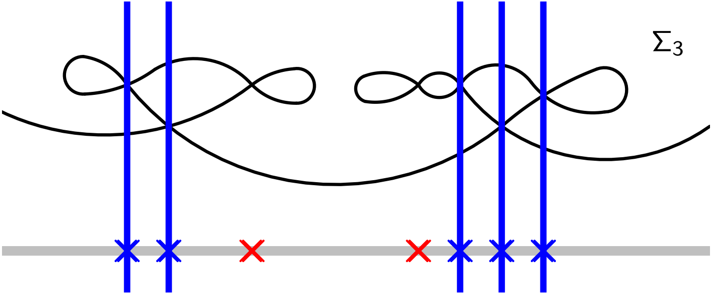

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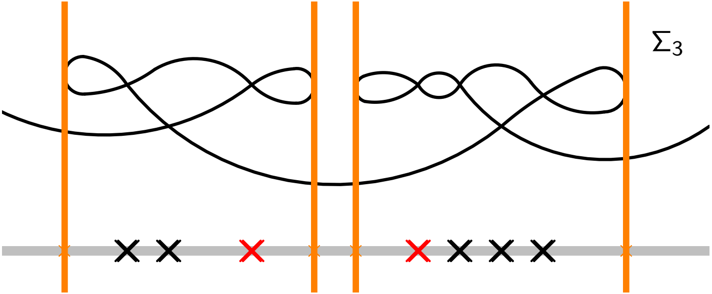

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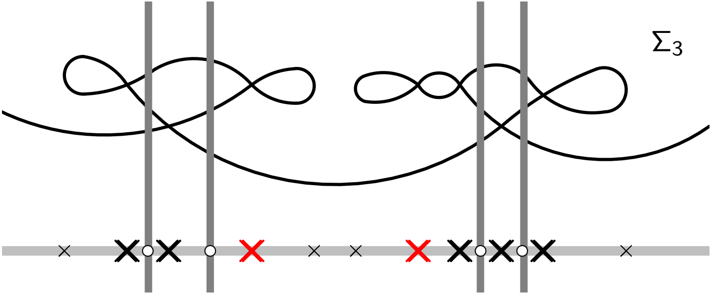

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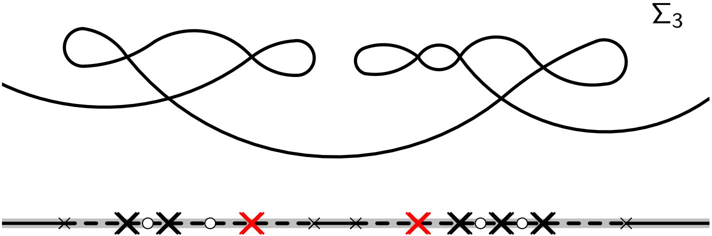
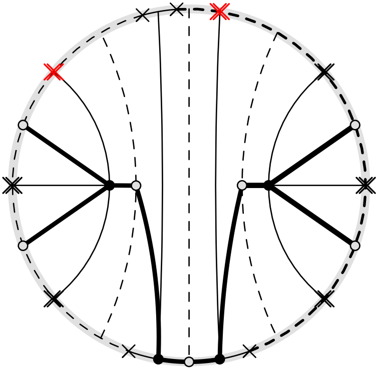
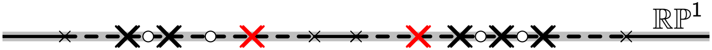
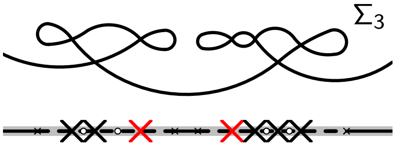
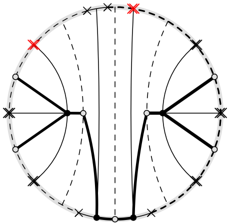


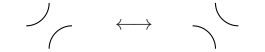



$\mathbb{RP}^1$

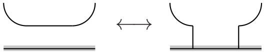

Σ_3



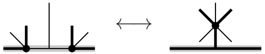
Elementary equivalence



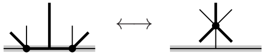
(a) Monochrome modification



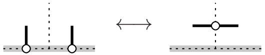
(b) Creating/destroying a bridge



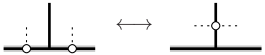
(c) •-in/•-out



(d) •-in/•-out

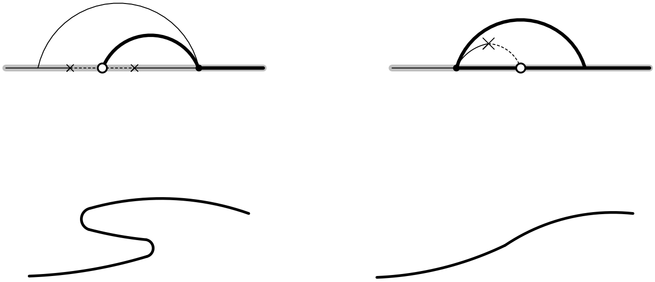


(e) ○-in/○-out

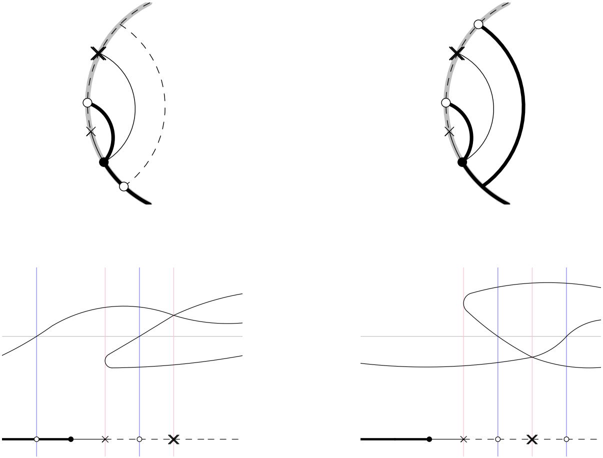


(f) ○-in/○-out

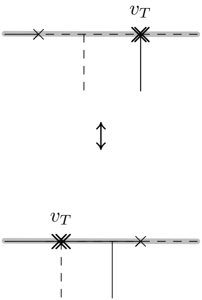
Weak equivalence

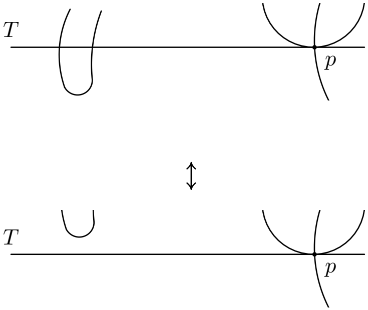


Very weak equivalence

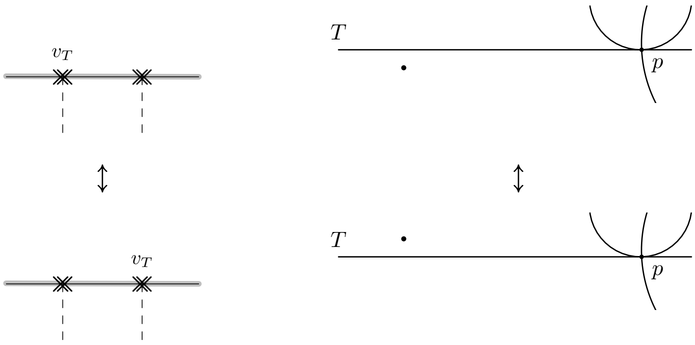


Additional equivalence relationships

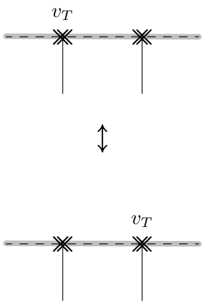


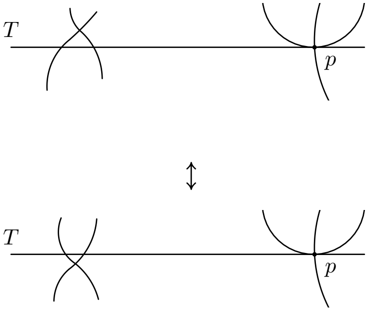


Additional equivalence relationships



Additional equivalence relationships





Theorem (JP)

There is a one-to-one correspondence between the rigid isotopy classes of marked nodal real rational curves of degree 5 and the equivalence classes of marked dessins on $\mathbb{D}^2$ with a fixed set of vertices.

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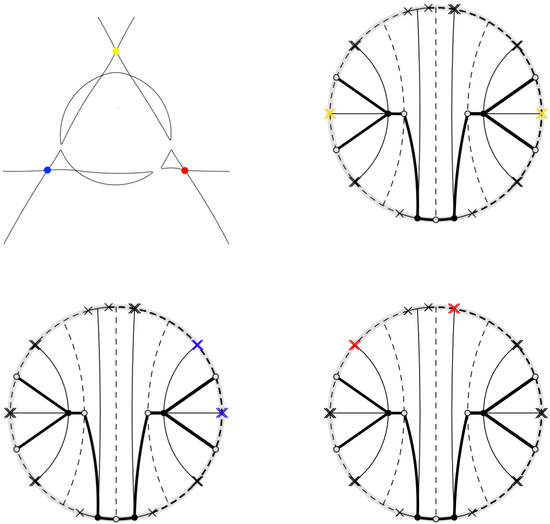
This gives rise to a classification of *marked* curves. From this we can easily deduce the rigid isotopy classification.



Proposition (JP)

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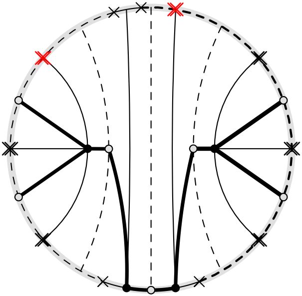


Classification of generic real pointed quartic curves

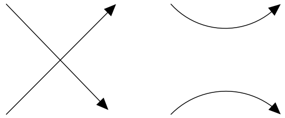
Theorem (JP)

There is a one-to-one correspondence between the weak equivalence classes of degree 6 uninodal toiles and the chambers of generic real pointed quartic curves in their moduli space.

In order to present the rigid isotopy classification we introduce the following notion.

Definition

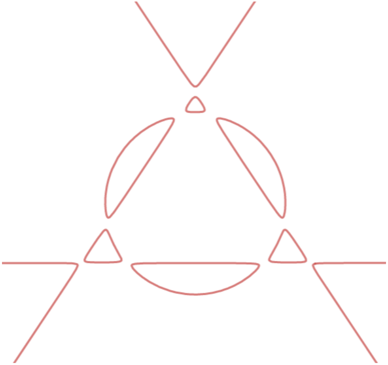
A *type* I *perturbation* X_0 of a nodal rational real curve X is a type I curve obtained by perturbing every elliptic node into an oval and every hyperbolic node into two real branches compatible with any orientation of X.



Definition

A nodal rational real curve of degree 5 is called $(M - s)$-perturbable if it has a type I perturbation with $7 - s$ connected components.

Non-singular maximal quintic



Rigid isotopy classification of M-perturbable real quintic rational curves

Theorem (JP)

If X is a maximally perturbable real nodal rational curve of degree 5 in $\mathbb{RP}^2$, then the rigid isotopy class of X is determined by its isotopy class and the position of its nodes with respect to the cyclic order of the ovals of a type I perturbation.

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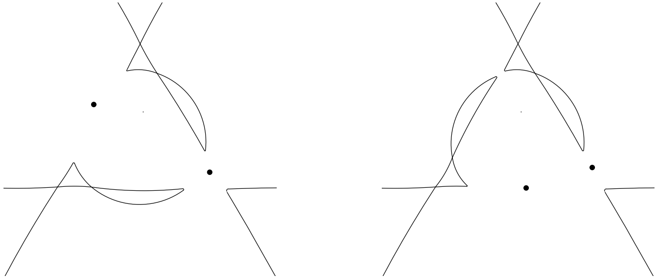
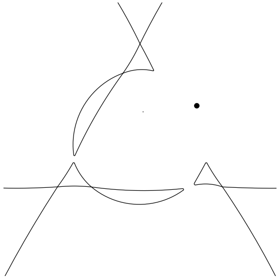
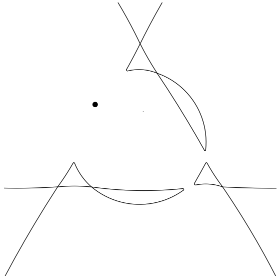
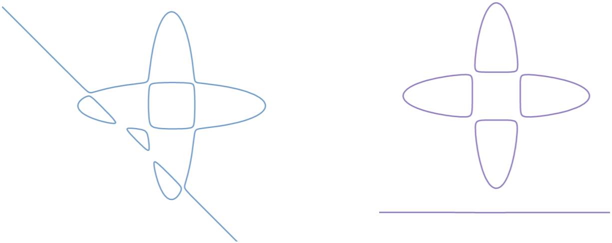




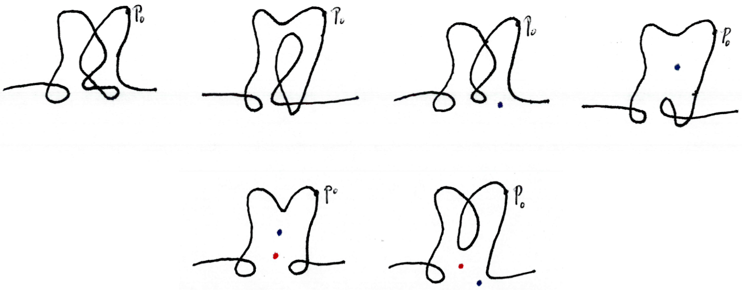




Let us denote by h_p the number of hyperbolic nodal points connecting the pseudoline to an oval in a type I perturbation of the curve.

Theorem (JP)

If $X \subset \mathbb{RP}^2$ is an $(M-2)$-perturbable nodal rational curve, its rigid isotopy class is determined by its isotopy class except for the following cases.


P_0
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Theorem (JP)

If $X \subset \mathbb{RP}^2$ is an $(M-4)$-perturbable nodal rational curve, its rigid isotopy class is determined by its isotopy class and the number $\sigma(C)$.

Figure: Rigid isotopy classes of nodal rational $(M-4)$-perturbable curves of degree 5 in $\mathbb{RP}^2$ with $\sigma = 0$.

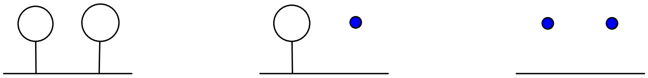

Figure: Rigid isotopy classes of nodal rational $(M-4)$-perturbable curves of degree 5 in $\mathbb{RP}^2$ with $\sigma = 2$.

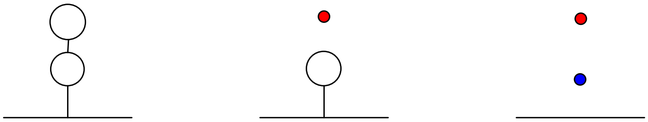

Figure: Rigid isotopy classes of nodal rational $(M-4)$-perturbable curves of degree 5 in $\mathbb{RP}^2$ with $\sigma = 4$.

Theorem (JP)

There is a unique rigid isotopy class of $(M-6)$-perturbable nodal rational curves in $\mathbb{RP}^2$.