Artin approximation.
The ordinary, the inverse, the left-right, and on quivers.

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Abstract
1. Artin approximation is useful.
2. What is this “Weakly-finite singularity type”?
3. Continue the sequence:
   $(\mathbb{K}^m, o) \leftrightarrow (\mathbb{K}^n, o) \odot \mathcal{R}$,
   $\mathcal{L} \odot (\mathbb{K}^m, o) \leftrightarrow (\mathbb{K}^n, o)$,
   $\mathcal{L} \odot (\mathbb{K}^m, o) \leftrightarrow (\mathbb{K}^n, o) \odot \mathcal{R}$, . . . ??

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Prologue (we often have to resolve equations)

Let \( k \) be \( \mathbb{R} \) or \( \mathbb{C} \). Consider equations \( F(x, y) = 0 \). Here:

\[
x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_m), \quad F = (f_1, \ldots, f_c), \quad \text{with } f_i \in k\{x, y\}.
\]

Want an analytic solution, \( F(x, y(x)) = 0 \). Usually no chance for explicit solutions.

An approach: resolve order-by-order.

Then take \( \hat{y}(x) := \lim_{d \to \infty} y^{(d)}(x) \in k[[x]] \).

- Is this an analytic solution? (\( \hat{y}(x) \in k\{x\} \)?)
- Suppose \( F(x, y) \in k[x, y] \). Is this a Nash solution? (\( \hat{y}(x) \in k\langle x \rangle \)?)

Example. \((k^n, o) \overset{f}{\rightarrow} (k^1, o)\) Take a perturbation, \( f + h \). Can this be undone by a coordinate change?

Namely, \( f(x) + h(x) = f(y(x)) \).

More generally, this question for \( X \overset{f}{ightarrow} Y \). (With various equivalence relations.)

For deformation theory. For vector fields/foliations. For dynamical systems. . .

Ruling out the trivial case. \( F(x, y) = 0 \). Suppose \( \partial_y F|_{(o, o)} \) is non-degenerate.

Then the (analytic/Nash) solution exists. (Implicit Function Theorem.)

Below \( \partial_y F|_{(o, o)} \) is always degenerate.
Trying to resolve the equation $F(x, y) = o$. (Always assume $F(o, o) = o$.)

Geometry. $(\mathbb{k}^n_x, o) \times (\mathbb{k}^m_y, o) \supset V(F(x, y)) \xrightarrow{\text{analytic}} (\mathbb{k}^n_x, o)$. Does there exist an analytic section?

Can this formal section be approximated by analytic sections?

Two main settings:

- (Analytic) $\mathbb{k}$ is a complete normed field. (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$.)
  $\mathbb{k}\{x\} = \{\text{locally convergent power series}\}$. $\mathbb{k}\{x, y\} \ni F(x, y) = 0$, analytic equations.

- (Nash) $\mathbb{k}$ is any field, $\mathbb{k}\langle x \rangle = \{\text{power series that satisfy polynomial equations}\}$
  $a_d(x)f^d + \cdots + a_1(x)f + a_0(x) = 0$, with $a_i(x) \in \mathbb{k}[x]$ and $a_d(o) \neq 0$.
  E.g. (for $\text{char}(\mathbb{k}) = 0$) $f(x) = \sqrt[1]{1+q(x)}$, for $q(x) \in (x) \subset \mathbb{k}[x]$.
  $\mathbb{k}\langle x, y \rangle \ni F(x, y) = 0$, Nash equations.

The question. Given a system of (analytic/Nash) equations, $F(x, y) = 0$. Given a formal solution, $F(x, \hat{y}(x)) = 0$, $\hat{y}(x) \in \mathbb{k}[[x]]$. Want to approximate it by ordinary (analytic/Nash) solutions. Namely, for any $d \in \mathbb{N}$ we want:

$y^{(d)}(x) \in \mathbb{k}\{x\}, \mathbb{k}\langle x \rangle$, such that: $F(x, y^{(d)}(x)) = 0$ and $\hat{y}(x) - y^{(d)}(x) \in (x)^d$.

(Artin, 1968, 1969) This approximation exists. (Name: the Artin approximation)
Artin approximation: Every formal solution is approximated by ordinary solutions.

Example 1. Take $f : (k^n, o) \to (k^m, o)$ (analytic/Nash). Take a perturbation, $f + g$. Suppose $f + g \overset{R}{\sim} f$. I.e. $f(x) + g(x) = f(\hat{y}(x))$, a formal coordinate change. Then $f + g \overset{R}{\sim} f$. Moreover, $\forall d$ can ensure $y^{(d)}(x) - \hat{y}(x) \in (x)^d$.

Example 2. Given a system of analytic/Nash equations $F(x, y) = 0$. Suppose it has the unique (formal) solution, $y(x)$. Then $y(x)$ is analytic/Nash.

Remark. Given a system of equations, physicists solve it up to order 3 or 4. Engineers solve it up to order 1 or 2. And ... somehow it works.

Question. Maybe it is enough to resolve $F(x, y) = 0$ up to a high enough order? (Do not need to construct a formal solution $\hat{y}(x) \in k[[x]]$.)

Strong Artin approximation. (Pfister-Popescu) Given $F(x, y)$, there exists a function $\beta : \mathbb{N} \to \mathbb{N}$ satisfying: if $F(x, y^{(d)}(x)) \in (x)^{\beta d}$ then exists a (analytic/Nash) solution, $F(x, y(x)) = 0$, and moreover $y(x) - y^{(d)}(x) \in (x)^d$.

How to find/to bound this $\beta$? It is large and complicated. Fact: $\beta$ depends only on $n, m$ and degrees of $F$. (Not on the coefficients of $F$.)
The inverse question
Artin approximation addresses equations of implicit function type, \( F(x, y) = 0 \). There are many other functional equations.

The inverse Artin question (Grothendieck, 1961).
Given \( y(x) = y_1(x), \ldots, y_m(x) \in \mathbb{k}\{x\}, \mathbb{k}\langle x \rangle \). Suppose \( \hat{F}(y(x)) = 0 \).
Is this \( \hat{F} \) approximated by analytic/Nash relations among \( y(x) \)?

A counterexample (Osgood 1916, Gabrielov, 1971) There exists an analytic map \((\mathbb{C}^2, o) \to (\mathbb{C}^4, o), x \to y(x)\), whose components satisfy a formal relation, \( \hat{F}(y_1(x), \ldots, y_4(x)) = 0 \), but do not satisfy any (non-trivial) analytic relation.

Geometry: The image \( y(\mathbb{C}^2, o) \subset (\mathbb{C}^4, o) \) lies inside a formal hypersurface germ. But it does not lie inside any analytic hypersurface germ.

Facts: 1. The inverse AP holds for algebraic power series, \( y(x) \in \mathbb{k}\langle x \rangle \). For any \( \mathbb{k} \).

2. (Shiota, 1998) The inverse AP holds for \( \mathbb{R} \)-analytic maps of finite singularity type.

I.e. the map \((\mathbb{R}^n, o) \xrightarrow{y(x)} (\mathbb{R}^m, o)\) is contact-finite.
I.e. the subscheme \( V(y(x))_\mathbb{C} \subset (\mathbb{C}^n, o) \) is either one-point or an ICIS.
Local structure of morphisms, Maps\((X, Y)\)

Here \(X = V(l_X) \subseteq (\mathbb{k}^n, o)\) and \(Y = V(l_Y) \subseteq (\mathbb{k}^m, o)\), germs of schemes. Analytic \((R_X = \mathbb{k}\{x\}/l_X, R_Y = \mathbb{k}\{y\}/l_Y)\) or Nash \((R_X = \mathbb{k}\langle x \rangle/l_X, R_Y = \mathbb{k}\langle y \rangle/l_Y)\).

They are studied up to automorphisms (over \(\mathbb{k}\)), \(\text{Aut}_X \circ X\), i.e. \(\text{Aut}_k(R_X) \circ R_X\). And similarly \(\text{Aut}_Y \circ Y\).

**Example.** The classic case: \(l_X = 0, l_Y = 0\). Then Maps\(((\mathbb{k}^n, o),(\mathbb{k}^m, o))\).

\(\text{Aut}_X\) = local coordinate changes in the source. \(\text{Aut}_Y\) = ............... in the target.

These define the left-right equivalence of morphisms, \(f \rightsquigarrow \Phi_Y \circ f \circ \Phi_X^{-1}\).

**Question (the left-right Artin approximation, L\(\mathcal{R}\)-AP)** Suppose \(\tilde{f} \sim \tilde{f}\), i.e. \(\tilde{f} = \Phi_Y \circ f \circ \Phi_X^{-1}\). Is this approximated by \(\tilde{f} = \Phi_Y \circ f \circ \Phi_X^{-1}\)?

**Shiota.1998\(\mathcal{L} \rightleftharpoons \mathcal{R}\)** **K.2023** any \(\mathbb{k}\): L\(\mathcal{R}\)-AP holds for Nash maps, \(\mathbb{k}\langle x \rangle/l_X, \mathbb{k}\langle y \rangle/l_Y\).

**Shiota.1998.** L\(\mathcal{R}\)-AP holds for analytic Maps\(((\mathbb{R}^n, o),(\mathbb{R}^m, o))\) of finite singularity type. i.e. \(V(f)_{\mathbb{C}} \subset (\mathbb{C}^n, o)\) is either a point or an ICIS.

**K.2023.** L\(\mathcal{R}\)-AP holds for analytic Maps\((X, Y)\) of weakly-finite singularity type.
Maps of weakly-finite singularity type  

Take $f : (k^n, o) \to (k^m, o)$, $n \geq m$. $f' \in \text{Mat}_{m \times n}$. Critical locus $\text{Crit} := V(l_m[f']) \subseteq (k^n, o)$.

Def. Let $X \xrightarrow{f} Y$ (dominant). The critical module $\mathcal{C} := \text{Der}(f^* T_Y, T_X)/\text{Der}_X f$. The critical locus $\text{Crit}_X f := \text{Supp}[\mathcal{C}] \subseteq X$.

Ex. $X \xrightarrow{f} (k^m, o)$. Then $\mathcal{C} = \mathbb{R}_X^+ / \text{Der}_X f$ and $\text{Crit}_X f = V(l_m[\text{Der}_X f])$.

\[ X \xrightarrow{f} Y \quad \text{Def. } f \text{ is of finite singularity type if } \text{Crit}_X \xrightarrow{f} \Delta_Y \text{ is finite.} \]

TFAE for $X \xrightarrow{f} (k^m, o)$:
- $f$ is of finite sing. type.
- $f$ is contact finite.
- $(\text{for } \text{char}(k) > 0)$ $V(f) \subseteq X$ is of dim=0 or an ICIS.

Suppose $f_1 : \text{Crit}_X \to \Delta_Y$ is not finite. Maybe $f_1 : \text{Crit}_X \to \Delta_Y$ is of finite sing. type? (I.e. $f_1 : \text{Crit}_{\text{Crit}_X} \to \Delta_{\Delta_Y}$ is finite.) If not, then maybe $f_1 : \text{Crit}_{\text{Crit}_X} \to \Delta_{\Delta_Y}$ is of finite sing. type? (I.e. $f_1 : \text{Crit}_{\text{Crit}_{\text{Crit}_X}} \to \Delta_{\Delta_{\Delta_Y}}$ is finite.) And so on...

Def. (Roughly) $f : X \to Y$ is of weakly finite singularity type if

- $\text{Crit}_{\text{Crit}_{\text{\ldots}}} := \text{Crit}_r \xrightarrow{f_1} \Delta_r := \Delta_{\Delta_{\ldots}}$ is finite for some $r$.
- $(\text{for } \text{char}(k) > 0)$ Certain logarithmic derivations of $X/\text{Crit}_X, X/\text{Crit}_X/\text{Crit}_{\text{Crit}_X}/\ldots$ are integrable.

Artin approximation on quivers

We spoke about several approximation problems for morphisms of scheme-germs

\[ Y \leftarrow X \odot \mathcal{R} \quad \mathcal{L} \odot Y \leftarrow X \quad \mathcal{L} \odot Y \leftarrow X \odot \mathcal{R} \]

\[ \tilde{f} = f \circ \Phi_X \quad \tilde{f} = \Phi_Y \circ f \quad \tilde{f} = \Phi_Y \circ f \circ \Phi_X \]

Artin approximation  Inverse Artin approximation  Left-Right Artin approximation

But \( X \) could be a multi-germ.
These all are some simple graphs.
Each graph encodes an approximation problem.

\[ \mathcal{R} \odot X_1 \quad \cdots \quad \rightarrow Y \odot \mathcal{L}. \]
\[ \mathcal{R} \odot X_k \]

**Def.** A quiver of map-germs: \((\Gamma, \{X_v\}_v, \{f_{wv}\}_{wv})\).

**Def.** A morphism of quivers: \(f_{wv} \circ \Phi_v = \Phi_w \circ \tilde{f}_{wv}\).

E.g. \( (\mathbb{k}^n, o) \xrightarrow{f + g} (\mathbb{k}^1, o) \)

\[ \Phi_X \downarrow \quad \text{Id}_{(\mathbb{k}^1, o)} \]

\( \mathcal{R} : \)

(\mathbb{k}^n, o) \xrightarrow{f} (\mathbb{k}^1, o)

**Def.** (\(\Gamma\)-AP) The Artin approximation holds for a quiver \((\Gamma, \{X_v\}_v, \{f_{wv}\}_{wv})\) if any formal morphism, \(\{\hat{\Phi}_v\}_v\), is approximated by analytic/Nash morphisms, \(\{\Phi_v\}_v\).

**K.2023.** \(\Gamma\)-AP holds in the Nash case for directed rooted trees.
Thanks for your attention!