Characteristic-free approach to unfoldings (and other results)

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In the 40's Whitney studied maps of C^{∞}-manifolds. When a map is not an immersion/submersion, one tries to deform it locally, in hope to make it 'generic'.
This approach has led to the rich theory of stable maps, developed by Thom, Mather and many others.
The main 'engine' was vector field integration. This chained the whole theory to the C^{∞}, or $\mathbb{R}/\mathbb{C}$-analytic setting.

I will present a purely algebraic approach, studying maps of germs of Noetherian schemes, in any characteristic. The relevant groups of equivalence admit 'good' tangent spaces. One has the theory of unfoldings (triviality and versality). Then I will discuss the new results on stable maps and theorems of Mather-Yau/Gaffney-Hauser.

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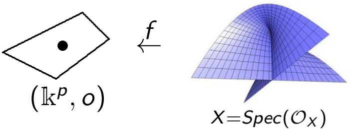

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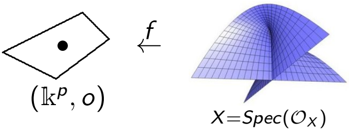

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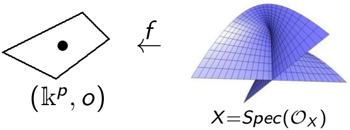

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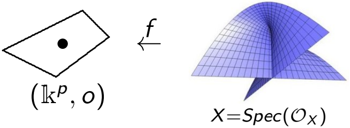

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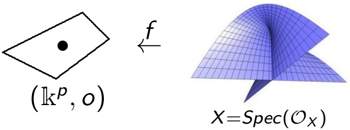

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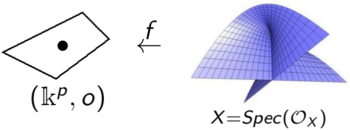

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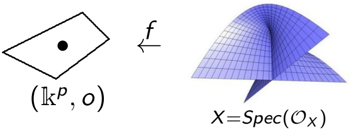

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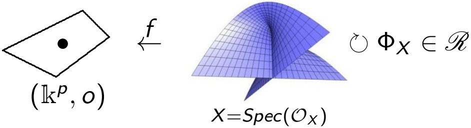

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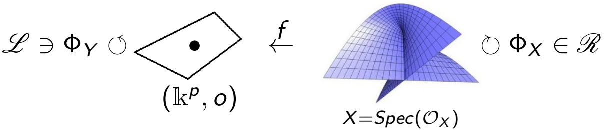

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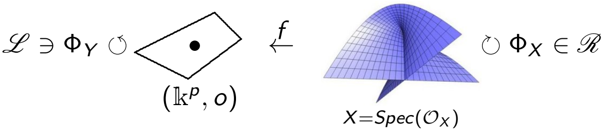

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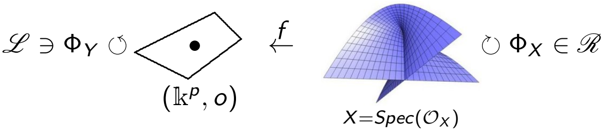

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2.*Suppose F is infinitesimally trivial and $\mathscr{G}$-separable. Then F is trivial.*

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This fails if $char(\mathbb{k})>0$. A modified version in [Greuel-Pham.19]. A stronger version:

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And a similar result for $\mathscr{A}$-equivalence.

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Thanks for your attention!

A bit about the proofs

1. The Implicit Function Theorem (IFT) holds over any field. But it is not directly applicable in Singularities, as the derivative $f'|_o$ is typically zero. Yet, there are some tricks (à la Tougeron's "implicit function theorem") to convert a system of equations into another form, where the IFT is applicable.
2. When the IFT "does not help", one can try to get an order-by-order solution. This will provide a formal solution to the problem. To ensure that the solution is (e.g.) analytic, one uses the Artin approximation. This works for the $\mathscr{R}$, $\mathscr{K}$ equivalences.
3. The $\mathscr{A}$-equivalence is more complicated, as the involved equations are not of implicit function type. Then one needs additional tools, e.g. the finite determinacy.

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