Artin approximation.

The ordinary, the inverse, the left-right, and on quivers.

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Abstract

1. Artin approximation is useful.
2. What is this "Weakly-finite singularity type"?
3. Continue the sequence:

$\underbrace{(\mathbb{k}^m,o)\leftarrow(\mathbb{k}^n,o)\circlearrowleft\mathscr{R}}_{Artin\ approximation},\quad \underbrace{\mathscr{L}\circlearrowright(\mathbb{k}^m,o)\leftarrow(\mathbb{k}^n,o)}_{Inverse\ Artin\ approximation},\quad \underbrace{\mathscr{L}\circlearrowright(\mathbb{k}^m,o)\leftarrow(\mathbb{k}^n,o)\circlearrowleft\mathscr{R}}_{Left-Right\ Artin\ approximation},\ \dots??$

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Prologue (we often have to resolve equations)

Let $\mathbb{k}$ be $\mathbb{R}$ or $\mathbb{C}$. Consider equations $F(x,y)=0$. Here:

$$x=(x_1,\dots,x_n),\qquad y=(y_1,\dots,y_m),\qquad F=(f_1,\dots,f_c),\qquad \text{with } f_i\in\mathbb{k}\{x,y\}.$$

Want an analytic solution, $F(x,y(x))=0$. Usually no chance for explicit solutions.

An approach: resolve order-by-order. $F(x,y^{(d)}(x))\in(x)^d$, for each $d\in\mathbb{N}$.
Then take $\hat{y}(x):=\lim\limits_{d\to\infty}y^{(d)}(x)\in\mathbb{k}[[x]]$. (Does this limit exist?)

- Is this an analytic solution? ($\hat{y}(x)\in\mathbb{k}\{x\}$?)
- Suppose $F(x,y)\in\mathbb{k}[x,y]$. Is this a Nash solution? ($\hat{y}(x)\in\mathbb{k}\langle x\rangle$?)

Example. $(\mathbb{k}^n,o)\xrightarrow{f}(\mathbb{k}^1,o)$ Take a perturbation, $f+h$. Can this be undone by a coordinate change? Namely, $f(x)+h(x)=f(y(x))$.

More generally, this question for $X\xrightarrow{f}Y$. (With various equivalence relations.)
For deformation theory. For vector fields/foliations. For dynamical systems. …

Ruling out the trivial case. $F(x,y)=0$. Suppose $\partial_y F|_{(o,o)}$ is non-degenerate.
Then the (analytic/Nash) solution exists. (Implicit Function Theorem.)
Below $\partial_y F|_{(o,o)}$ is always degenerate.

Trying to resolve the equation $F(x,y)=o$. (Always assume $F(o,o)=o$.)

Geometry. $(\mathbb{k}^n_x,o)\times(\mathbb{k}^m_y,o)\supset V(F(x,y)) \overset{analytic}{\to} (\mathbb{k}^n_x,o)$. Does there exist an analytic section?

singular ← a formal section $\hat{y}(x)$

Can this formal section be approximated by analytic sections?

Two main settings:

- (Analytic) $\mathbb{k}$ is a complete normed field. (e.g. $\mathbb{R},\mathbb{C},\mathbb{Q}_p$.)
  $\mathbb{k}\{x\}=$(locally convergent power series). $\mathbb{k}\{x,y\}\ni F(x,y)=0$, analytic equations.
- (Nash) $\mathbb{k}$ is any field, $\mathbb{k}\langle x\rangle=\{$power series that satisfy polynomial equations$\}$
  $a_d(x)f^d+\cdots+a_1(x)f+a_0(x)=0$, with $a_i(x)\in\mathbb{k}[x]$ and $a_d(o)\neq 0$.
  E.g. (for $char(\mathbb{k})=0$) $f(x)=\sqrt[d]{1+q(x)}$, for $q(x)\in(x)\subset\mathbb{k}[x]$.
  $\mathbb{k}\langle x,y\rangle\ni F(x,y)=0$, Nash equations.

The question. Given a system of (analytic/Nash) equations, $F(x,y)=0$. Given a formal solution, $F(x,\hat{y}(x))=0$, $\hat{y}(x)\in\mathbb{k}[[x]]$. Want to approximate it by ordinary (analytic/Nash) solutions. Namely, for any $d\in\mathbb{N}$ we want:

$$y^{(d)}(x)\in\mathbb{k}\{x\},\mathbb{k}\langle x\rangle,\ \text{such that:}\quad F(x,y^{(d)}(x))=0\quad\text{and}\ \hat{y}(x)-y^{(d)}(x)\in(x)^d.$$

(Artin, 1968, 1969) *This approximation exists.* (Name: the Artin approximation)

Artin approximation: *Every formal solution is approximated by ordinary solutions.*

Example 1. Take $f:(\mathbb{k}^n,o)\to(\mathbb{k}^m,o)$ (analytic/Nash). Take a perturbation, $f+g$.
Suppose $f+g \overset{\widehat{\mathscr{R}}}{\approx} f$. I.e. $f(x)+g(x)=f(\hat{y}(x))$, a formal coordinate change.
Then $f+g \overset{\mathscr{R}}{\approx} f$. Moreover, $\forall d$ can ensure $y^{(d)}(x)-\hat{y}(x)\in(x)^d$.

Example 2. Given a system of analytic/Nash equations $F(x,y)=0$.
Suppose it has the unique (formal) solution, $y(x)$. Then $y(x)$ is analytic/Nash.

Remark. Given a system of equations, physicists solve it up to order 3 or 4.
Engineers solve it up to order 1 or 2. And ... somehow it works.

Question. Maybe it is enough to resolve $F(x,y)=0$ up to a high enough order?
(Do not need to construct a formal solution $\hat{y}(x)\in\mathbb{k}[[x]]$.)

Strong Artin approximation. (Pfister-Popescu) Given $F(x,y)$, there exists a function $\beta:\mathbb{N}\to\mathbb{N}$ satisfying: if $F(x,y^{(d)}(x))\in(x)^{\beta_d}$ then exists a (analytic/Nash) solution, $F(x,y(x))=0$, and moreover $y(x)-y^{(d)}(x)\in(x)^d$.

How to find/to bound this β? It is large and complicated.
Fact: β depends only on n,m and degrees of F. (Not on the coefficients of F.)

Artin approximation addresses equations of implicit function type, $F(x,y)=0$.
There are many other functional equations.

The inverse Artin question (Grothendieck, 1961).
Given $y(x)=y_1(x),\dots,y_m(x)\in\mathbb{k}\{x\},\mathbb{k}\langle x\rangle$. Suppose $\hat{F}(y(x))=0$.
Is this $\hat{F}$ approximated by analytic/Nash relations among $y(x)$?

A counterexample (Osgood 1916, Gabrielov, 1971) There exists an analytic map $(\mathbb{C}^2,o)\to(\mathbb{C}^4,o)$, $x\to y(x)$, whose components satisfy a formal relation, $\hat{F}(y_1(x),\dots,y_4(x))=0$, but do not satisfy any (non-trivial) analytic relation.

Geometry: The image $y(\mathbb{C}^2,o)\subset(\mathbb{C}^4,o)$ lies inside a formal hypersurface germ. But it does not lie inside any analytic hypersurface germ.

Facts: 1. The inverse AP holds for algebraic power series, $y(x)\in\mathbb{k}\langle x\rangle$. For any $\mathbb{k}$.

2.(Shiota,1998)The inverse AP holds for $\mathbb{R}$-analytic maps of finite singularity type.

I.e. the map $(\mathbb{R}^n,o)\stackrel{y(x)}{\to}(\mathbb{R}^m,o)$ is contact-finite.
I.e. the subscheme $V(y(x))_{\mathbb{C}}\subset(\mathbb{C}^n,o)$ is either one-point or an ICIS.

Local structure of morphisms, $Maps(X,Y)$

Here $X = V(I_X) \subseteq (\mathbb{k}^n, o)$ and $Y = V(I_Y) \subseteq (\mathbb{k}^m, o)$, germs of schemes.
Analytic ($R_X = \mathbb{k}\{x\}/_{I_X}$, $R_Y = \mathbb{k}\{y\}/_{I_Y}$) or Nash ($R_X = \mathbb{k}\langle x\rangle/_{I_X}$, $R_Y = \mathbb{k}\langle y\rangle/_{I_Y}$).

They are studied up to automorphisms (over $\mathbb{k}$), $Aut_X \circlearrowright X$, i.e. $Aut_{\mathbb{k}}(R_X) \circlearrowright R_X$.
And similarly $Aut_Y \circlearrowright Y$.

Example. The classic case: $I_X = 0$, $I_Y = 0$. Then $\mathrm{Maps}\big((\mathbb{k}^n, o),(\mathbb{k}^m, o)\big)$.
Aut_X =local coordinate changes in the source.
Aut_Y =............ in the target.
These define the left-right equivalence
of morphisms, $f \rightsquigarrow \Phi_Y \circ f \circ \Phi_X^{-1}$.

$$\begin{array}{ccc} X & \stackrel{f}{\to} & Y \\ \circlearrowright & & \circlearrowright \\ \mathscr{R}:=Aut_X & & \mathscr{L}:=Aut_Y \end{array} \qquad \begin{array}{ccc} R_X & \stackrel{f^\sharp}{\to} & R_Y \\ \Phi_X^\sharp \downarrow & & \downarrow \Phi_Y^\sharp \\ R_X & \stackrel{\tilde{f}^\sharp}{\to} & R_Y \end{array}$$

Question (the left-right Artin approximation, $\mathscr{LR}$-AP) Suppose $\tilde{f} \stackrel{\widehat{\mathscr{LR}}}{\sim} f$,
i.e. $\tilde{f} = \hat{\Phi}_Y \circ f \circ \hat{\Phi}_X^{-1}$. Is this approximated by $\tilde{f} = \Phi_Y \circ f \circ \Phi_X^{-1}$?

Shiota.1998$_{\mathbb{k}=\mathbb{R}}$ **K.2023**$_{any\ \mathbb{k}}$: $\mathscr{LR}$-AP holds for Nash maps, $\mathbb{k}\langle x\rangle/_{I_X}, \mathbb{k}\langle y\rangle/_{I_Y}$.

Shiota.1998. $\mathscr{LR}$-AP holds for analytic $Maps((\mathbb{R}^n,o),(\mathbb{R}^m,o))$ of finite singularity type. i.e. $V(f)_{\mathbb{C}} \subset (\mathbb{C}^n, o)$ is either a point or an ICIS.

K.2023. $\mathscr{LR}$-AP holds for analytic $Maps(X,Y)$ of weakly-finite singularity type.

(Analytic) Maps of weakly-finite singularity type Take $f\colon(\mathbb{k}^n,o)\to(\mathbb{k}^m,o)$, $n\geq m$. $f'\in Mat_{m\times n}$. Critical locus $Crit:=V(I_m[f'])\subseteq(\mathbb{k}^n,o)$.

Def. Let $X\xrightarrow{f}Y$ (dominant). The critical module $\mathcal{C}:={}^{Der(f^*T_Y,T_X)}/_{Der_X f}$.
The critical locus $Crit_X f:=Supp[\mathcal{C}]\subseteq X$. (set-theoretically)

Ex. $X\xrightarrow{f}(\mathbb{k}^m,o)$. Then $\mathcal{C}={}^{R_X^{\oplus m}}/_{Der_X f}$ and $Crit_X f=V(I_m[Der_X f])$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup & & \cup \\ Crit_X & \xrightarrow{f_|} & \Delta_Y \end{array}$$
discriminant

Def. f is of finite singularity type if $Crit_X\xrightarrow{f_|}\Delta_Y$ is finite.
TFAE for $X\xrightarrow{f}(\mathbb{k}^m,o)$:
- f is of finite sing. type.
- f is contact finite.
- (for $\mathbb{k}=\bar{\mathbb{k}}$) $V(f)\subset X$ is of $dim=0$ or an ICIS.

Suppose $f_|\colon Crit_X\to\Delta_Y$ is not finite. Maybe $f_|\colon Crit_X\to\Delta_Y$ is of finite sing. type? (I.e. $f_|\colon Crit_{Crit_X}\to\Delta_{\Delta_Y}$ is finite.) If not, then maybe $f_|\colon Crit_{Crit_X}\to\Delta_{\Delta_Y}$ is of finite sing. type? (I.e. $f_|\colon Cirt_{Crit_{Crit_X}}\to\Delta_{\Delta_{\Delta_Y}}$ is finite.) And so on...

Def. (Roughly) $f:X\to Y$ is of weakly finite singularity type if

1. $Cirt_{Crit_{\dots}}=:Crit_r\xrightarrow{f_|}\Delta_r:=\Delta_{\Delta_{\dots}}$ is finite for some r.
2. (for $char(\mathbb{k})>0$) Certain logarithmic derivations of $X/Crit_X, X/Crit_X/Crit_{Crit_X}/\dots$ are integrable.

K.2023. *$\mathscr{L}\mathscr{R}$-AP holds for analytic $Maps(X,Y)$ of weakly-finite singularity type.*

Artin approximation on quivers

We spoke about several approximation problems for morphisms of scheme-germs

$Y \leftarrow X \circlearrowleft \mathscr{R}$	$\mathscr{L} \circlearrowright Y \leftarrow X$	$\mathscr{L} \circlearrowright Y \leftarrow X \circlearrowleft \mathscr{R}$
$\tilde{f} = f \circ \Phi_X$	$\tilde{f} = \Phi_Y \circ f$	$\tilde{f} = \Phi_Y \circ f \circ \Phi_X$
Artin approximation	Inverse Artin approximation	Left-Right Artin approximation

But X could be a multi-germ.
These all are some simple graphs.
Each graph encodes an approximation problem.

$$\begin{array}{l} \mathscr{R} \circlearrowright X_1 \searrow \\ \quad \cdots \quad \rightarrow Y \circlearrowleft \mathscr{L}. \\ \mathscr{R} \circlearrowright X_k \nearrow \end{array}$$

Def. A quiver of map-germs: $(\Gamma, \{X_v\}_v, \{f_{wv}\}_{wv})$.

$$\begin{array}{c} \searrow \\ \rightarrow \\ \nearrow \end{array} X_v \xrightarrow{f_{wv}} X_w \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array}$$

Def. A morphism of quivers:
$f_{wv} \circ \Phi_v = \Phi_w \circ \tilde{f}_{wv}$

$$\begin{array}{ccccc} \rightrightarrows & \tilde{X}_v & \xrightarrow{\tilde{f}_{wv}} & \tilde{X}_w & \rightrightarrows \\ & \Phi_v \downarrow & & \downarrow \Phi_w & \\ \rightrightarrows & X_v & \xrightarrow{f_{wv}} & X_w & \rightrightarrows \end{array}$$

E.g. $\mathscr{R}$:

$$\begin{array}{ccc} (\mathbb{k}^n, o) & \xrightarrow{f+g} & (\mathbb{k}^1, o) \\ \Phi_X \downarrow & & Id_{(\mathbb{k}^1, o)} \| \\ (\mathbb{k}^n, o) & \xrightarrow{f} & (\mathbb{k}^1, o) \end{array}$$

Def. (Γ-AP) The Artin approximation holds for a quiver $(\Gamma, \{X_v\}_v, \{f_{wv}\}_{wv})$ if any formal morphism, $\{\hat{\Phi}_v\}_v$, is approximated by analytic/Nash morphisms, $\{\Phi_v\}_v$.

K.2023. Γ-AP holds in the Nash case for directed rooted trees.

Thanks for your attention!