Generalized Cusps on Convex projective Manifolds

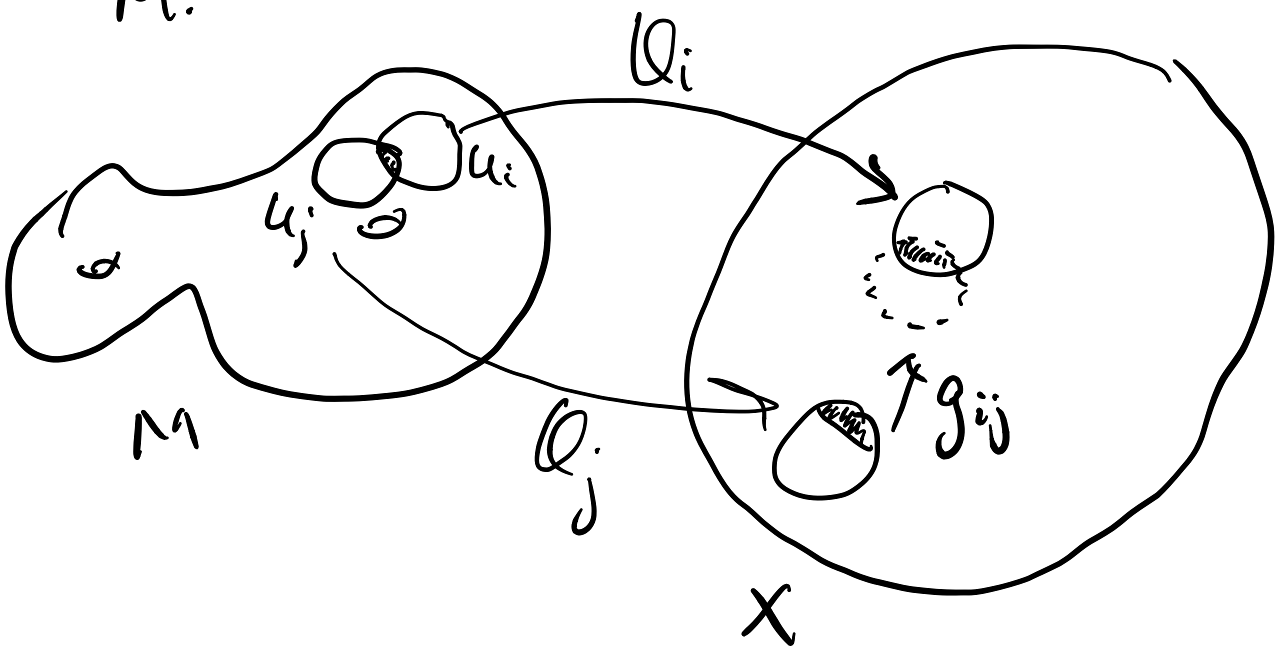
Plan:

1. Review Geom. Struc & Motivation
   Rev proj geom.
2. Motivate prop conv Geom.
3. Do ex of cusps on Surf and on 3 mnfd
4. Hyp. cusps
5. Classification Thm & Realizability
6. Deformation sp. for 3 mnfd.

Geom Struc & Motivation

Topological Sp. Come equipped w/ a pattern or Structure, and we can try to study a space by fitting a Structure on it

<u>Def</u>: G = Lie gp X = conn mnfd
$G \curvearrowright X$ trans. & analy.
A (G, X) mnfd is a mnfd, M,
w/ a collection of G compatible
coord charts whose domains cover
M.

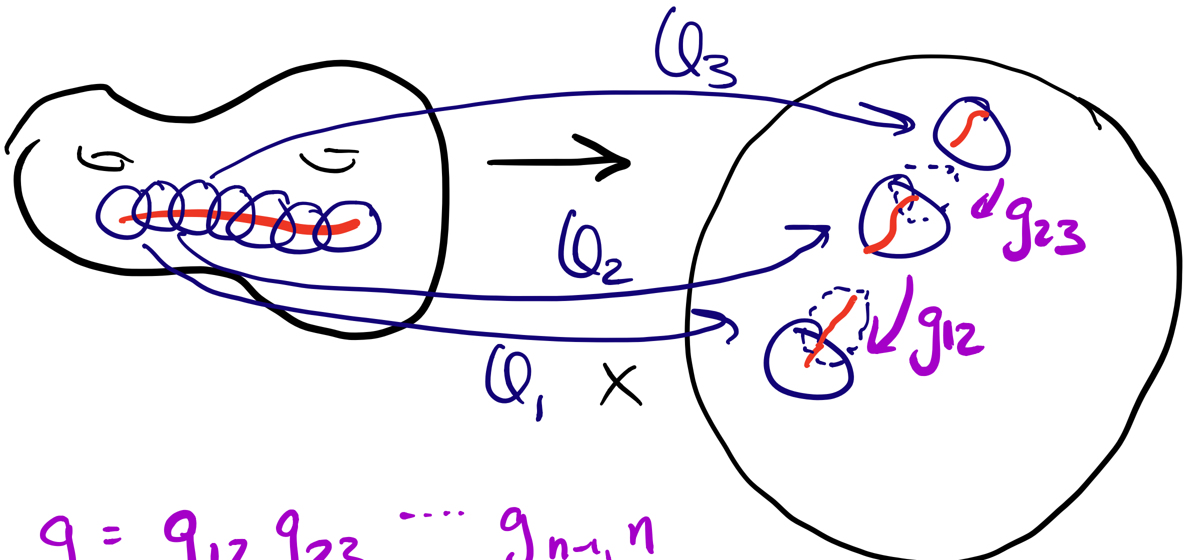


<u>Ex</u> Torus $(\mathbb{R}^2, \mathrm{Isom}(\mathbb{R}^2))$ mnfd
Sphere $(O(n+1), S^n)$ mnfd

no hyp str. on torus

Analytic Cont.

takes paths in $M \longrightarrow X$



$g = g_{12}\, g_{23} \cdots g_{n-1,n}$

patches path together

Developing map of (G,X) mnfd

$$D: \tilde{M} \longrightarrow X$$

local diffeo

univ cover = paths

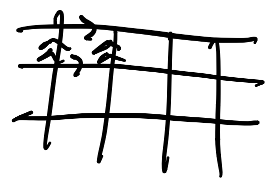
M complete $\Rightarrow$ D covering map

X simply conn $\Rightarrow$ $\tilde{M} \simeq X$

Holonomy $H: \pi_1 M \to G$ $\quad H(\sigma) = g_\sigma$

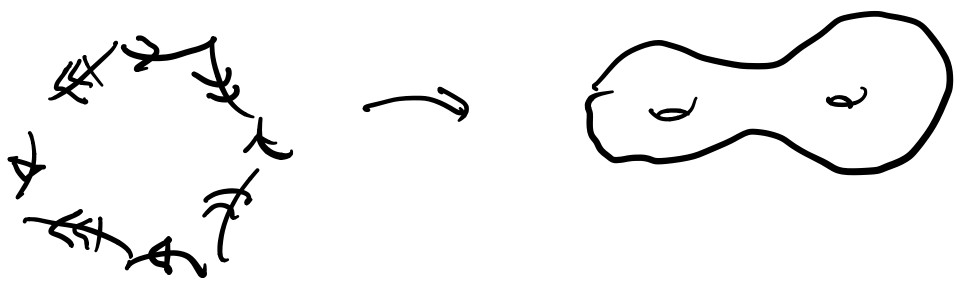
Prop G = gp of analytic diffeo of simply conn X, then any complete (G,X) mnfd can be reconstr. from holonomy Γ as $M = X/\Gamma$.

Ex Torus $\mathbb{R}^2/\mathbb{Z}^2$



$$\Gamma = \langle a_1, \dots a_g, b_1, \dots b_g \mid \prod [a_i, b_i] = 1 \rangle$$

$\mathbb{H}^2/\Gamma$ = genus g surf



Thm: (Uniformization) Every simply conn Riem surf is conformally equiv to one of open unit disk, cplx plane, Riem sph.

Thm (Geometrization) [Thurston, Perelman]
Every cpt prime 3 dim'l mnfd can be decomp. into pieces each w/ one kind of geom: S^3, $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL_2 \mathbb{R}}$, Nil, Sol

Want to discuss mnfd as a whole and not in pieces.

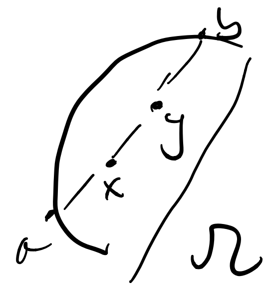
Projective Geometry

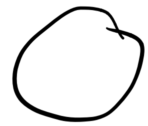
$$\underset{\text{aff patch}}{\mathbb{R}P^n = \mathbb{R}^n} \sqcup \underset{\text{hyperplane}}{\mathbb{R}P^{n-1}}$$

$\Omega \subset \mathbb{R}P^n$ is properly convex if it is a bounded convex subset of an affine patch

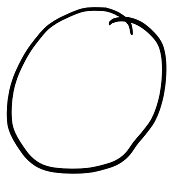
Hilbert metric on Ω:

$$d_\Omega(x,y) = \frac{1}{2} \log CR[a,x,y,b]$$

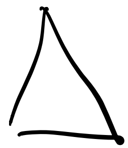


Get hyperbolic metric when $\Omega =$ 

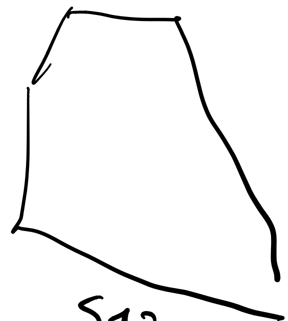
$$PGL(\Omega) = \{ A \in PGL_{n+1}(\mathbb{R}) \mid A(\Omega) = \Omega \}$$



$PO(2,1)$



$\text{diag}_3 \rtimes \text{Sym}_3$



$\{I\}$

Generically trivial

A properly convex projective mnfd

is $M = \Omega / \Gamma$ Ω = prop conv

$\Gamma \subset PGL(\Omega)$ discr tor free

Are there interesting prop conv mnfd?

- Hyperbolic
- Marquis, Benoist bending along tot geod hyp surf
- Deformations

Mostow Rigidity $\exists$ for $n \geq 3$ unique hyp str.

Thm (Koszul) M = closed n mnfd

$\mathrm{Dev}_c(M, \mathbb{P}) = \{ \mathrm{dev} \in \mathrm{Dev}(M, PGL_{n+1}(\mathbb{R}), \mathbb{R}P^n)$

dev$(\tilde{M})$ prop conv & dev inj $\}$

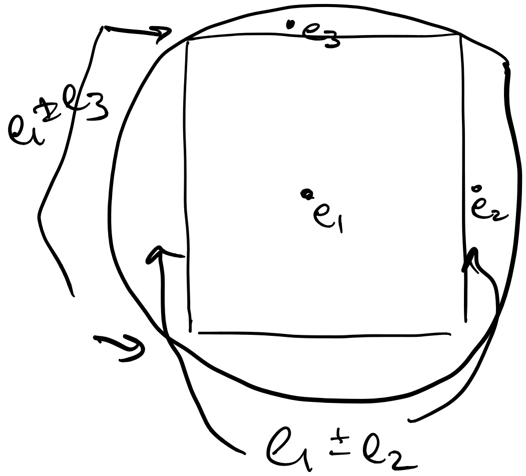
Then $\mathrm{Dev}_c(M, \mathbb{P}) \to \mathrm{Hom}(\pi_1 M, PGL_{n+1}(\mathbb{R}))$

is open

(Skip)

Ex: Cone torus $\to$ hyp cusp torus

Klein model



$$A = \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & 1 & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix},$$

$$B = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$ABA^{-1}B^{-1}$ describes Vertex

$t = \frac{2\pi}{k}$ cone pts

$t \to \infty$ parb.

Cone angle $\to 0$ Cusp

Tells us we need some conditions on the ends of mnfd.

Generalized Cusps

Cooper - Long - Tillmann: If ends of M are generalized cusps then can deform to new prop convex proj mnfd.

Before saying what a gen cusp is we'll review hyp cusps

Paraboloid model of hyp sp:

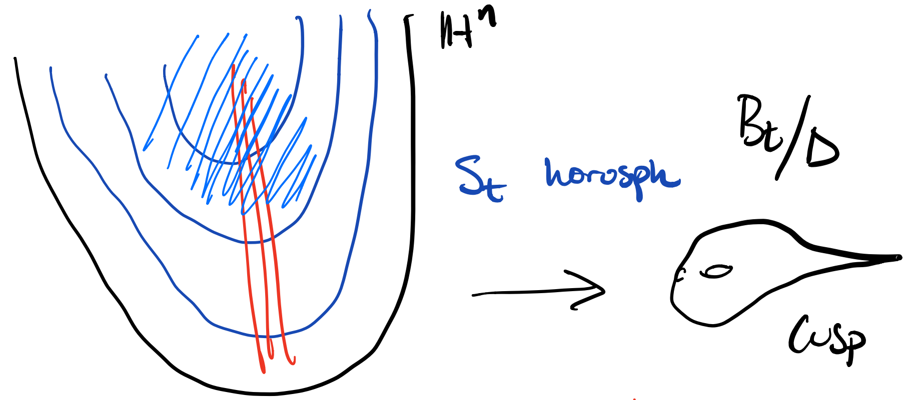
$$\mathbb{H}^n = \{ (z, v) \in \underset{\text{vertical}}{\mathbb{R}} \times \underset{\text{horiz.}}{\mathbb{R}^{n-1}} \mid z > \tfrac{1}{2}|v|^2 \}$$

$$\subseteq \mathbb{R}^n \subset \mathbb{RP}^n$$

hyp geod are segments of proj lines

horoball $B_t = \{ (z, v) \in \mathbb{H}^n \mid z > \frac{1}{2}|v|^2 + t \}$

horosphere $S_t = \partial B_t \qquad t > 0$



orthog fol by vert lines

$G = \mathrm{Isom}(B_t) = T \rtimes O(n-1) \qquad T \simeq \mathbb{R}^{n-1}$

$\Delta \subset G$ discr tor free

fol on $\mathbb{H}^n$ cover fol on C

topol $C \cong [0,\infty) \times \partial C$

in dim 3 $\quad C = T^2 \times [0,\infty)$

Generalized cusps

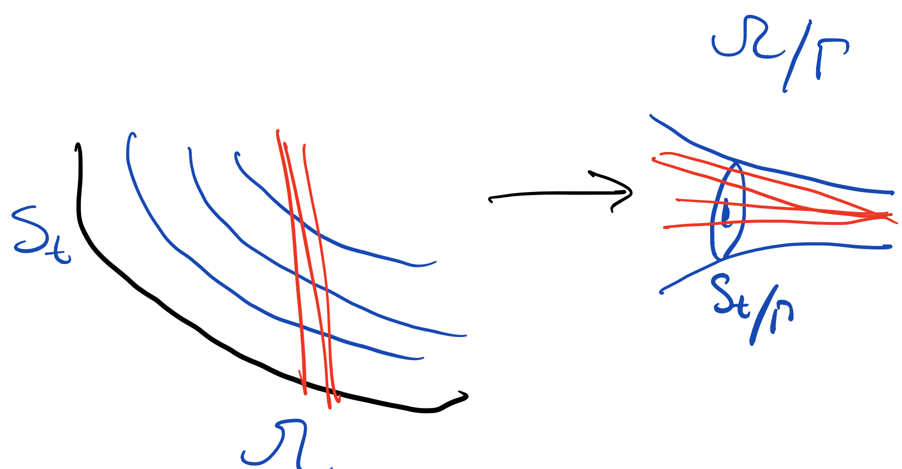
Def: A prop conv proj mnfd C is a gen. cusp if

- $C \cong \Sigma \times [0,\infty) \qquad \Sigma$ cpt
- ∂C = str. conv hypersurf
- Δ virt nilp. (BCL $\Rightarrow$ virt abel)

Thm (Cooper-Long-Tillmann)

A gen asp $C = \Omega'/\Delta$ Satisfies

- $\exists$ prop conv dom $\Omega \subset \Omega'$ w/ Str. conv bdry
- $G = PGL(\Omega) \curvearrowright \partial\Omega$ trans
- Ω has foliation by Str. conv hypersurf (S_t)
- Ω has fol by vert geod transv to S_t
- G pres both foliations
- $\Delta \subset G$ lattice



Thm (Ballas-Cooper-Leitner)

- $G = T \rtimes O$ $\quad T = \mathbb{R}^{n-1}$ $\quad O =$ pt stab

· G inv't Eucl metr on S_t

~ Look at handout page import ~

$$W_n = \{(\lambda_1, \dots, \lambda_{n-1} \mid 0 \le \lambda_1 \cdots \le \lambda_{n-1}\} \subseteq \mathbb{R}^n$$

Weyl chamber

Parabolic rank $\quad p = \max\{i \mid \lambda_i = 0\}$

Semisimple rk $\quad s = n-p-1$

$$f : \mathbb{R}^p \times \mathbb{R}^+_s \longrightarrow \mathbb{R}$$

$$(x_1, \dots, x_p, y_1, \dots, y_s) \longmapsto \underbrace{\sum_{i=1}^{n} \tfrac{1}{2} x_i^2}_{\text{hyp}} - \underbrace{\sum \lambda_j^{-1} \log y_j}_{\text{diag}}$$

$$\Omega = \{ z, (x,y) \in \underset{\text{vert}}{\mathbb{R}} \times \underbrace{\mathbb{R}^p \times \mathbb{R}^s_+}_{\text{horiz}} \mid z > f(x,y) \}$$

foliated by horospheres and transv vert geod

$$T_\lambda = \left\{ \begin{pmatrix} I & u & 0 & f(u,v) \\ & I_p & 0 & u \\ & & \mathrm{Dev} & 0 \\ & & & 1 \end{pmatrix} \middle| \underset{\begin{pmatrix} e^{v_1} & & \\ & \ddots & \\ & & e^{v_s} \end{pmatrix}}{\overset{\mathrm{Dev}}{\shortparallel}} \right\}$$

$$O_\lambda = O(p) \rtimes (\text{coord perms in } \lambda_i = \lambda_j)$$

$\Gamma < T_\lambda \rtimes O_\lambda$ then Ω/Γ is a gen csp.

<u>Thm</u> (BCL) $C = \Omega/\Gamma$ gen csp then $\exists\, \lambda \in W_n$ unique up to scaling s.t.

- Γ conjugate into a lattice $\Gamma' \subset G_\lambda$
- C def retrs along vert fol to a submnfd proj equiv to Ω_λ / Γ'

Realizability

Bobb realized all but diag attached to convex proj mnfd.

Resid fin / Sep & bending $\Rightarrow$ take large covers, cannot control number of csps or isolate a csp.

Deformation Space for Gen csps on 3 mnfds

Gen csp is uniquely det by $q = x^2 + y^2$ and $c = ax^3 + by^3 + cx^2y + dxy^2$

In dim 3 Cusp $= T^2 \times [0,\infty)$

$\rightsquigarrow$ pt in moduli space given by
Similarity struc on torus
$\rightsquigarrow$ pt in upper half plane
$\rightsquigarrow$ quadratic diff.

Action by $SL^{\pm}(2,\mathbb{R})$ pres similarity struc.
So pres up to scaling $\rightsquigarrow$ quad form.

Want to understand how gen cusps fiber over these quad forms

Claim fiber at every pt looks the same and is a homog cubic poly

We show cusp $\longrightarrow \{q, c\}$ is a homeo

Given $C = \Omega/\Gamma$ can pick pt in $\partial\Omega$ and hyperplane and compute Taylor series.

C convex $\Rightarrow q$ pos def.
We show truncation to cubic determines cusp (determines λ_i) and that this is a homeo.

Which cubics can we get?
We used $SL_2(\mathbb{R})$ action to normalize q
So we are left w/ an $SO(2)$ action

on Cubics.

4 diml vect sp w/ $SO(2)$ action

2 invt subsp:

lin poly $\cdot (x^2+y^2)$ — harmonic

$x(x^2+y^2),\ y(x^2+y^2)$ — $x(x^2-3y^2),\ y(y^2-3x^2)$

Compute norm on each invt subsp

$\Rightarrow\ a^2+b^2 \leq c^2+d^2$

cone on a solid torus.

interior = diag cusps

type 2 = bdry torus cone

type = $(1,3)$ curve ($SO(2)$ action)

cone pt = type 0

Thm (BCL): Marked moduli sp is a contractible
Semi-alg set of dim n^2-n
made by gluing together strata.

Thm (BCL): 3 equiv descriptions
of marked moduli sp.
all homeo

- Compute invt $(\chi, [\beta])$
  (character of holon & β = pos def quad form)
- quad diff + cubic
- $[\beta]$ + Lie alg wts w/ geom constr.