Counting Curves in $\mathbb{P}^r$

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Tel Aviv University, January 8, 2026

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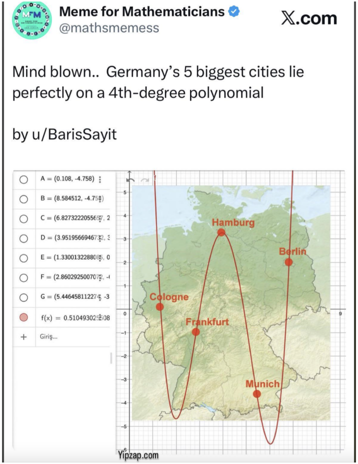

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Denote the answer by $\mathrm{Tev}^{\mathbb{P}^r}_{g,n,d}$, the **geometric Tevelev degree** of $\mathbb{P}^r$.

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Assume

$$n = \frac{r+1}{r} \cdot d - g + 1$$

to expect a finite answer.

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- Bertram-Daskalopoulos-Wentworth (1996) enumerated f virtually, recovered by Siebert-Tian, Marian-Oprea, Buch-Pandharipande

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(If $d > 2g - 2$), then

$$\overline{\mathrm{Mor}}_d(C, \mathbb{P}^r) \to \mathrm{Pic}^d(C)$$

is a projective bundle with fibers

$$\mathbb{P}(H^0(C, \mathcal{L})^{r+1}) = \mathbb{P}^{(r+1)(d-g+1)-1},$$

hence $\mathrm{vir}\dim(\overline{\mathrm{Mor}}_d(C, \mathbb{P}^r)) = g + (r + 1)(d - g + 1) - 1.$

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In fact, $\overline{\mathrm{Mor}}_d(C,\mathbb{P}^r) \cong \mathrm{Quot}(\mathcal{O}_C^{r+1},1,d)$.

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cut out by r linear equations.

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Then,

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However, this calculation is incorrect in general! Rather, it computes instead the "virtual Tevelev degrees" of $\mathbb{P}^r$:

$$\mathrm{vTev}^{\mathbb{P}^r}_{g,n,d} = (r+1)^g,$$

where

$$\mathrm{vTev}^{X}_{g,n,\beta} := \mathrm{virdeg}(\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n} \times X^n).$$

Virtual count $\neq$ actual count

Issue

There is excess intersection:

$$\mathrm{Tev}^{\mathbb{P}^r}_{g,n,d} \neq \# \bigcap_{i=1}^{n} \mathrm{Inc}(p_i, x_i) \neq \int_{[\overline{\mathrm{Mor}}_d(C,\mathbb{P}^r)]^{\mathrm{vir}}} \prod_{i=1}^{n} [\mathrm{Inc}(p_i, x_i)]$$

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However, this can only happen if $d \geq n$, and is in some sense the worst behavior:

Theorem (Farkas-L. 2021)

If $d \geq rg + r$, then $\mathrm{Tev}^{\mathbb{P}^r}_{g,n,d} = (r+1)^g$.

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Hope: resolve excess intersection by blowing up $\overline{\mathrm{Mor}}_d(C, \mathbb{P}^r)$ along loci where f_j are dependent. This gives the space of "complete collineations" (Vainsencher, Thaddeus, ...)

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Consider instead intersection of proper transforms:

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A direct calculation on $\mathrm{Coll}_d(C, \mathbb{P}^r)$ is difficult. Instead, proceed by *degeneration*.

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Completes ideas of Gillespie–Reimer-Berg (2021):

Schubert calculus $\to$ count of Young tableaux $\xrightarrow{\text{RSK}}$ count of words

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  We know how to do the virtual enumeration in many cases (e.g. Grassmannians, hypersurfaces, some toric varieties), but removing the excess intersections would require better Brill-Noether results.