Nodes on quintic spectrahedra

Khazhgali Kozhasov (Universität Osnabrück)
(based on a j.w.w. Taylor Brysiewicz and Mario Kummer)

Seminar in Real and Complex Geometry,
Tel Aviv University

July 15, 2021

Outline

Outline

- Real algebraic geometry

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves
- What are spectrahedra?

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**
    - Restrictions on types

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**
    - Restrictions on types
    - Numerical algebraic geometry

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**
  - Restrictions on types
  - Numerical algebraic geometry
  - Generic collisions/repulsions of singularities

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**
  - Restrictions on types
  - Numerical algebraic geometry
  - Generic collisions/repulsions of singularities
  - Hill-climbing algorithm

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**
    - Restrictions on types
    - Numerical algebraic geometry
    - Generic collisions/repulsions of singularities
    - Hill-climbing algorithm
    - Certification

Outline

- Real algebraic geometry
- Effective bounds on topology, geometry, ... of real varieties
- Example: topology of real plane curves

- What are spectrahedra?
- Examples and applications
- Symmetroids $\sim$ algebraic boundaries of spectrahedra
- Singular points
- Reality questions, definition of a (combinatorial) type
- **Classification of types of quintic spectrahedra**
  - Restrictions on types
  - Numerical algebraic geometry
  - Generic collisions/repulsions of singularities
  - Hill-climbing algorithm
  - Certification
- Conclusion

Real algebraic geometry

Real algebraic geometry

Study properties of real polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$,

Real algebraic geometry

Study properties of real polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$, real varieties

$X(\mathbb{R}) \;=\; \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0\}, \; f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n],$

Real algebraic geometry

Study properties of real polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$, real varieties

$$X(\mathbb{R}) \; = \; \{x \in \mathbb{R}^n \,:\, f_1(x) = \cdots = f_k(x) = 0\}, \;\; f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n],$$

and mappings between real varieties.

Real algebraic geometry

Study properties of real polynomials $f \in \mathbb{R}[x_1, \dots, x_n]$, real varieties

$$X(\mathbb{R}) = \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0\}, \; f_1, \dots, f_k \in \mathbb{R}[x_1, \dots, x_n],$$

and mappings between real varieties.

Nash-Tognoli theorem, 1976

Real algebraic geometry

Study properties of real polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$, real varieties

$X(\mathbb{R}) \; = \; \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0\}, \; f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n],$

and mappings between real varieties.

Nash-Tognoli theorem, 1976

Any smooth closed manifold is diffeomorphic to a real algebraic variety.

Real algebraic geometry

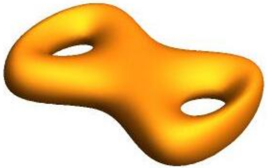
Study properties of real polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$, real varieties

$$X(\mathbb{R}) = \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0\}, \; f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n],$$

and mappings between real varieties.

Nash-Tognoli theorem, 1976

Any smooth closed manifold is diffeomorphic to a real algebraic variety.



Real algebraic geometry

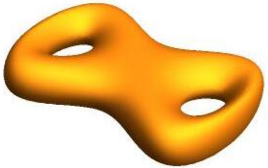
Study properties of real polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$, real varieties

$$X(\mathbb{R}) \; = \; \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0\}, \; f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n],$$

and mappings between real varieties.

Nash-Tognoli theorem, 1976

Any smooth closed manifold is diffeomorphic to a real algebraic variety.



$$25(x_1^4 - x_1^2 + x_2^2)^2 + 225x_3^2 - 1 = 0$$

Effective bounds / Classification

Consider a family $\mathcal{F} \subset \mathbb{R}[x_1, \ldots, x_n]^k$ of systems $\mathbf{f} = (f_1, \ldots, f_k)$.

Problem

Find effective bounds on some topological or geometric quantity of $\{\mathbf{f} = 0\} \subset \mathbb{R}^n$, for a (general) element $\mathbf{f} \in \mathcal{F}$. Understand all members $\mathbf{f} \in \mathcal{F}$ that are maximal with respect to this bound.

Topology of plane curves

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

,

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$

,

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$ satisfies

$$\frac{1-(-1)^d}{2} \leq \; c \; \leq \; \frac{(d-1)(d-2)}{2} + 1$$

,

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$ satisfies

$$\frac{1-(-1)^d}{2} \leq c \leq \frac{(d-1)(d-2)}{2} + 1$$

- For any d these bounds are optimal (Harnack).

,

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$ satisfies

$$\frac{1-(-1)^d}{2} \leq \; c \; \leq \; \frac{(d-1)(d-2)}{2} + 1$$

- For any d these bounds are optimal (Harnack).
- When $c = \frac{(d-1)(d-2)}{2} + 1$, $X_f(\mathbb{R})$ is called an M-curve.

,

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$ satisfies

$$\frac{1-(-1)^d}{2} \leq c \leq \frac{(d-1)(d-2)}{2} + 1$$

- For any d these bounds are optimal (Harnack).
- When $c = \frac{(d-1)(d-2)}{2} + 1$, $X_f(\mathbb{R})$ is called an M-curve.

- If the complex locus $X_f = \{f = 0\} \subset \mathbb{C}\mathrm{P}^2$ is non-singular,

Topology of plane curves

Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$ satisfies

$$\frac{1-(-1)^d}{2} \leq c \leq \frac{(d-1)(d-2)}{2} + 1$$

- For any d these bounds are optimal (Harnack).
- When $c = \frac{(d-1)(d-2)}{2} + 1$, $X_f(\mathbb{R})$ is called an M-curve.

- If the complex locus $X_f = \{f = 0\} \subset \mathbb{C}\mathrm{P}^2$ is non-singular, it is a (Riemann) surface of genus $\frac{(d-1)(d-2)}{2}$.

Topology of plane curves

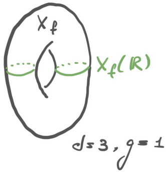
Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a homogeneous polynomial of degree d.

Harnack's curve theorem, 1976

The number c of connected components of the (real) plane curve $X_f(\mathbb{R}) = \{(x_1 : x_2 : x_3) \in \mathbb{R}\mathrm{P}^2 : f(x_1, x_2, x_3) = 0\}$ satisfies

$$\frac{1-(-1)^d}{2} \leq c \leq \frac{(d-1)(d-2)}{2} + 1$$

- For any d these bounds are optimal (Harnack).
- When $c = \frac{(d-1)(d-2)}{2} + 1$, $X_f(\mathbb{R})$ is called an M-curve.
- If the complex locus $X_f = \{f = 0\} \subset \mathbb{C}\mathrm{P}^2$ is non-singular, it is a (Riemann) surface of genus $\frac{(d-1)(d-2)}{2}$.



Hilbert's 16th problem

Hilbert's 16th problem

When non-singular,

Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals

Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals plus (if d is odd) a pseudo-line positioned in a certain way in $\mathbb{R}\mathrm{P}^2$.

Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals plus (if d is odd) a pseudo-line positioned in a certain way in $\mathbb{R}\mathrm{P}^2$.

Hilbert's 16th problem

Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals plus (if d is odd) a pseudo-line positioned in a certain way in $\mathbb{R}\mathrm{P}^2$.

Hilbert's 16th problem

For a fixed d classify configurations of connected components of M-curves of degree d.

Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals plus (if d is odd) a pseudo-line positioned in a certain way in $\mathbb{R}P^2$.

Hilbert's 16th problem

For a fixed d classify configurations of connected components of M-curves of degree d.

- Solved up to degree $d = 7$ (Hilbert, Rohn, Petrovsky, Rokhlin, Gudkov, Nikulin, Kharlamov, Viro).

Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals plus (if d is odd) a pseudo-line positioned in a certain way in $\mathbb{R}\mathrm{P}^2$.

Hilbert's 16th problem

For a fixed d classify configurations of connected components of M-curves of degree d.

- Solved up to degree $d = 7$ (Hilbert, Rohn, Petrovsky, Rokhlin, Gudkov, Nikulin, Kharlamov, Viro).
- For $d = 6$ there are 56 classes of curves, 3 of them are M-curves with 11 components each (Gudkov).

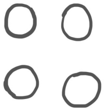
Hilbert's 16th problem

When non-singular, the real curve $X_f(\mathbb{R})$ is a collection of ovals plus (if d is odd) a pseudo-line positioned in a certain way in $\mathbb{R}\mathrm{P}^2$.

Hilbert's 16th problem

For a fixed d classify configurations of connected components of M-curves of degree d.

- Solved up to degree $d = 7$ (Hilbert, Rohn, Petrovsky, Rokhlin, Gudkov, Nikulin, Kharlamov, Viro).
- For $d = 6$ there are 56 classes of curves, 3 of them are M-curves with 11 components each (Gudkov).
- For $d = 4$ there is only one M-curve.



In this talk

Optimal bound on the number of singular points on the boundary of a quintic spectrahedron

$$S_A = \{x \in \mathbb{R}^3 : \mathbb{1} + x_1A_1 + x_2A_2 + x_3A_3 \succcurlyeq 0\},$$

where A_1, A_2, A_3 are general 5×5 real symmetric matrices.

What are spectrahedra?

What are spectrahedra?

$$\mathbb{S}^d \; = \; \{d \times d \text{ real symmetric matrices}\}$$

What are spectrahedra?

$$\mathbb{S}^d \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^d_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

What are spectrahedra?

$$\mathbb{S}^d \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^d_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

The convex cone of positive semidefinite matrices :

$$\mathbb{S}^d_{\succcurlyeq} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, A \succcurlyeq 0\} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, \boldsymbol{v}^T A \boldsymbol{v} \geq 0,\ \boldsymbol{v} \in \mathbb{R}^d\}$$

What are spectrahedra?

$$\mathbb{S}^{d} \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^{d}_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

The convex cone of positive semidefinite matrices :

$$\mathbb{S}^{d}_{\succcurlyeq} \;=\; \{A \in \mathbb{S}^{d}_{\mathbb{R}} \,:\, A \succcurlyeq 0\} \;=\; \{A \in \mathbb{S}^{d}_{\mathbb{R}} \,:\, \boldsymbol{v}^{T} A \boldsymbol{v} \geq 0,\ \boldsymbol{v} \in \mathbb{R}^{d}\}$$

A spectrahedron is the slice of $\mathbb{S}^{d}_{\succcurlyeq}$ by an affine-linear subspace

What are spectrahedra?

$$\mathbb{S}^d \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^d_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

The convex cone of positive semidefinite matrices :

$$\mathbb{S}^d_{\succeq} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, A \succeq 0\} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, \boldsymbol{v}^T A \boldsymbol{v} \geq 0,\; \boldsymbol{v} \in \mathbb{R}^d\}$$

A spectrahedron is the slice of $\mathbb{S}^d_{\succeq}$ by an affine-linear subspace

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 + x_1 A_1 + \cdots + x_n A_n \,:\, \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \mathbb{S}^d :$$

What are spectrahedra?

$$\mathbb{S}^d \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^d_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

The convex cone of positive semidefinite matrices :

$$\mathbb{S}^d_{\succcurlyeq} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, A \succcurlyeq 0\} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, \boldsymbol{v}^T A \boldsymbol{v} \geq 0,\; \boldsymbol{v} \in \mathbb{R}^d\}$$

A spectrahedron is the slice of $\mathbb{S}^d_{\succcurlyeq}$ by an affine-linear subspace

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 + x_1 A_1 + \cdots + x_n A_n \,:\, \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \mathbb{S}^d :$$

$$S_A \;=\; \mathbb{S}^d_{\succcurlyeq} \cap L \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^n \,:\, \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}.$$

What are spectrahedra?

$$\mathbb{S}^d \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^d_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

The convex cone of positive semidefinite matrices :

$$\mathbb{S}^d_{\succcurlyeq} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, A \succcurlyeq 0\} \;=\; \{A \in \mathbb{S}^d_{\mathbb{R}} \,:\, \boldsymbol{v}^T A \boldsymbol{v} \geq 0,\ \boldsymbol{v} \in \mathbb{R}^d\}$$

A spectrahedron is the slice of $\mathbb{S}^d_{\succcurlyeq}$ by an affine-linear subspace

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 + x_1 A_1 + \cdots + x_n A_n \,:\, \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \mathbb{S}^d :$$

$$S_A \;=\; \mathbb{S}^d_{\succcurlyeq} \cap L \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^n \,:\, \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}.$$

In particular, spectrahedra are convex sets.

Examples

Examples

A spectrahedron is a “nonlinear” generalization of a polyhedron:

Examples

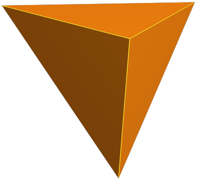
A spectrahedron is a "nonlinear" generalization of a polyhedron:

$$\boldsymbol{A}(\boldsymbol{x}) \;=\; \begin{pmatrix} a_1^0 & & 0 \\ & \ddots & \\ 0 & & a_d^0 \end{pmatrix} + x_1 \begin{pmatrix} a_1^1 & & 0 \\ & \ddots & \\ 0 & & a_d^1 \end{pmatrix} + x_n \begin{pmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_d^n \end{pmatrix} \;=\; \begin{pmatrix} a_1(\boldsymbol{x}) & & 0 \\ & \ddots & \\ 0 & & a_d(\boldsymbol{x}) \end{pmatrix},$$

Examples

A spectrahedron is a “nonlinear” generalization of a polyhedron:

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} a_1^0 & & 0 \\ & \ddots & \\ 0 & & a_d^0 \end{pmatrix} + x_1 \begin{pmatrix} a_1^1 & & 0 \\ & \ddots & \\ 0 & & a_d^1 \end{pmatrix} + x_n \begin{pmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_d^n \end{pmatrix} = \begin{pmatrix} a_1(\boldsymbol{x}) & & 0 \\ & \ddots & \\ 0 & & a_d(\boldsymbol{x}) \end{pmatrix},$$

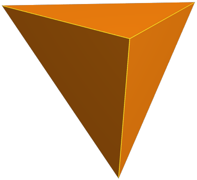


$$\{\boldsymbol{x} \in \mathbb{R}^n \,:\, a_1(\boldsymbol{x}), \ldots, a_d(\boldsymbol{x}) \geq 0\}$$

Examples

A spectrahedron is a "nonlinear" generalization of a polyhedron:

$$\boldsymbol{A}(\boldsymbol{x}) \;=\; \begin{pmatrix} a_1^0 & & 0 \\ & \ddots & \\ 0 & & a_d^0 \end{pmatrix} + x_1 \begin{pmatrix} a_1^1 & & 0 \\ & \ddots & \\ 0 & & a_d^1 \end{pmatrix} + x_n \begin{pmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_d^n \end{pmatrix} \;=\; \begin{pmatrix} a_1(\boldsymbol{x}) & & 0 \\ & \ddots & \\ 0 & & a_d(\boldsymbol{x}) \end{pmatrix},$$



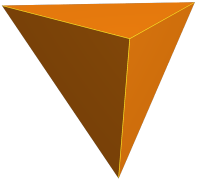
$$\{\boldsymbol{x} \in \mathbb{R}^n \;:\; a_1(\boldsymbol{x}), \ldots, a_d(\boldsymbol{x}) \geq 0\}$$

$$\boldsymbol{A}(\boldsymbol{x}) \;=\; \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix},$$

Examples

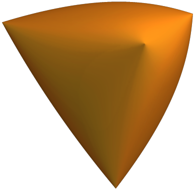
A spectrahedron is a "nonlinear" generalization of a polyhedron:

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} a_1^0 & & 0 \\ & \ddots & \\ 0 & & a_d^0 \end{pmatrix} + x_1 \begin{pmatrix} a_1^1 & & 0 \\ & \ddots & \\ 0 & & a_d^1 \end{pmatrix} + x_n \begin{pmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_d^n \end{pmatrix} = \begin{pmatrix} a_1(\boldsymbol{x}) & & 0 \\ & \ddots & \\ 0 & & a_d(\boldsymbol{x}) \end{pmatrix},$$



$$\{\boldsymbol{x} \in \mathbb{R}^n \,:\, a_1(\boldsymbol{x}), \dots, a_d(\boldsymbol{x}) \geq 0\}$$

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix},$$



Elliptope

$$\{\boldsymbol{x} \in \mathbb{R}^3 \,:\, 1 - x_1^2,\ 1 - x_2^2,\ 1 - x_3^2,\ 1 - x_1^2 - x_2^2 - x_3^2 + 2x_1x_2x_3 \geq 0\}$$

Spectrahedra, SOS and optimization

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

$$1 + 2t + 3t^2 + 5t^4 \; = \; \begin{pmatrix} 1 & t & t^2 \end{pmatrix} A \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \; A \; = \; \begin{pmatrix} 1 & 1 & a \\ 1 & b & 0 \\ a & 0 & 5 \end{pmatrix} \succcurlyeq 0$$

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

$$1 + 2t + 3t^2 + 5t^4 \; = \; \begin{pmatrix} 1 & t & t^2 \end{pmatrix} A \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \; A \; = \; \begin{pmatrix} 1 & 1 & a \\ 1 & b & 0 \\ a & 0 & 5 \end{pmatrix} \succcurlyeq 0$$

- **Semidefine programming**

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

$$1 + 2t + 3t^2 + 5t^4 = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} A \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 1 & a \\ 1 & b & 0 \\ a & 0 & 5 \end{pmatrix} \succcurlyeq 0$$

- **Semidefine programming**

$$\min_{\boldsymbol{x} \in S_A} \langle \boldsymbol{\ell}, \boldsymbol{x} \rangle = \min\{\langle \boldsymbol{\ell}, \boldsymbol{x} \rangle : \boldsymbol{x} \in \mathbb{R}^n, \; A_0 + x_1 A_1 + \cdots + x_n A_n \succcurlyeq 0\},$$

Spectrahedra are feasible regions of semidefinite programs (polyhedra) (linear)

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

$$1+2t+3t^2+5t^4 = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} A \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix},\ A = \begin{pmatrix} 1 & 1 & a \\ 1 & b & 0 \\ a & 0 & 5 \end{pmatrix} \succcurlyeq 0$$

- **Semidefine programming**

$$\min_{\boldsymbol{x} \in S_A} \langle \boldsymbol{\ell}, \boldsymbol{x} \rangle = \min\{\langle \boldsymbol{\ell}, \boldsymbol{x} \rangle : \boldsymbol{x} \in \mathbb{R}^n,\ A_0 + x_1 A_1 + \cdots + x_n A_n \succcurlyeq 0\},$$

Spectrahedra are feasible regions of semidefinite programs
(polyhedra) (linear)

Particular case:

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

$$1 + 2t + 3t^2 + 5t^4 \;=\; \begin{pmatrix} 1 & t & t^2 \end{pmatrix} A \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix},\; A \;=\; \begin{pmatrix} 1 & 1 & a \\ 1 & b & 0 \\ a & 0 & 5 \end{pmatrix} \succcurlyeq 0$$

- **Semidefine programming**

$$\min_{\boldsymbol{x} \in S_A} \langle \boldsymbol{\ell}, \boldsymbol{x} \rangle \;=\; \min\{\langle \boldsymbol{\ell}, \boldsymbol{x} \rangle \;:\; \boldsymbol{x} \in \mathbb{R}^n,\; A_0 + x_1 A_1 + \cdots + x_n A_n \succcurlyeq 0\},$$

Spectrahedra (polyhedra) are feasible regions of semidefinite programs (linear)

Particular case: **SOS approach to polynomial optimization**

Spectrahedra, SOS and optimization

- **Sum of squares representations of a real polynomial**

$$1 + 2t + 3t^2 + 5t^4 = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} A \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 1 & a \\ 1 & b & 0 \\ a & 0 & 5 \end{pmatrix} \succcurlyeq 0$$

- **Semidefine programming**

$$\min_{\boldsymbol{x} \in S_A} \langle \boldsymbol{\ell}, \boldsymbol{x} \rangle = \min\{\langle \boldsymbol{\ell}, \boldsymbol{x} \rangle : \boldsymbol{x} \in \mathbb{R}^n, \; A_0 + x_1 A_1 + \cdots + x_n A_n \succcurlyeq 0\},$$

Spectrahedra (polyhedra) are feasible regions of semidefinite programs (linear)

Particular case: **SOS approach to polynomial optimization**

$$\min_{\boldsymbol{t} \in \mathbb{R}^m} f(\boldsymbol{t}) \geq \max \left\{ \lambda \in \mathbb{R} : f(\boldsymbol{t}) - \lambda = \begin{pmatrix} 1 & t_1 & \ldots \end{pmatrix} A \begin{pmatrix} 1 \\ t_1 \\ \vdots \\ \vdots \end{pmatrix}, A \succcurlyeq 0 \right\}$$

Projective setting

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^{n+1} \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^{n+1} \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Assumption

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^{n+1} \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Assumption

$\boldsymbol{A}(\boldsymbol{x}) \succ 0$ for some $\boldsymbol{x} \in \mathbb{R}^{n+1}$
($L \cap \mathbb{S}^d_{\succcurlyeq} \subset \mathbb{R}^{n+1}$ is full-dimensional)

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^{n+1} \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Assumption

$\boldsymbol{A}(\boldsymbol{x}) \succ 0$ for some $\boldsymbol{x} \in \mathbb{R}^{n+1} \Rightarrow$ can take $A_0 = \mathbb{1}$.
($L \cap \mathbb{S}^d_{\succcurlyeq} \subset \mathbb{R}^{n+1}$ is full-dimensional) (orthogonal congruence applied to L)

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^{n+1} \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Assumption

$\boldsymbol{A}(\boldsymbol{x}) \succ 0$ for some $\boldsymbol{x} \in \mathbb{R}^{n+1} \Rightarrow$ can take $A_0 = \mathbb{1}$.

($L \cap \mathbb{S}^d_{\succcurlyeq} \subset \mathbb{R}^{n+1}$ is full-dimensional) (orthogonal congruence applied to L)

Projective spectrahedron:

Projective setting

Linear space $L \subseteq \mathbb{S}^d$:

$$L \;=\; \{\boldsymbol{A}(\boldsymbol{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\boldsymbol{x} \in \mathbb{R}^{n+1} \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Assumption

$\boldsymbol{A}(\boldsymbol{x}) \succ 0$ for some $\boldsymbol{x} \in \mathbb{R}^{n+1}$ $\Rightarrow$ can take $A_0 = \mathbb{1}$.
($L \cap \mathbb{S}^d_{\succcurlyeq} \subset \mathbb{R}^{n+1}$ is full-dimensional) (orthogonal congruence applied to L)

Projective spectrahedron:

$$S_A \;=\; \mathbb{P}(L \cap \mathbb{S}^d_{\succcurlyeq}) \;\simeq\; \{[\boldsymbol{x}] \in \mathbb{R}\mathrm{P}^n \;:\; \boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0 \text{ or } -\boldsymbol{A}(\boldsymbol{x}) \succcurlyeq 0\}$$

Algebraic boundary of spectrahedra

Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \ = \ \{[\boldsymbol{x}] \in S_A \ : \ \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \;=\; \{[\boldsymbol{x}] \in S_A \,:\, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$

Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \;=\; \{[\boldsymbol{x}] \in S_A \,:\, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_A \subset \mathbb{C}\mathrm{P}^n$

Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \; = \; \{[\boldsymbol{x}] \in S_A \, : \, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_A \subset \mathbb{C}\mathrm{P}^n$ is a degree d hypersurface,

Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \;=\; \{[x] \in S_A \,:\, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_A \subset \mathbb{C}\mathrm{P}^n$ is a degree d hypersurface, called *(spectrahedral) symmetroid*:

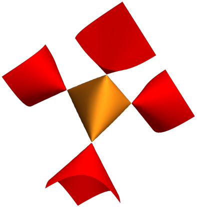
Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \;=\; \{[\boldsymbol{x}] \in S_A \,:\, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_A \subset \mathbb{C}\mathrm{P}^n$ is a degree d hypersurface, called *(spectrahedral) symmetroid*:

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$



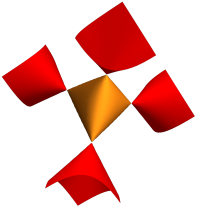
Algebraic boundary of spectrahedra

The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \;=\; \{[\boldsymbol{x}] \in S_A \,:\, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_A \subset \mathbb{C}\mathrm{P}^n$ is a degree d hypersurface, called *(spectrahedral) symmetroid*:

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$



$S_A \subset \mathbb{R}\mathrm{P}^n$ is a polyhedron
($A_1,\ldots,A_n$ are diagonal)

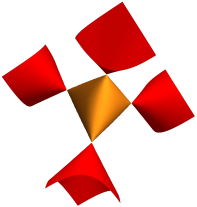
Algebraic boundary of spectrahedra

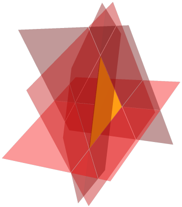
The Euclidean boundary of $S_A \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_A \;=\; \{[\boldsymbol{x}] \in S_A \,:\, \det(\boldsymbol{A}(\boldsymbol{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_A \subset \mathbb{C}\mathrm{P}^n$ is a degree d hypersurface, called *(spectrahedral) symmetroid*:

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$





$S_A \subset \mathbb{R}\mathrm{P}^n$ is a polyhedron $\Rightarrow$ $X_A \subset \mathbb{C}\mathrm{P}^n$ is a set of hyperplanes.
($A_1,\ldots,A_n$ are diagonal)

Singular points

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \; = \; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \, : \, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3 :

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \;:\; \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\boxed{\binom{d+1}{3} \;=\; \#\mathrm{Sing}(X) \;=\; \deg(\Sigma)}$$

singular points,

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \;:\; \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\boxed{\binom{d+1}{3} \;=\; \#\mathrm{Sing}(X) \;=\; \deg(\Sigma)}$$

singular points, which are all nodal (of multiplicity 2).

Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \;:\; \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\boxed{\binom{d+1}{3} \;=\; \#\mathrm{Sing}(X) \;=\; \deg(\Sigma)}$$

singular points, which are all nodal (of multiplicity 2).

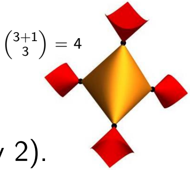
Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \;:\; \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\boxed{\binom{d+1}{3} \;=\; \#\mathrm{Sing}(X) \;=\; \deg(\Sigma)}$$



singular points, which are all nodal (of multiplicity 2).

Singular points

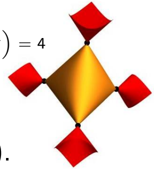
The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A = \{[\mathbf{x}] \in \mathbb{C}\mathrm{P}^n : \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\binom{d+1}{3} = \#\mathrm{Sing}(X) = \deg(\Sigma)$$

$\binom{3+1}{3} = 4$



singular points, which are all nodal (of multiplicity 2).

Combinatorial type (ρ, σ), $\sigma \leq \rho$

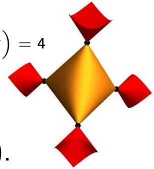
Singular points

The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \;:\; \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\binom{d+1}{3} \;=\; \#\mathrm{Sing}(X) \;=\; \deg(\Sigma)$$



singular points, which are all nodal (of multiplicity 2).

Combinatorial type (ρ, σ), $\sigma \leq \rho$

$\rho = \#(\mathrm{Sing}(X) \cap \mathbb{R}\mathrm{P}^3)$ is the number of real singularities of X.

Singular points

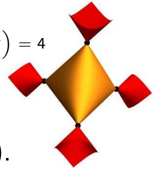
The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank ≥ 2 has codimension 3.
For $n \geq 3$ any symmetroid

$$X_A \;=\; \{[\boldsymbol{x}] \in \mathbb{C}\mathrm{P}^n \;:\; \det(x_0\mathbb{1} + x_1 A_1 + \cdots + x_n A_n) = 0\}$$

contains points of corank ≥ 2 and hence is singular.
n=3: surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\boxed{\binom{d+1}{3} \;=\; \#\mathrm{Sing}(X) \;=\; \deg(\Sigma)}$$

$\binom{3+1}{3} = 4$



singular points, which are all nodal (of multiplicity 2).

Combinatorial type $(\rho, \sigma),\ \sigma \leq \rho$

$\rho = \#(\mathrm{Sing}(X) \cap \mathbb{R}\mathrm{P}^3)$ is the number of real singularities of X.
$\sigma = \#(\mathrm{Sing}(X) \cap \partial S)$ is the number of singularities on $\partial S \subset X$.

n=3

Questions about real geometry

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3:

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

Degtyarev and Itenberg, 2011

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}P^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

Degtyarev and Itenberg, 2011

(ρ, σ) is a combinatorial type of a generic quartic d=4 spectrahedron

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

Degtyarev and Itenberg, 2011

(ρ, σ) is a combinatorial type of a generic quartic d=4 spectrahedron iff $0 \leq \sigma \leq \rho \leq 10$, both are even and $2 \leq \rho$.

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}P^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

Degtyarev and Itenberg, 2011

(ρ, σ) is a combinatorial type of a generic quartic d=4 spectrahedron iff $0 \leq \sigma \leq \rho \leq 10$, both are even and $2 \leq \rho$.

In 2015 Ottem, Ranestad, Sturmfels and Vinzant gave an alternative proof.

n=3

Questions about real geometry

Does there exist $X_A \subset \mathbb{C}P^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$

- with $\binom{d+1}{3}$ **real** singular points?
- with $\binom{d+1}{3}$ real singular points, all lying on ∂S_A?
- with a given combinatorial type (ρ, σ)?

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

Degtyarev and Itenberg, 2011

(ρ, σ) is a combinatorial type of a generic quartic d=4 spectrahedron iff $0 \leq \sigma \leq \rho \leq 10$, both are even and $2 \leq \rho$.

In 2015 Ottem, Ranestad, Sturmfels and Vinzant gave an alternative proof.

$d = 5$: Does a similar classification hold for quintic spectrahedra?

Main result

Main result

Brysiewicz, K. and Kummer, 2020

(ρ, σ) is a combinatorial type of a generic quintic $\boxed{\text{d=5}}$ spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

Main result

Brysiewicz, K. and Kummer, 2020

(ρ, σ) is a combinatorial type of a generic quintic $\boxed{\text{d=5}}$ spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

Proof strategy:

Main result

Brysiewicz, K. and Kummer, 2020

(ρ, σ) is a combinatorial type of a generic quintic $\boxed{\text{d=5}}$ spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

Proof strategy:

- Understand restrictions on (ρ, σ)

$\boxed{\text{65 possible types}}$

Main result

Brysiewicz, K. and Kummer, 2020

(ρ, σ) is a combinatorial type of a generic quintic $\boxed{\text{d=5}}$ spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

Proof strategy:

- Understand restrictions on (ρ, σ) $\boxed{\text{65 possible types}}$
- Find explicit representatives for each (ρ, σ) numerically

Main result

Brysiewicz, K. and Kummer, 2020

(ρ, σ) is a combinatorial type of a generic quintic $\boxed{\text{d=5}}$ spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

Proof strategy:

- Understand restrictions on (ρ, σ) $\boxed{\text{65 possible types}}$
- Find explicit representatives for each (ρ, σ) numerically
- Certify the numerical answers

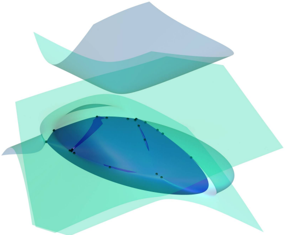
Main result

Brysiewicz, K. and Kummer, 2020

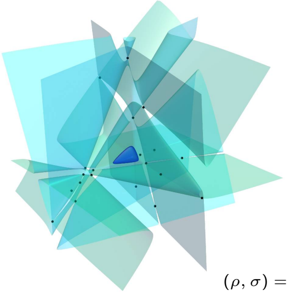
(ρ, σ) is a combinatorial type of a generic quintic $\boxed{\text{d}=5}$ spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

Proof strategy:

- Understand restrictions on (ρ, σ) $\boxed{\text{65 possible types}}$
- Find explicit representatives for each (ρ, σ) numerically
- Certify the numerical answers



$(\rho, \sigma) = (20, 20)$



$(\rho, \sigma) = (20, 0)$

Restrictions on combinatorial types

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$,

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1 \bmod 8$.

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1$ mod 8.

  (topological argument a lá Radon-Hurwitz)

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1 \bmod 8$.

  (topological argument a lá Radon-Hurwitz)

Corollary d=5

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1 \bmod 8$.

  (topological argument a lá Radon-Hurwitz)

Corollary d=5

If (σ, ρ) is a combinatorial type of a quintic spectrahedron,

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1 \bmod 8$.

  (topological argument a lá Radon-Hurwitz)

Corollary d=5

If (σ, ρ) is a combinatorial type of a quintic spectrahedron, then $0 \leq \sigma \leq \rho \leq 20$,

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1 \bmod 8$.

  (topological argument a lá Radon-Hurwitz)

Corollary d=5

If (σ, ρ) is a combinatorial type of a quintic spectrahedron, then $0 \leq \sigma \leq \rho \leq 20$, both σ and ρ are even

Restrictions on combinatorial types

Lemma

If (ρ, σ) is the combinatorial type of the surface $X_A \subset \mathbb{C}\mathrm{P}^3$ defined by generic matrices $A_1, A_2, A_3 \in \mathbb{S}^d$, then

- $0 \leq \sigma \leq \rho \leq \binom{d+1}{3}$,
- ρ is of the same parity as $\binom{d+1}{3} = \#\mathrm{Sing}(X_A)$,

  (non-real nodes come in complex conjugate pairs since X_A is real and generic),
- σ is even,

  (there exist X_A with $\sigma = 0$ and (generically) nodes can come to S_A (or leave it) only in pairs)
- $\rho > 0$ (for all X_A) if and only if $d \neq -1, 0, 1 \bmod 8$.

  (topological argument a lá Radon-Hurwitz)

Corollary d=5

If (σ, ρ) is a combinatorial type of a quintic spectrahedron, then $0 \leq \sigma \leq \rho \leq 20$, both σ and ρ are even and $2 \leq \rho$.

Numerical algebraic geometry

Numerical algebraic geometry

Singular points of $X_A \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

Numerical algebraic geometry

Singular points of $X_A \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

$$F_A: \quad \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) \;=\; 0, \quad i = 0, \ldots, 3.$$

Numerical algebraic geometry

Singular points of $X_A \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

$$F_A: \quad \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) \ = \ 0, \quad i = 0, \ldots, 3.$$

For a generic triple $A = (A_1, A_2, A_3) \in (\mathbb{S}^5_{\mathbb{C}})^3$ of **complex** symmetric matrices, F_A has 20 zeros.

Numerical algebraic geometry

Singular points of $X_A \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

$$F_A : \quad \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) \;=\; 0, \quad i = 0, \ldots, 3.$$

For a generic triple $A = (A_1, A_2, A_3) \in (\mathbb{S}^5_{\mathbb{C}})^3$ of **complex** symmetric matrices, F_A has 20 zeros.If we know (can easily find) solutions to a reference system $F_{A'}$,

Numerical algebraic geometry

Singular points of $X_A \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

$$F_A : \quad \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) \ = \ 0, \quad i = 0, \ldots, 3.$$

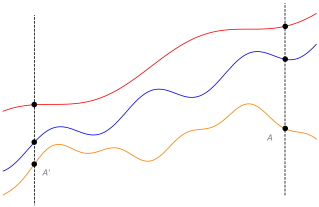
For a generic triple $A = (A_1, A_2, A_3) \in (\mathbb{S}^5_{\mathbb{C}})^3$ of **complex** symmetric matrices, F_A has 20 zeros.If we know (can easily find) solutions to a reference system $F_{A'}$,can solve the desired system F_A using the method of homotopy continuation:

Numerical algebraic geometry

Singular points of $X_A \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

$$F_A: \quad \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) \;=\; 0, \quad i = 0, \ldots, 3.$$

For a generic triple $A = (A_1, A_2, A_3) \in (\mathbb{S}^5_{\mathbb{C}})^3$ of **complex** symmetric matrices, F_A has 20 zeros.If we know (can easily find) solutions to a reference system $F_{A'}$,can solve the desired system F_A using the method of homotopy continuation:



Neighboring types

Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically the type can change in one of the following 4 ways:

Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically the type can change in one of the following 4 ways:

$(+,+)$: $(\rho,\sigma) \to (\rho{+}2, \sigma{+}2)$

$(-,-)$: $(\rho,\sigma) \to (\rho{-}2, \sigma{-}2)$

Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically the type can change in one of the following 4 ways:

$(+,+)$: $(\rho,\sigma) \to (\rho{+}2, \sigma{+}2)$ $(+,0)$: $(\rho,\sigma) \to (\rho+2, \sigma)$

$(-,-)$: $(\rho,\sigma) \to (\rho{-}2, \sigma{-}2)$ $(-,0)$: $(\rho,\sigma) \to (\rho-2, \sigma)$

Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically the type can change in one of the following 4 ways:

$(+,+)$: $(\rho,\sigma) \to (\rho+2,\sigma+2)$ $(+,0)$: $(\rho,\sigma) \to (\rho+2,\sigma)$

$(-,-)$: $(\rho,\sigma) \to (\rho-2,\sigma-2)$ $(-,0)$: $(\rho,\sigma) \to (\rho-2,\sigma)$

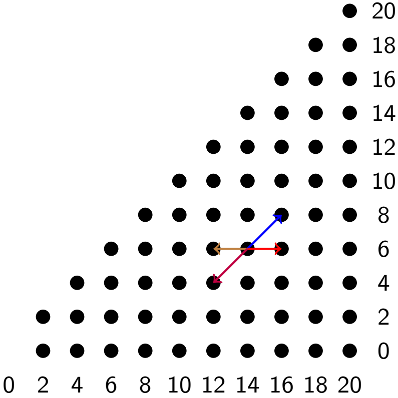
Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically the type can change in one of the following 4 ways:

$(+,+)$: $(\rho,\sigma) \to (\rho+2,\sigma+2)$ $(+,0)$: $(\rho,\sigma) \to (\rho+2,\sigma)$

$(-,-)$: $(\rho,\sigma) \to (\rho-2,\sigma-2)$ $(-,0)$: $(\rho,\sigma) \to (\rho-2,\sigma)$



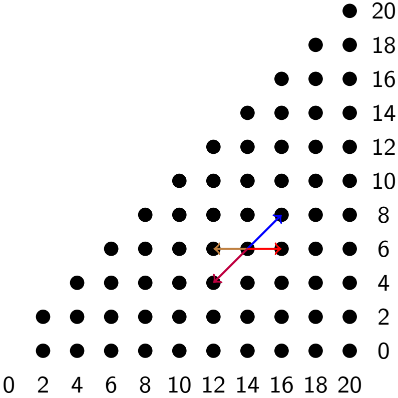
Neighboring types

How to find $X_A \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically the type can change in one of the following 4 ways:

$(+,+)$: $(\rho,\sigma) \to (\rho+2,\sigma+2)$ $\quad$ $(+,0)$: $(\rho,\sigma) \to (\rho+2,\sigma)$

$(-,-)$: $(\rho,\sigma) \to (\rho-2,\sigma-2)$ $\quad$ $(-,0)$: $(\rho,\sigma) \to (\rho-2,\sigma)$



If we are able to find all possible neighboring types for each (ρ,σ), then all 65 types can be found.

We achieve this goal using the hill-climbing algorithm

Hill-climbing algorithm

Hill-climbing algorithm

Given $A \in (\mathbb{S}^5)^3$,

Hill-climbing algorithm

Given $A \in (\mathbb{S}^5)^3$, let

- $S_{\mathbb{R}_+}(A)$ (resp. $S_{\mathbb{R}_-}(A)$) be the set of definite (indefinite) real nodes $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{S}^5$ of $X_A \subset \mathbb{C}\mathrm{P}^3$,

Hill-climbing algorithm

Given $A \in (\mathbb{S}^5)^3$, let

- $S_{\mathbb{R}_+}(A)$ (resp. $S_{\mathbb{R}_-}(A)$) be the set of definite (indefinite) real nodes $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{S}^5$ of $X_A \subset \mathbb{C}\mathrm{P}^3$,
- $S_{\mathbb{C}_+}(A)$ (resp. $S_{\mathbb{C}_-}(A)$) be the set of those non-real nodes $\boldsymbol{A}(\boldsymbol{x})$ of X_A with definite (resp. indefinite) real part $\mathbb{R}\mathrm{e}(\boldsymbol{A}(\boldsymbol{x}))$.

Hill-climbing algorithm

Given $A \in (\mathbb{S}^5)^3$, let

- $S_{\mathbb{R}_+}(A)$ (resp. $S_{\mathbb{R}_-}(A)$) be the set of definite (indefinite) real nodes $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{S}^5$ of $X_A \subset \mathbb{C}\mathrm{P}^3$,
- $S_{\mathbb{C}_+}(A)$ (resp. $S_{\mathbb{C}_-}(A)$) be the set of those non-real nodes $\boldsymbol{A}(\boldsymbol{x})$ of X_A with definite (resp. indefinite) real part $\mathbb{R}\mathrm{e}(\boldsymbol{A}(\boldsymbol{x}))$.

If $X_{A'}$ has type (ρ, σ), go for the 4 neighboring types as follows:

Hill-climbing algorithm

Given $A \in (\mathbb{S}^5)^3$, let

- $S_{\mathbb{R}_+}(A)$ (resp. $S_{\mathbb{R}_-}(A)$) be the set of definite (indefinite) real nodes $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{S}^5$ of $X_A \subset \mathbb{C}\mathrm{P}^3$,
- $S_{\mathbb{C}_+}(A)$ (resp. $S_{\mathbb{C}_-}(A)$) be the set of those non-real nodes $\boldsymbol{A}(\boldsymbol{x})$ of X_A with definite (resp. indefinite) real part $\mathrm{Re}(\boldsymbol{A}(\boldsymbol{x}))$.

If $X_{A'}$ has type (ρ, σ), go for the 4 neighboring types as follows:

- randomly sample a few A near A',

Hill-climbing algorithm

Given $A \in (\mathbb{S}^5)^3$, let

- $S_{\mathbb{R}_+}(A)$ (resp. $S_{\mathbb{R}_-}(A)$) be the set of definite (indefinite) real nodes $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{S}^5$ of $X_A \subset \mathbb{C}\mathrm{P}^3$,
- $S_{\mathbb{C}_+}(A)$ (resp. $S_{\mathbb{C}_-}(A)$) be the set of those non-real nodes $\boldsymbol{A}(\boldsymbol{x})$ of X_A with definite (resp. indefinite) real part $\mathbb{R}\mathrm{e}(\boldsymbol{A}(\boldsymbol{x}))$.

If $X_{A'}$ has type (ρ, σ), go for the 4 neighboring types as follows:

- randomly sample a few A near A',
- for each neighboring type that is not found in the sample, as a new A' choose that sampled A with the smallest:

$$
\begin{aligned}
(+,+): &\quad \min\{\|\mathbb{I}\mathrm{m}(\boldsymbol{A}(\boldsymbol{x}))\| \,:\, \boldsymbol{A}(\boldsymbol{x}) \in S_{\mathbb{C}_+}(A)\} \\
(+,0): &\quad \min\{\|\mathbb{I}\mathrm{m}(\boldsymbol{A}(\boldsymbol{x}))\| \,:\, \boldsymbol{A}(\boldsymbol{x}) \in S_{\mathbb{C}_-}(A)\} \\
(-,-): &\quad \min\{\|\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{A}(\tilde{\boldsymbol{x}})\| \,:\, \boldsymbol{A}(\boldsymbol{x}),\ \boldsymbol{A}(\tilde{\boldsymbol{x}}) \in S_{\mathbb{R}_+}(A)\} \\
(-,0): &\quad \min\{\|\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{A}(\tilde{\boldsymbol{x}})\| \,:\, \boldsymbol{A}(\boldsymbol{x}),\ \boldsymbol{A}(\tilde{\boldsymbol{x}}) \in S_{\mathbb{R}_-}(A)\}
\end{aligned}
$$

- repeat until all neighboring types of (ρ, σ) are found

Certification

Certifying the correctness and reality of solutions to a **square** polynomial system can be usually done via the command $\boxed{\text{certify}}$ in *HomotopyContinuation.jl*:

- its input is a collection S of approximate solutions,
- (when successful) the output is the set of $\#(S)$ bounding boxes, each containing a unique **true** solution.

Since our system F_A is not square we consider a different one:

$$G_A : \quad \begin{cases} \frac{\partial}{\partial x_i}\det(x_0\mathbb{1} + x_1A_1 + x_2A_2 + x_3A_3) \;=\; 0, \quad i = 1,2,3, \\ \langle \boldsymbol{\ell}, \boldsymbol{x}\rangle = l, \\ \det(\boldsymbol{A}(\boldsymbol{x})) = \delta, \\ [\boldsymbol{A}(\boldsymbol{x})]_I = M_I, \quad I \subset \{1,\ldots,5\},\; \#(I) = 1,2,3, \end{cases}$$

where $[\boldsymbol{A}(\boldsymbol{x})]_I$ are the principal minors indexed by 25 many sets I.

G_A : $30 = 3 + 1 + 1 + 25$ equations in 30 unknowns $\boldsymbol{x}, \delta, M_I$.

Certification

$$G_A : \quad \begin{cases} \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) = 0, \quad i = 1, 2, 3, \\ \langle \boldsymbol{\ell}, \boldsymbol{x} \rangle = l, \\ \det(\boldsymbol{A}(\boldsymbol{x})) = \delta, \\ [\boldsymbol{A}(\boldsymbol{x})]_I = M_I, \quad I \subset \{1, \dots, 5\}, \ \#(I) = 1, 2, 3, \end{cases}$$

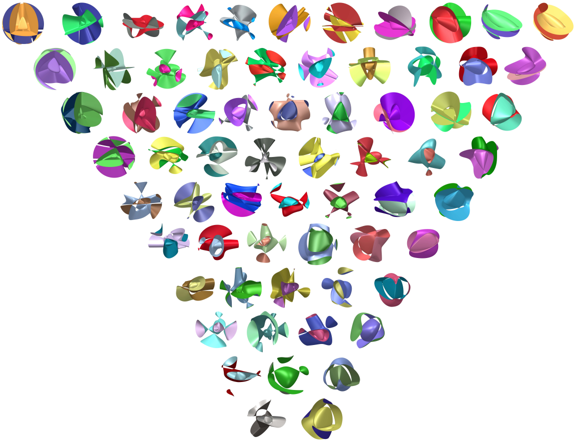
The first 4 equations have $64 = 4 \cdot 4 \cdot 4 \cdot 1$ solutions, the value of variables δ and M_I are then uniquely determined. 20 out of 64 solutions are nodes of $X_A \subset \mathbb{C}\mathrm{P}^3$. To certify the type of X_A:

- Solve G_A to obtain the set of 64 solutions
- Run *certify* to obtain 64 bounding boxes
- Delete those solutions which do not contain 0 in the δ-box
- Put ρ to be the number of real boxes
- For each real box determine the signs of the M_I-coordinates
- Put σ to be the number of real boxes with positive M_I-boxes

If all this is successful, (ρ, σ) is the certified type of $X_A \subset \mathbb{C}\mathrm{P}^3$.

Conclusion

- We identified 65 combinatorial types of quintic spectrahedra,
- Numerically found representatives, using the hill-climbing algorithm,
- Certified types of the found representatives,
- Produced plots of the associated surfaces.



References

- A. Degtyarev and I. Itenberg, *On real determinantal quartics*, Proceedings of the Gökova Geometry-Topology Conference 2010, Int. Press, Somerville, MA, 110–128, 2011
- J. C. Ottem, K. Ranestad, B. Sturmfels and C. Vinzant, *Quartic spectrahedra*, Math. Program., 151 (2, Ser. B), 585–612, 2015
- P. Breiding and S. Timme, *HomotopyContinuation.jl: A package for homotopy continuation in Julia*, 458–465 in Mathematical software — ICMS 2018 (South Bend, IN, 2018), edited by J.H. Davenport et al., Lecture Notes in Computer Science 10931, Springer, 2018
- T. Brysiewicz, Kh. Kozhasov and M. Kummer, *Nodes on quintic spectrahedra*, arXiv:2011.13860 [math.AG], 2020

Thank you!