# Higher order Toda brackets

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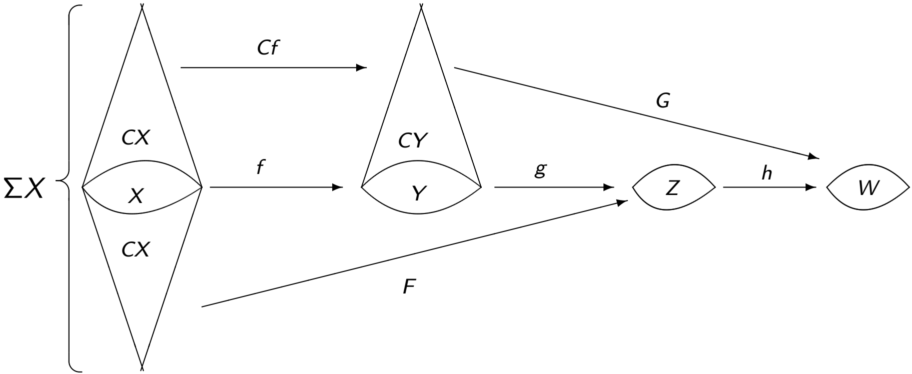

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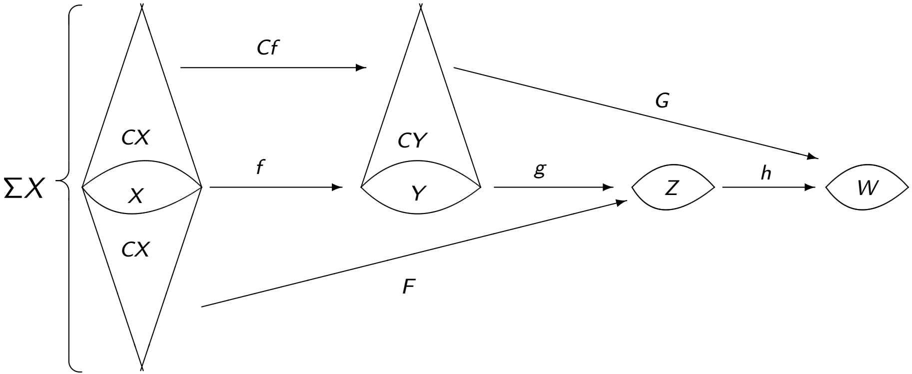


The nullhomotopies $h \circ F$ and $G \circ Cf$ induce a map

$$\langle f, g, h, (F, G) \rangle : \Sigma X \to W$$

# Some background

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• In cubically enriched categories (i.e., $\infty$-categories).

• In pointed model categories, where they serve as obstructions to rectification.

# Toda bracket in terms of homotopy cofiber

If we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $F : CX \to Z$ a nullhomotopy for $g \circ f$, then we have the following commutative square:

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The composition equal to $\langle f, g, h, (F, G) \rangle : \Sigma X \to W$.

# Toda bracket and rectification of linear diagrams

**Definition:** Given a diagram

$X_* = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots X_{n-1} \xrightarrow{f_{n-1}} X_n)$ where $f_{j+1} \circ f_j \sim *$,

$X_*$ called *rectifiable* if we have:

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where $g_{j+1} \circ g_j = *$.

**Theorem:** For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, where $g \circ f \sim *$ and $h \circ g \sim *$, If there are null-homotopies $F : CX \to Z$, and $G : CY \to W$, such that $\langle f, g, h, (F, G) \rangle$ is nullhomotopic, then the diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ is rectifiable.

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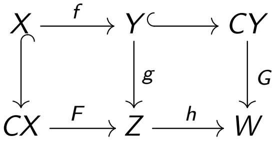
- **<u>Recursive definition</u>** Using the definition of $\alpha$ and $\beta$, which allows us to generalize the Rectification Theorem.
- Show that these two approaches yield equivalent notions of higher Toda brackets.

In order to do so, we give a third **diagrammatic description** of Toda brackets, more specifically, we can translate the data of the primary toda bracket into:




which we think of as a sequence of two horizontal maps of vertical 1-cubes.

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• Given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$ with nullhomotopies $F : CX \to Z$, $G : CY \to W$, $H : CZ \to V$, and second-order nullhomotopies $F^{(2)} : C^2X \to W$, $G^{(2)} : C^2Y \to V$ with

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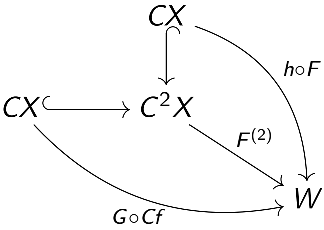

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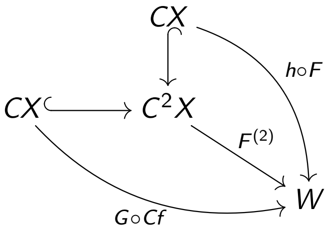


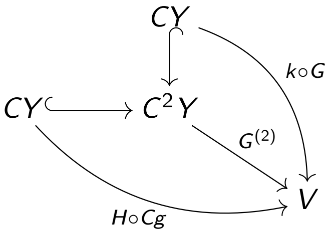

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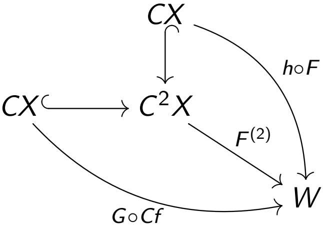


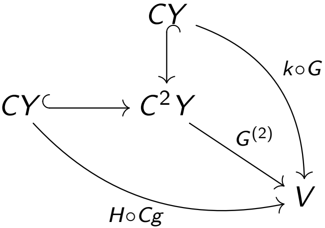


yielding $k \circ F^{(2)}$ and $G^{(2)} \circ C^2 f$ from $C^2X$ to $V$

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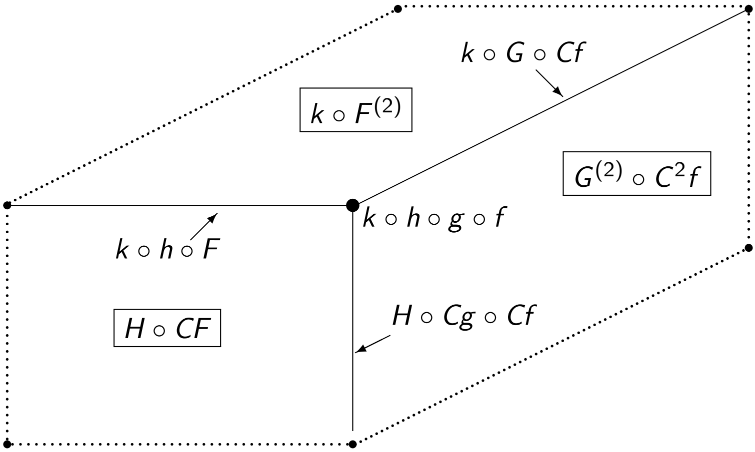

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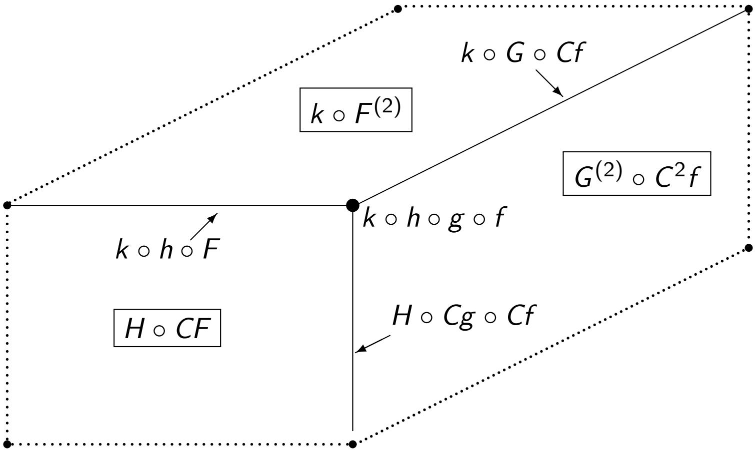


we obtain a map $\langle f, g, h, k, (F, G, H, F^{(2)}, G^{(2)}) \rangle : L^2 X \to V$ with $L^2 X \simeq \Sigma^2 X$.

# Cubical Toda system

**Definition:** a sequence of maps

$$X_* : \; X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

with a nullhomotopies $F_* : \; F, G, H, F^{(2)}, G^{(2)}$ as before called (second order) *cubical Toda system*, and denoted by $(X_*, F_*)$.

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We will denote the Toda bracket of the system $(X_*, F_*)$ by $T(X_*, F_*) := \langle f, g, h, (F, G, H, F^{(2)}, G^{(2)})\rangle$.

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Later we will define an equivalence relation between such Toda systems where two equivalent systems have equivalent Toda brackets.

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CX & \overset{F}{\hookrightarrow} & Z \\
\downarrow & & \downarrow \\
\widetilde{\Sigma} X & \overset{\alpha_1}{\hookrightarrow} & \mathrm{cof}(g)
\end{array}
\qquad
\begin{array}{ccccc}
Y & \overset{g}{\hookrightarrow} & Z & \longrightarrow & \mathrm{cof}(g) \\
\downarrow & & \downarrow h & & \downarrow \beta_2 \\
CY & \overset{G}{\hookrightarrow} & W & = & W \\
\downarrow & & \downarrow & & \\
\widetilde{\Sigma} Y & \overset{\alpha_2}{\hookrightarrow} & \mathrm{cof}(h) & &
\end{array}
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\begin{array}{ccccc}
Z & \overset{h}{\hookrightarrow} & W & \longrightarrow & \mathrm{cof}(h) \\
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$$\begin{array}{ccc} \widetilde{\Sigma}X & \xrightarrow{\alpha_1} & \mathrm{cof}(g) \\ \downarrow & & \downarrow \beta_2 \\ C\widetilde{\Sigma}X & \xrightarrow{\widetilde{F}^{(2)}} & W \end{array} \qquad \begin{array}{ccc} \widetilde{\Sigma}Y & \xrightarrow{\alpha_2} & \mathrm{cof}(h) \\ \downarrow & & \downarrow \beta_3 \\ C\widetilde{\Sigma}Y & \xrightarrow{\widetilde{G}^{(2)}} & V \end{array}$$

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$$\widetilde{\Sigma}^2 X \xrightarrow{\alpha_1^{(2)}} \mathrm{cof}(\beta_2) = \mathrm{cof}(\alpha_2) \xrightarrow{\beta_2^{(2)}} \mathrm{cof}(\widetilde{G}^{(2)})$$

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with a nullhomotopies $\widetilde{F}_* : \ F, G, H, \widetilde{F}^{(2)}, \widetilde{G}^{(2)}$ as before called (second order) *recursive Toda system*, and denoted by $(X_*, \widetilde{F}_*)$.

We will denote the corresponding Toda bracket of the system $(X_*, \widetilde{F}_*)$ by $\widetilde{T}(X_*, \widetilde{F}_*) : \widetilde{\Sigma}^2 X \to \mathrm{cof}(\widetilde{G}^{(2)})$

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Later we will define an equivalence relation between such a Toda systems where two equivalent systems have equivalent Toda brackets.

**Theorem:** If we have a recursive system $(X_*, \widetilde{F}_*)$ with a nullhomotopic $\widetilde{T}(X_*, \widetilde{F}_*)$ then $X_*$ is rectifiable .

# The obvious difference between the two definitions

- **The construction**

In the cubical definition we have all the nullhomotopies, we compose and glue to get the Toda bracket.
In the recursive definition we first construct the ordinary Toda brackets, then choose second order nullhomotopies for them to continue.

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Second order nullhomotopies in cubical Toda system:
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# Diagrammatic description for a cubical Toda system

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

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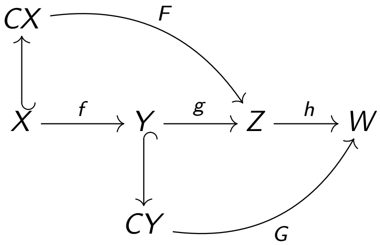

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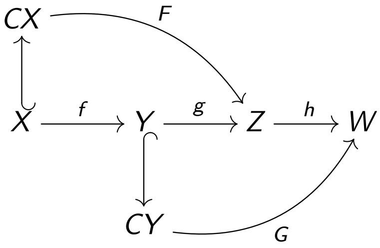


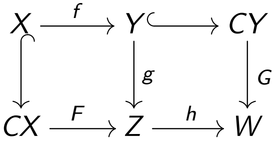

# Diagrammatic description for a cubical Toda system(cont.)

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

$F, G, H, \quad F^{(2)}, G^{(2)}.$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \hookrightarrow & CY \\ \big\downarrow & & \big\downarrow g & & \big\downarrow G \\ CX & \xrightarrow{F} & Z & \xrightarrow{h} & W \end{array} \qquad\qquad \begin{array}{ccccc} X & \xrightarrow{g} & Z & \hookrightarrow & CZ \\ \big\downarrow & & \big\downarrow h & & \big\downarrow H \\ CY & \xrightarrow{G} & W & \xrightarrow{k} & V \end{array}$$

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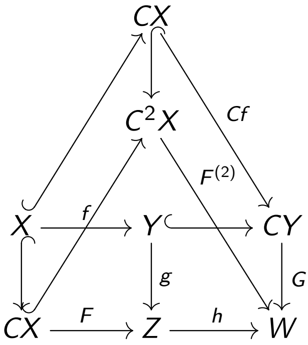


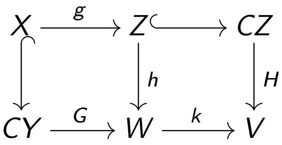

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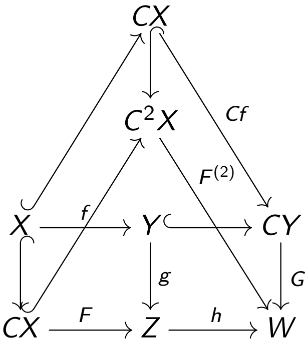


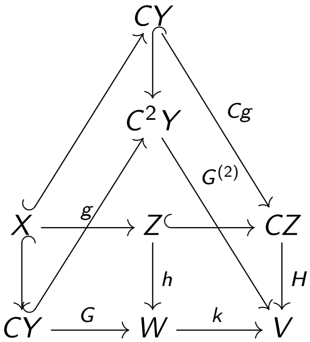

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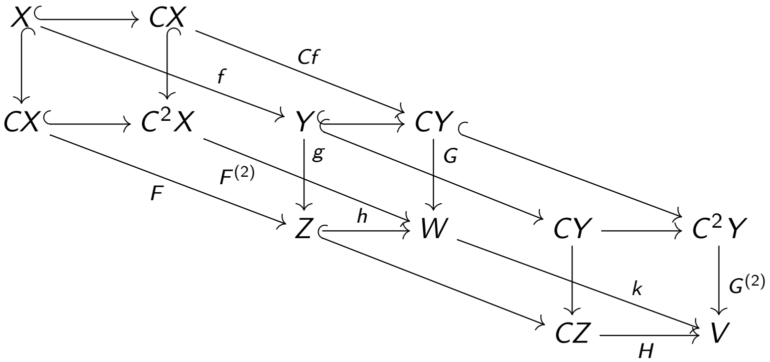

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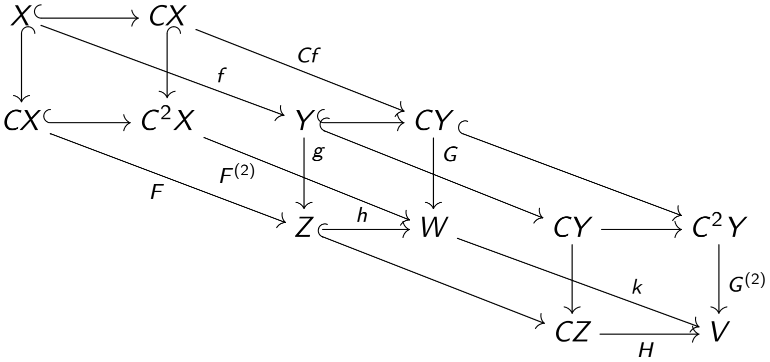
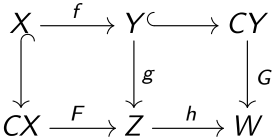
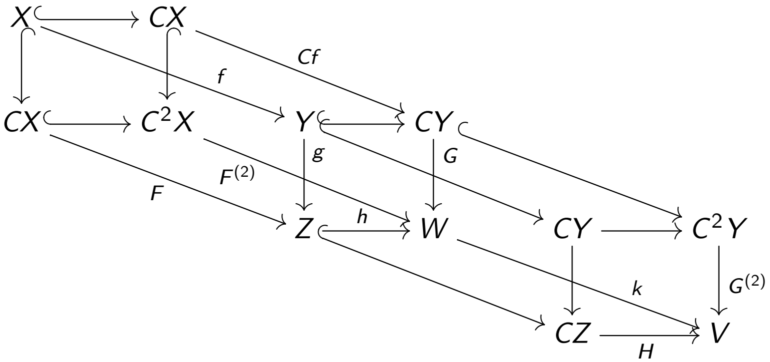


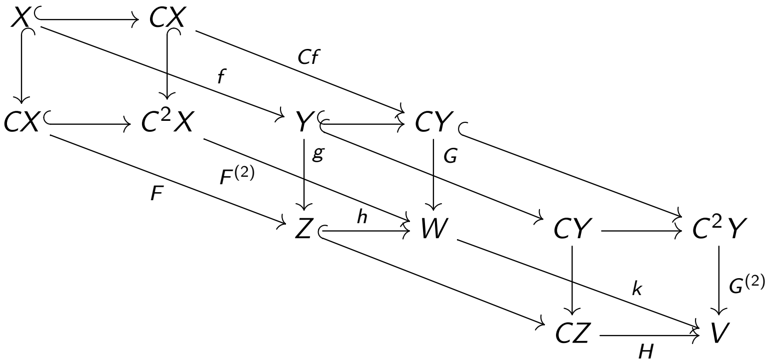



# Diagrammatic description for a cubical Toda system (II)




The data of the cubical Toda system $(X_*, F_*)$ is encoded in this commutative diagram.

# Diagrammatic description for a cubical Toda system (II)




The data of the cubical Toda system $(X_*, F_*)$ is encoded in this commutative diagram.
We will use the following notations:

$$\mathbb{C}^{(2)}X \xrightarrow{\mathfrak{A}^{(2)}(X_*,F_*)} \mathbb{M}^{(2)}(X_*, F_*) \xrightarrow{\mathfrak{B}^{(2)}(X_*,F_*)} \mathbb{V}^{(2)}(X_*, F_*)$$

# Equivalence relation between cubical Toda systems

**Definition** Two cubical Toda systems $(X_*, F_*)$ and $(X'_*, G_*)$ are *equivalent* (written $(X_*, F_*) \approx (X'_*, G_*)$) if there is a zig-zag of weak equivalences between the sequences

$$\mathbb{C}^{(2)}X \xrightarrow{\mathfrak{A}^{(2)}(X_*,F_*)} \mathbb{M}^{(2)}(X_*, F_*) \xrightarrow{\mathfrak{B}^{(n)}(X_*,F_*)} \mathbb{V}^{(2)}(X_*, F_*)$$

and

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**Proposition:** If $(X_*, F_*) \approx (X'_*, G_*)$, then

$$T(X_*, F_*) \approx T(X'_*, G_*)$$

# How To get the cubical Toda bracket from the diagram?

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Recall that applying the homotopy cofiber functor to

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yields

$$\Sigma X \xrightarrow{\alpha} \mathrm{hcof}(g) \xrightarrow{\beta} W$$

and the corresponding Toda bracket is $\beta \circ \alpha$.

# Homotopy cofiber of a square

The *cofiber* of the square

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle f} & & {\scriptstyle g}\downarrow \\ Y & \xrightarrow[k]{} & W \end{array}$$

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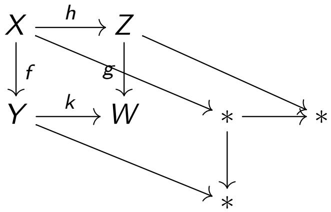

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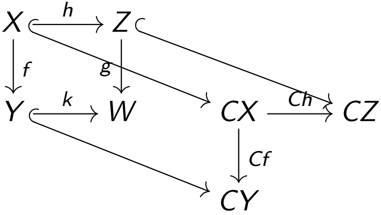

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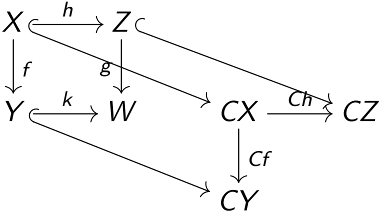
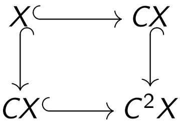


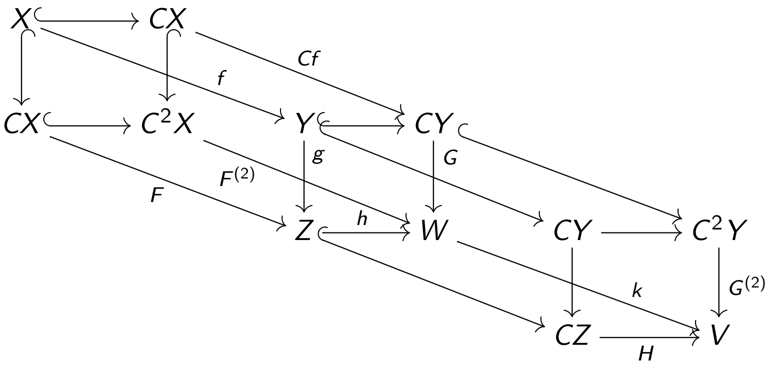

**Example:** For the following square:




the cofiber is $\widetilde{\Sigma}^2 X$, and the homotopy cofiber is $L^2 X$.

# Cubical definition in terms of homotopy cofiber

Applying homotopy cofiber of squares to

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$$L^2X \longrightarrow \mathrm{hcof}(\mathbb{M}^{(2)}(X_*, F_*)) \longrightarrow V$$

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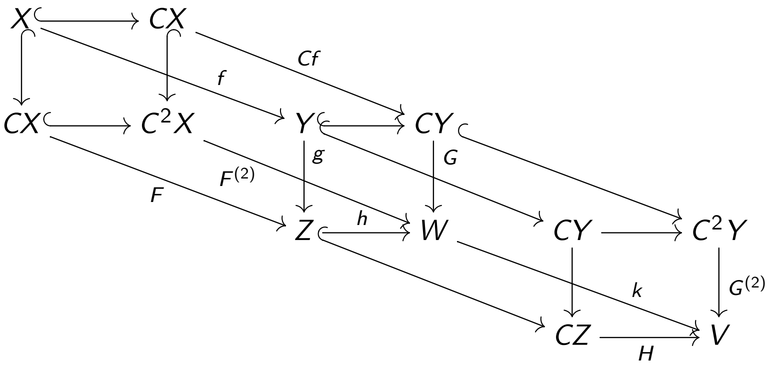


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The composite equals $T(X_*, F_*) : L^2X \to V$.

# A diagrammatic description for recursive Toda systems

Recall that a recursive Toda system is a sequence of maps

$$X_* : \; X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

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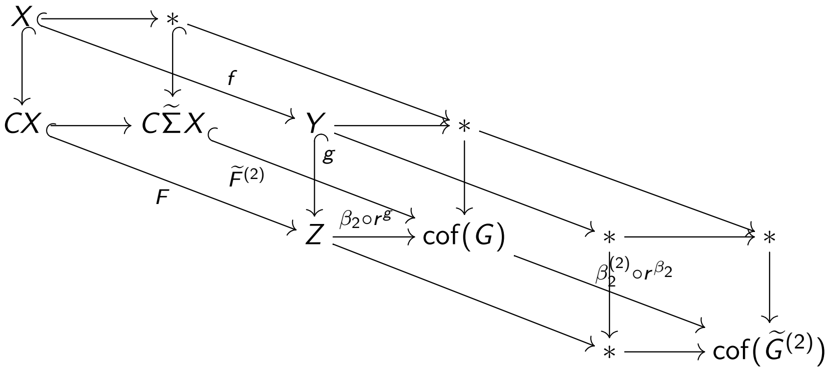
One can encodes this data in the following commutative diagram:



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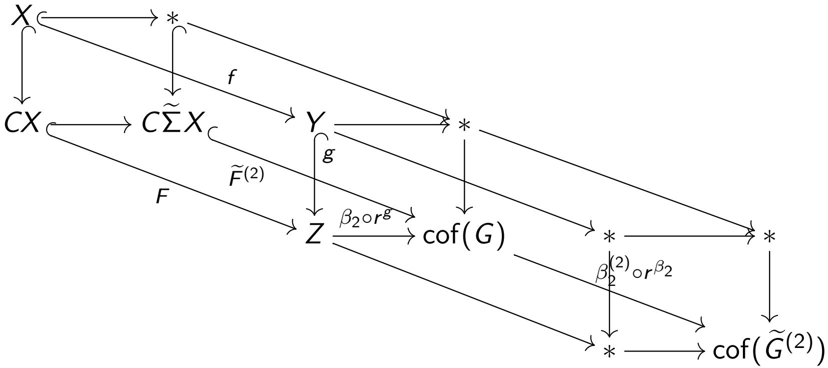


yields

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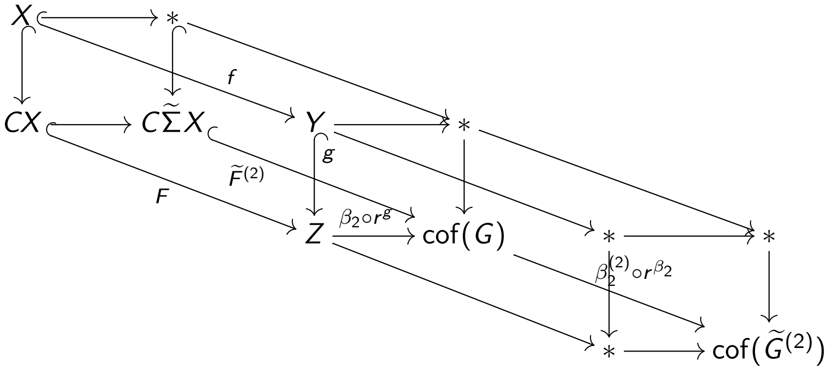


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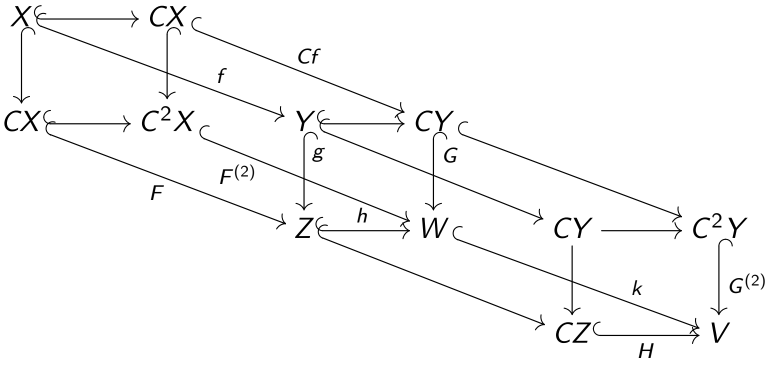
# From cubical system to recursive system



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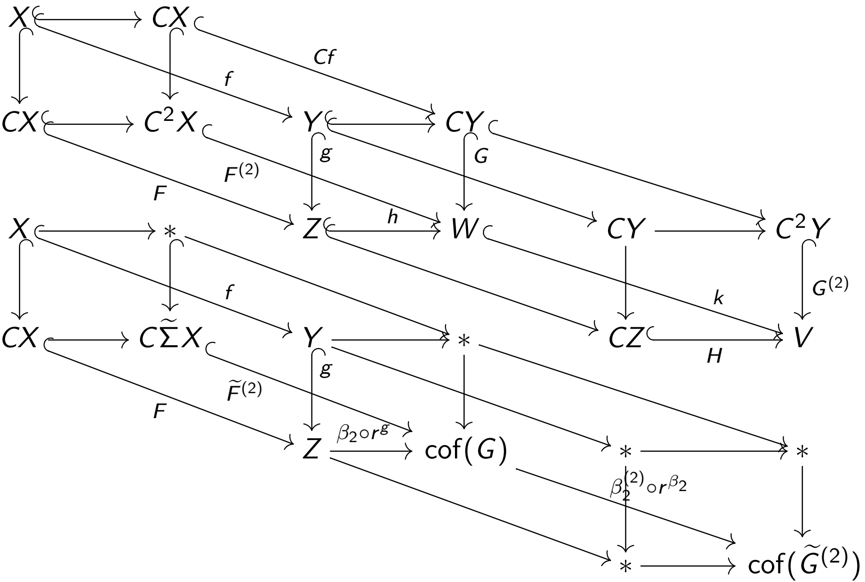

**Theorem 1:** If we have a cubical Toda system $(X_*, F_*)$, then we get a recursive Toda system $(X_*, \widetilde{F}_*) = R(X_*, F_*)$ where:

$$\begin{array}{ccc} L^2X & \xrightarrow{T(X_*,F_*)} & V \\ \simeq\downarrow & & \downarrow\simeq \\ \widetilde{\Sigma}^2X & \xrightarrow{\widetilde{T}(X_*,\widetilde{F}_*)} & \mathrm{cof}(\widetilde{G}^2) \end{array}$$

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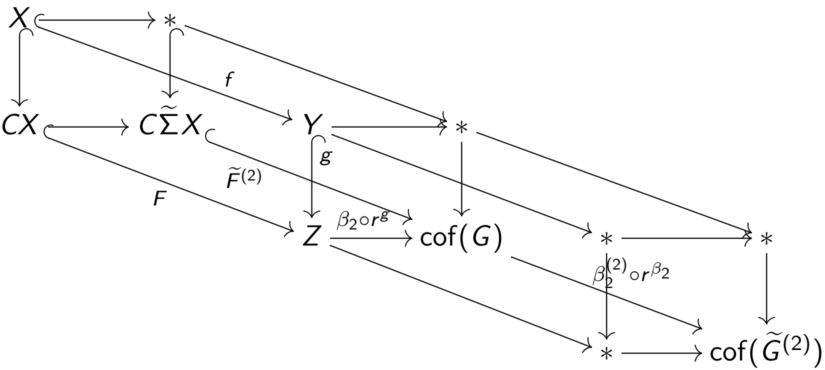
Note that $R$ preserve the equivalence relation between Toda systems.

# From recursive system to cubical system



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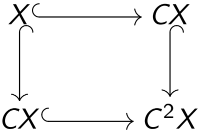


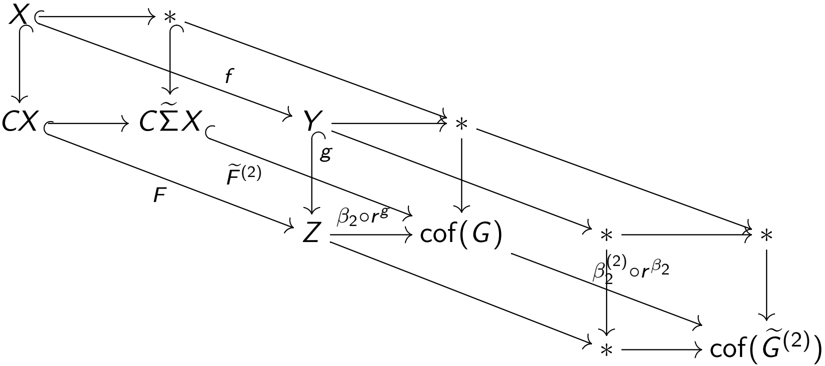

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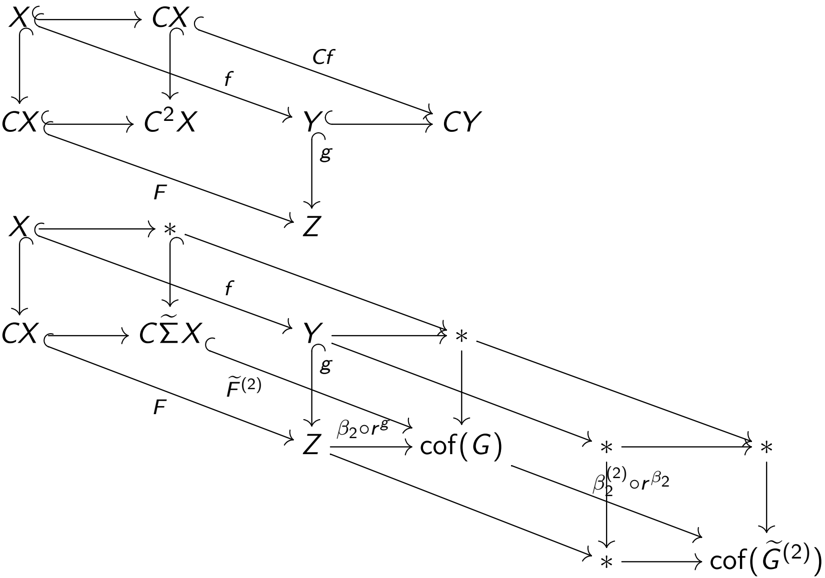

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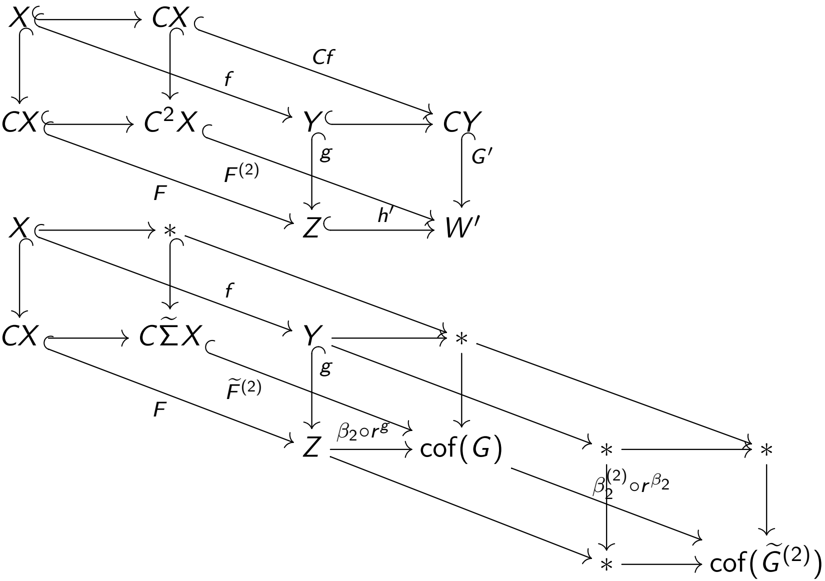

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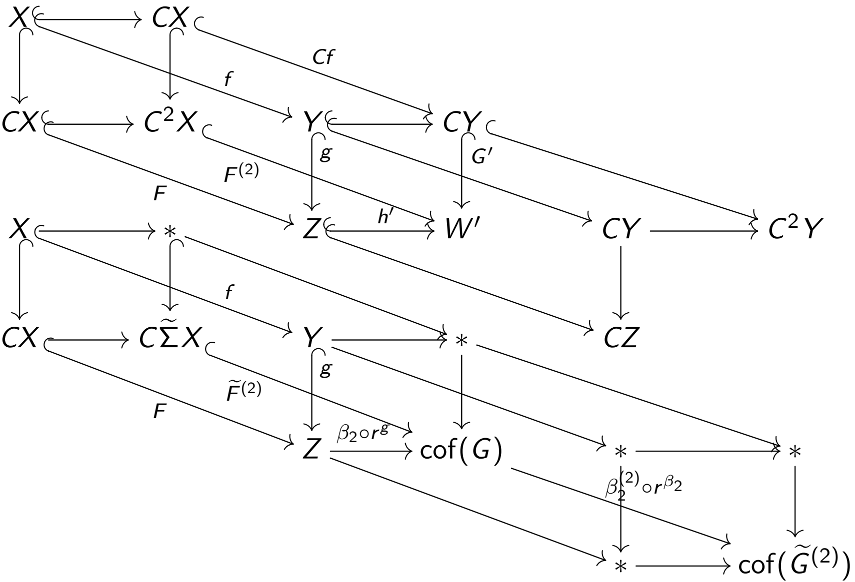

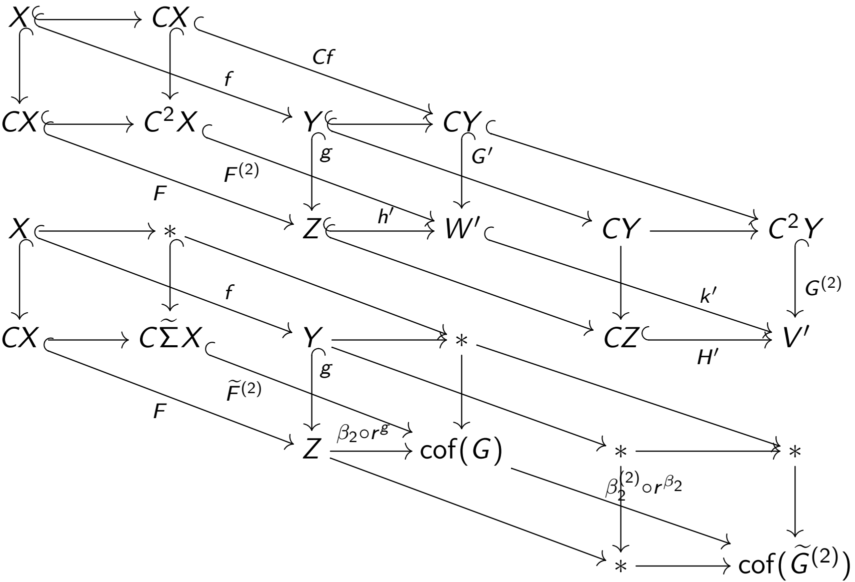
$X$
$CX$
$Cf$
$f$
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$C^2X$
$Y$
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$g$
$G'$
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$F$
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$F$
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$Z$
$\mathrm{cof}(G)$
$*$
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$*$
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# From recursive system to cubical system

**Theorem 2:** If we have a recursive Toda system $(X_*, \widetilde{F}_*)$, then there is a cubical Toda system $(X'_*, F_*)$ where:

$$\begin{array}{ccc} L^2X & \xrightarrow{T(X'_*, F_*)} & V' \\ \simeq \downarrow & & \downarrow \simeq \\ \widetilde{\Sigma}^2 X & \xrightarrow{\widetilde{T}(X_*, \widetilde{F}_*)} & \mathrm{cof}(\widetilde{G}^2) \end{array}$$

In addition $R(X'_*, F_*) \simeq (X_*, \widetilde{F}_*)$, and if we have a cubical Toda system $(X_*, F_*)$, then a corresponding cubical Toda system for $R(X_*, F_*)$ can be $(X_*, F_*)$.