The stabilized ellipsoidal symplectic embedding problem and scattering diagrams

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Plan of lecture:

I will explain joint work with Kyler Siegel that solves the stabilized symplectic embedding problem for ellipsoids into rigid Fano surfaces. The key is to translate this into a problem of constructing suitable cuspidal curves, and then to investigate this via almost toric fibrations, scattering diagrams and their symmetries.
The talk will be based on the papers arXiv:2404.00561 and arXiv:2412.14702. (all other references can be found there.)

1. The stabilized ellipsoidal embedding capacity function
2. Relevance of cuspidal curves
3. ATFs (almost toric fibrations) and Looijenga pairs
4. Finding cuspidal curves via scattering diagrams

Embedding ellipsoids

- $E(a_1, \dots, a_n) = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_i \frac{\pi |z_i|^2}{a_i} \le 1\right\}$.

- When does $E(a_1, \dots, a_n)$ embed symplectically in $E(b_1, \dots, b_n)$?

- We have a complete answer in dimension 4, but not even a guess as to the correct answer in dimensions > 4

- In dim 4 we know exactly when $E(a_1, a_2)$ embeds into a toric manifold or toric domain in terms of ECH capacities (work of McDuff, Hutchings, Cristofaro-Gardiner).

- Or we can use the fact that for $a/b \in \mathbb{Q}$, we can cut $E(a, b)$ into a union of balls $\bigsqcup_{i=1}^k B(w_i)$ in such a way that $E(a, b) \overset{s}{\hookrightarrow} \mathbb{C}P^2(\mu)$

iff there is a symplectic form ω on $\mathrm{Bl}^k(\mathbb{C}P^2(\mu))$, the k-fold blowup of $\mathbb{C}P^2(\mu)$, with blowup divisors E_i of sizes w_i,

iff the class $\alpha = PD(\mu L - \sum w_i E_i)$ takes positive values on *all* exceptional divisors in $\mathrm{Bl}^k(\mathbb{C}P^2)$.

Thus the embedding obstructions come from exceptional curves in $\mathrm{Bl}^k(\mathbb{C}P^2)$.

(exceptional curves $=$ symplectically embedded 2-sphere of self-intersection -1)

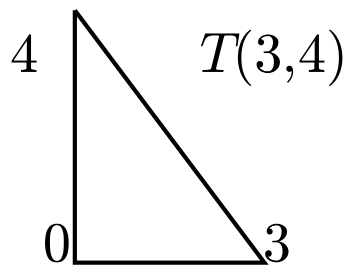
The cutting procedure: Toric models

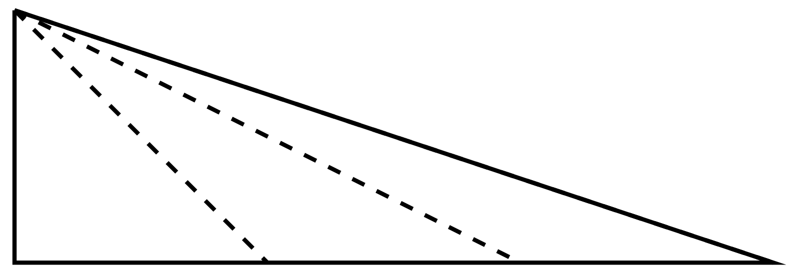
Using coordinates $(t = \pi|z|^2, \theta = \arg z)$ and forgetting θ gives a map

$$\mathbb{C}^2 \to \mathbb{R}^2, \quad (z_1, z_2) \mapsto (t_1, t_2) := (\pi|z_1|^2,\ \pi|z_2|^2).$$

It takes the ellipsoid $E(a,b) = \left\{\pi\frac{|z_1|^2}{a} + \pi\frac{|z_2|^2}{b} \le 1\right\}$ to the triangle

$$T(a,b) := \ \left\{(t_1,t_2) \in \mathbb{R}^2_+ :\ 0 \le t_1, t_2, \frac{t_1}{a} + \frac{t_2}{b} \le 1\right\}.$$





The ball $B(1)$ maps to the standard triangle $T(1,1)$, with various affine equivalent images given by integral changes of basis of $\mathbb{C}^2$ (eg use $(z_1, z_1 + z_2)$). Thus $T(1,3)$ can be cut into three standard triangles; cf. diagram on right.

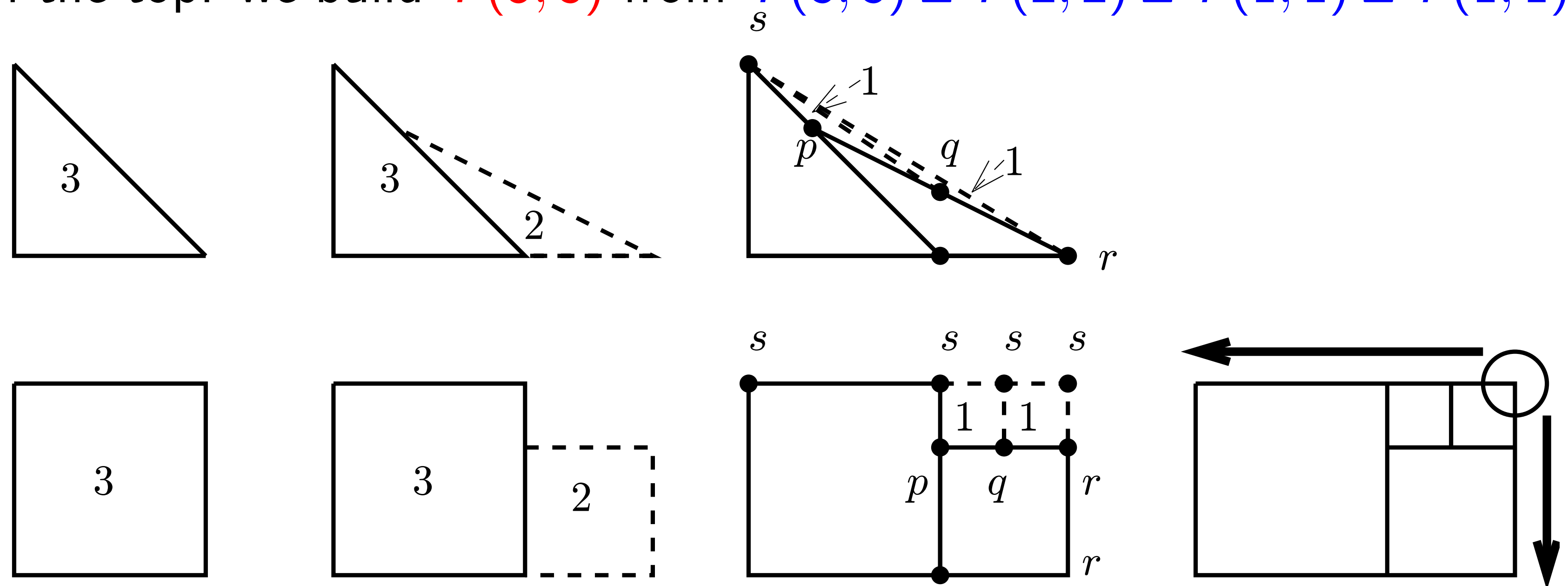
Hence $E(1,3)$ contains three disjoint balls $B(1) \sqcup B(1) \sqcup B(1)$.
and one can prove that $E(1,3)$ embeds into $\mathbb{C}P^2(\mu)$ iff $B(1) \sqcup B(1) \sqcup B(1)$ does iff $\exists$ symplectic form on $\mathbb{C}P^2(\mu)\#3\overline{\mathbb{C}P}^2$ in which each blowup has size 1.

(blowup of size a= cut out ball of size a and collapse bdry to S^2 via Hopf flow)

General Triangle decompositions

In fact any triangle $T(a,b)$ with $a,b \in \mathbb{Z}$ can be decomposed into standard $\triangle$s of different sizes, (a standard triangle is affine equiv to some $T(c,c)$)

On the top: we build $T(5,3)$ from $T(3,3) \sqcup T(2,2) \sqcup T(1,1) \sqcup T(1,1)$.



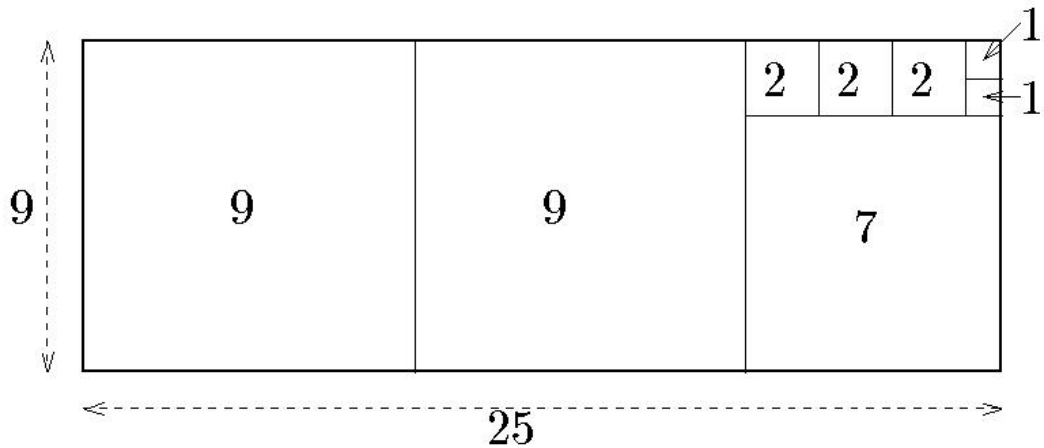
On bottom: we build a rectangle from squares. Combinatorially these decomp are same, but the second is much easier to understand.

To get the triangles from the rectangles, remove the top right point and collapse top and right sides to points in affine way.

This pattern of blowups (or cutting) is precisely what is required to resolve the (5,3)-cusp singularity.

The combinatorics of this decomposition of a triangle into standard triangles (or of a rectangle into squares), is the same as that governing the weight expansion $W(\frac{p}{q})$ of a rational fraction $\frac{p}{q}$. e.g.
$W(\frac{25}{9}) = (\underbrace{9, 9}, 7, \underbrace{2, 2, 2}, \underbrace{1, 1})$

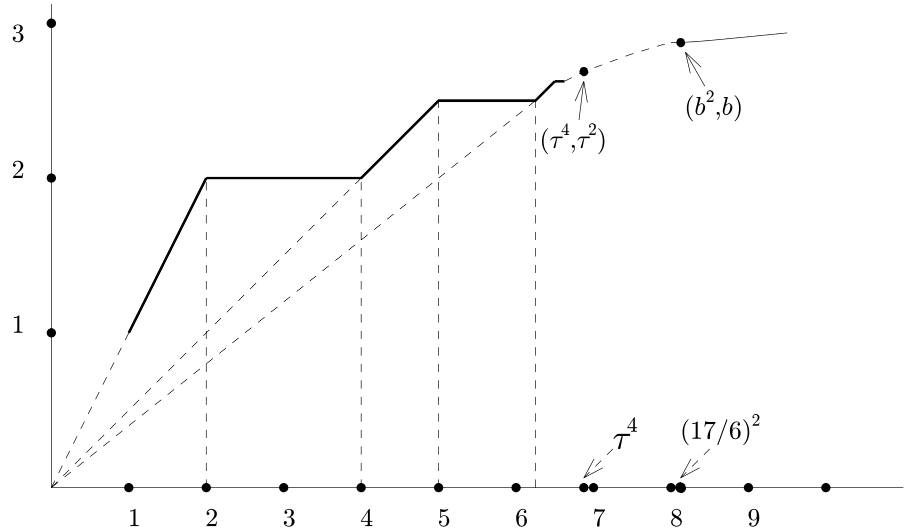
corresponds to



$$\frac{25}{9} = [2; 1, 3, 2] = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{2}}} = 2 + \cfrac{1}{1 + \frac{2}{7}} = 2 + \tfrac{7}{9}.$$

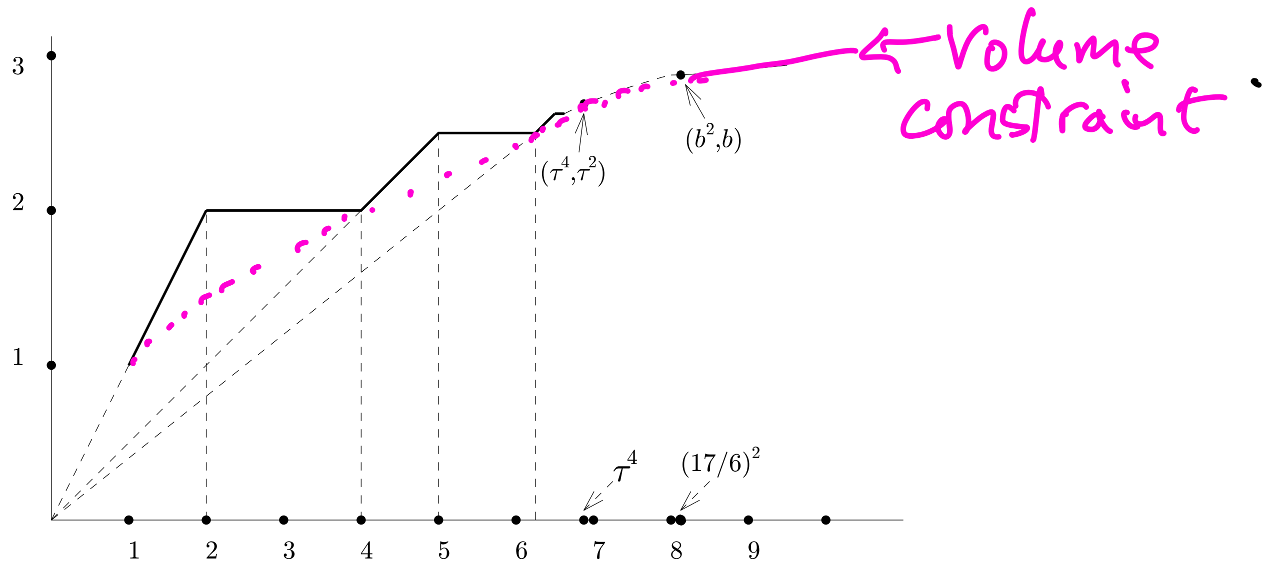
The entries of the continued fraction are the multiplicities of the weights

For $x \geq 1$ define $c(x) := \inf\{\mu : E(1,x)$ embeds sympl. in $B^4(\mu)\}$.
This function was calculated by McDuff–Schlenk (2012).



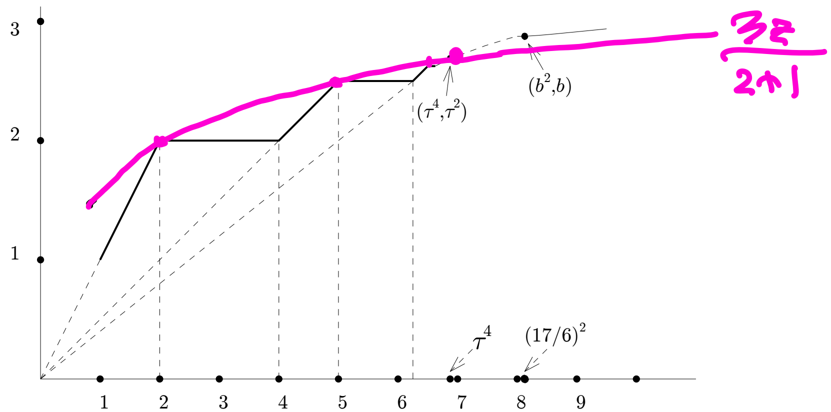
- For $x < \tau^4 \approx 6.7$ (where $\tau = \frac{1+\sqrt{5}}{2}$) there is an infinite staircase (with numerics based on the Fibonacci numbers),
- for $x \geq 8\frac{1}{36} = \left(\frac{17}{6}\right)^2$, $c(x) = \sqrt{x}$ – no obstruction except for volume
- $\tau^4 < x < 8\frac{1}{36}$ is a transitional region;

Fibonacci staircase continued



- Schlenk and I found that the exceptional classes in blow ups of $\mathbb{C}P^2$ that gave the sharpest obstruction to embedding $E(p,q)$ had the form $dL - \sum m_i E_i$,where $(m_i) = W(p/q)$ and d, p, q are odd placed Fibonacci numbers. These are called perfect classes.
- Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- For example at $p/q = 2/1$ with $W(2) = (1^{\times 2})$ the obstruction is $L - E_{12} = L - E_1 - E_2$ (blow up of a line through 2 generic points.
- at $p/q = 5/1$ with $W(5) = (1^{\times 5})$ the obstruction is $2L - E_{1\cdots 5}$ (blow up of a conic through 5 generic points.
- at $p/q = 13/2$, we have $W(13/2) = (2^{\times 6}, 1^{\times 6})$ and the obstructive class is $5L - 2E_{1\cdots 6} - E_{78}$.

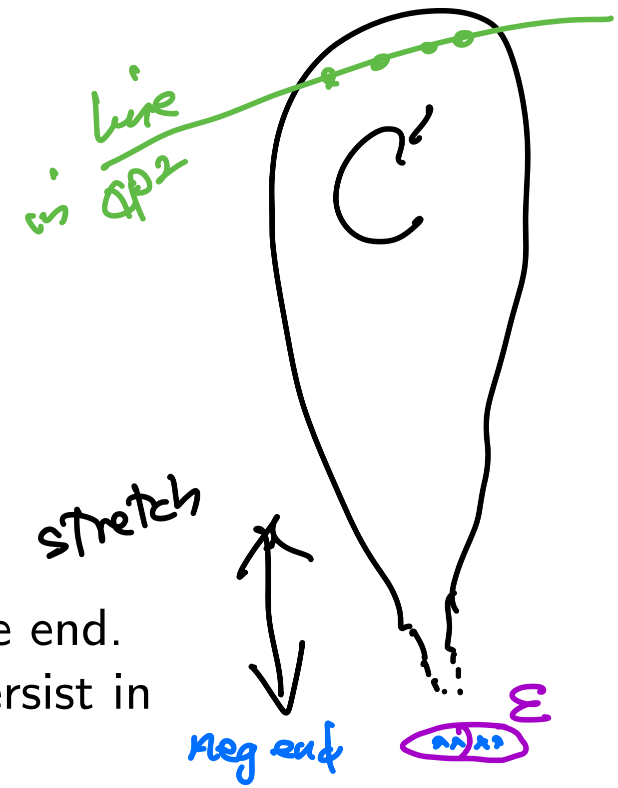
The stabilized embedding problem (for the ball)



- work of Richard Hind and Ely Kerman shows that the Fibonacci staircase obstructions stabilize (i.e. persist when domain and target is multiplied by $\mathbb{R}^2$ (or $\mathbb{R}^{2k}$.) The outer corners (z, y) all satisfy $y = \frac{3z}{z+1}$.
- Hind developed a very clever version of the folding construction that allowed him to embed $E(1, z, S)$ into $\mathbb{C}P^2(\frac{3z}{z+1} + \varepsilon) \times \mathbb{R}^2$ for all $S > 0$.
- work of Pelayo–Vu Ngoc shows that one can therefore embed $E(1, z) \times \mathbb{R}^2$ into $\mathbb{C}P^2(\frac{3z}{z+1} + \varepsilon) \times \mathbb{R}^2$.
- the graph of the folding curve $y = \frac{3z}{z+1}$ crosses the volume constraint $y = \sqrt{z}$ precisely at the accumulation point (τ^4, τ^2) of the staircase.
- the natural conjecture is that the stabilized embedding function for $z > \tau^4$ is given by the folding curve $\frac{3z}{z+1}$.
- to prove this we need to find enough embedding obstructions. These will be curves in 4-dimensions that persist in higher dimensions.

The stabilized embedding problem (for the ball)

Construction method: to find a representative for the exceptional divisor in class $E = dL - \sum_{i=1}^{k} W_i E_i$, where W_i are the integral weights of p/q.



- embed a small copy $\mathcal{E}$ of $E(1, p/q + \varepsilon)$ into $\mathbb{C}P^2$. (where $\varepsilon > 0$ is very small)
- blow up k points inside this ellipsoid and let C be a J-holomorphic curve in class E.
- then stretch the neck around the boundary of the ellipsoid to obtain a curve C' of degree d in X = completion of $\mathbb{C}P^2 \backslash \mathcal{E}$.
- one can show that C' has one negative end that winds p times around the short orbit of the negative end. By Hind–Kerman these curves stabilize, i.e. they persist in $X \times \mathbb{R}^{2k}$ and remain J-holomorphic as J varies. So one can change the size and the embedding of the ellipsoid in $X \times \mathbb{R}^{2k}$ and this curve does not disappear. Since it must have positive symplectic area, the ellipsoid cannot grow. too big
- one can show that the obstruction is precisely $3p/(p+q)$, i.e. it lies on the folding curve

  ie the area of C'

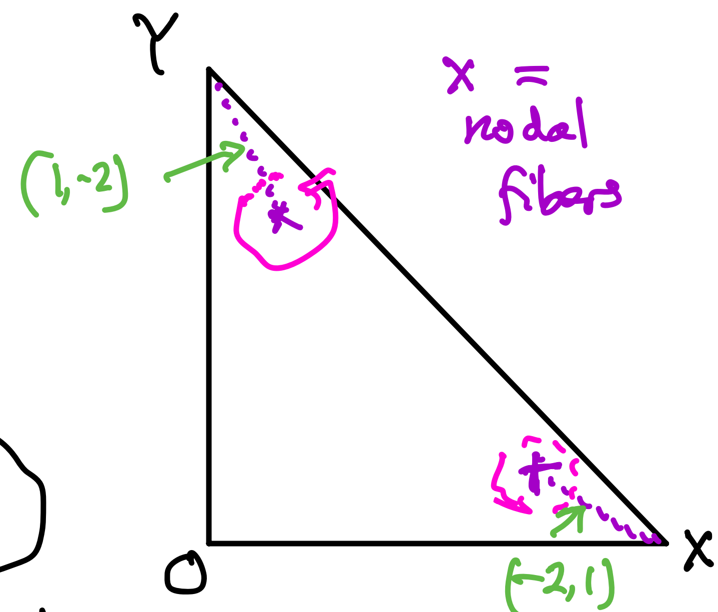
The stabilized embedding problem (for the ball)

- The key point of the stabilization argument is that the curve must be genus zero, regular (i.e. persists under deformation), with Fredholm index zero, and one negative end on the ellipsoid; it may have double points.
- Siegel constructs a diffeomorphism ϕ between a neighbourhood of 0 in $\mathbb{C}^2 \setminus \{0\}$ and the negative end of $X = \mathbb{C}P^2 \setminus \delta E(1, p/q + \varepsilon)$ that takes the standard complex structure on $\mathbb{C}^2 \setminus \{0\}$ to an adapted almost complex structure J_X near the neg end of X. The inverse ϕ^{-1} takes a J_X-hol. curve with one neg end going p-times round the short orbit to a holomorphic curve with a p, q-cusp at 0. – we call these sesquicuspidal curves.
- So the problem becomes: how to find these curves?
- It turns out that one can see all the curves responsible for the Fibonacci staircase via almost toric structures (ATFs).
- Also one can transform the ATF picture (which is symplectic) to a holomorphic model called a Looijenga pair, and from there use scattering diagrams to construct all the needed curves.
- This approach also works for the other rigid Fano surfaces (with their monotone symplectic form) i.e. $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the blowup of $\mathbb{C}P^2$ by up to 3 points.

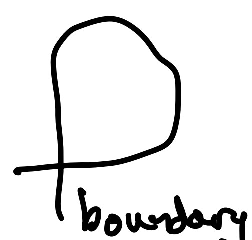
Almost toric fibrations

Two groups (Casals and Vianna, and Cristofaro-Gardiner, Holm, Mandini and Pires) showed how to use almost toric fibrations (ATFs) for construct the (full) fillings giving the inner corners of the Fibonacci staircase, while Siegel and I show that the obstructive curves giving the outer corners can also be seen from this perspective.

An almost toric fibration $\pi : M \to B$ on a symplectic 4-manifold is a map whose generic fiber is a Lagrangian torus, and with singular fibers that are either points and circles (as in the toric case) or a pinched (Lagrangian) torus. There is no global T^2 action because there is monodromy round the singular fiber. The monodromy matrix is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with fixed vector equal to the nodal ray.

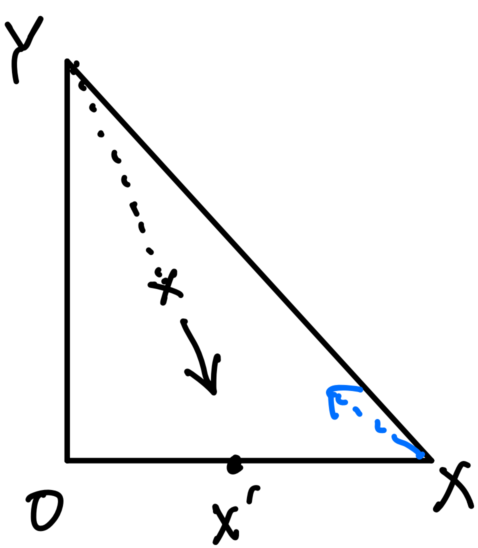


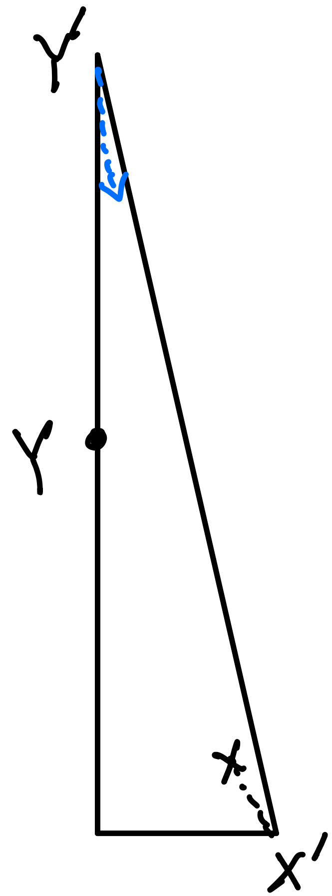
The monodromy action at Y fixes the nodal ray $(1,-2)$ and takes the vector YV to the vertical. Despite appearances, there is no vertex at X or Y so that the boundary is a 2-sphere with one self-intersection (at 0) that represents the anticanonical divisor.



Mutations of almost toric fibrations

The operation is: slide the node (near Y say) in the direction of the nodal ray, and then when one is close to another edge, switch the sign of the nodal ray so that it hits that new edge at X' say; then the base of the ATF has no vertex at Y, but does have a new vertex at X'. To get the new base diagram we apply the mutation matrix M to one half of the diagram.





Monodromy M

$$M = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$$

$$M(\overrightarrow{X'X}) = \overrightarrow{X'Y'}$$

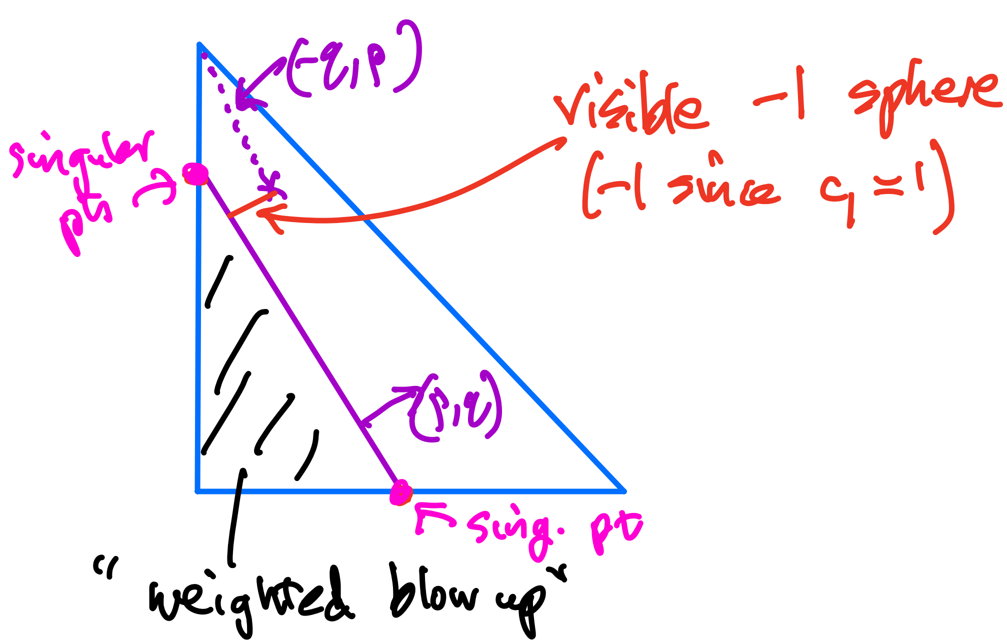
$$M(\overrightarrow{YX'}) = \overrightarrow{YX'}$$

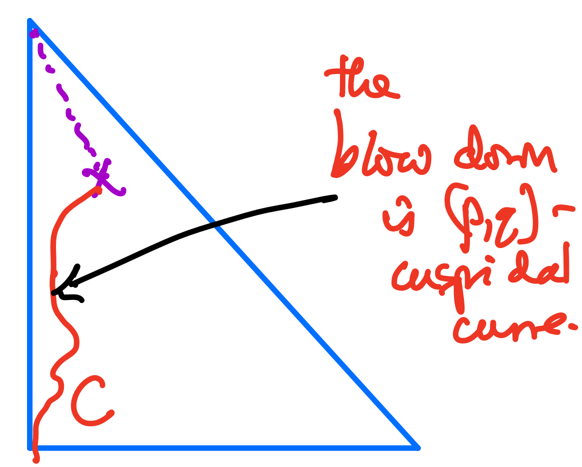
$$M(\overrightarrow{XY}) = \overrightarrow{Y'Y}$$

new nodal ray at Y' is M(nodal ray at X)

Visible symplectic curves in almost toric fibrations I

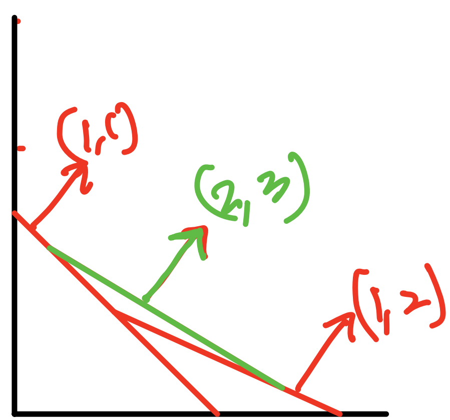
If there is a nodal ray in direction $(q, -p)$ at a vertex adjacent to a smooth vertex 0, do a weighted blowup at 0 with normal (p, q). Then the resulting ATF contains a visible symplectic 2-sphere C with $c_1(C) = 1$ whose projection to the base is in the direction of this normal, and meets the node perpendicularly to the nodal ray. The inverse image of any point on the line is the fixed circle that collapses at the nodal point. This is a symplectically embedded sphere, with $c_1(C) =$ intersection number with the boundary, (by Symington).
After blowing down at the origin, we obtain a curve with a (p, q)-cusp that intersects the anticanonical divisor $\mathcal{N}$ only at 0. We say it is (p, q)-**well-placed**: it intersects one branch of $\mathcal{N}$ to order p and the other to order q.





Visible symplectic curves in almost toric fibrations II

- All the exceptional curves that give the outer corners of the Fibonacci staircase can be constructed in this way from mutations of the standard ATF for $\mathbb{C}P^2$.



To resolve a (2,3) cusp you blow up 3 times

$(2,3)$

$t \mapsto ([t^2,t^3], t^2, t^3) = ([1,t], t^2, t^3)$

coords (t,t^2) $(1,2)$

$t \mapsto ([t,t^2], t, t^2) = ([1,t], t, t^2)$

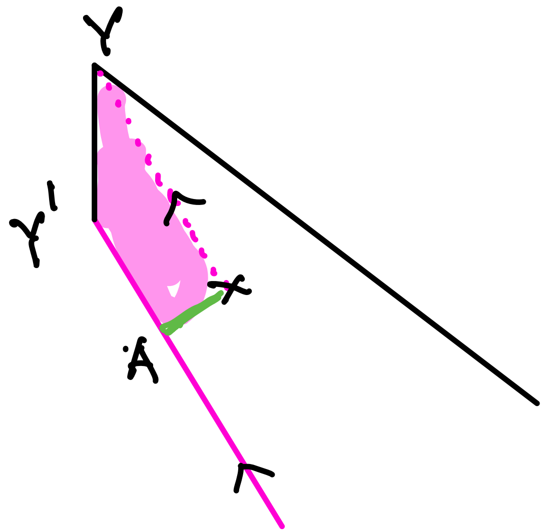
coords (t,t) $(1,1)$

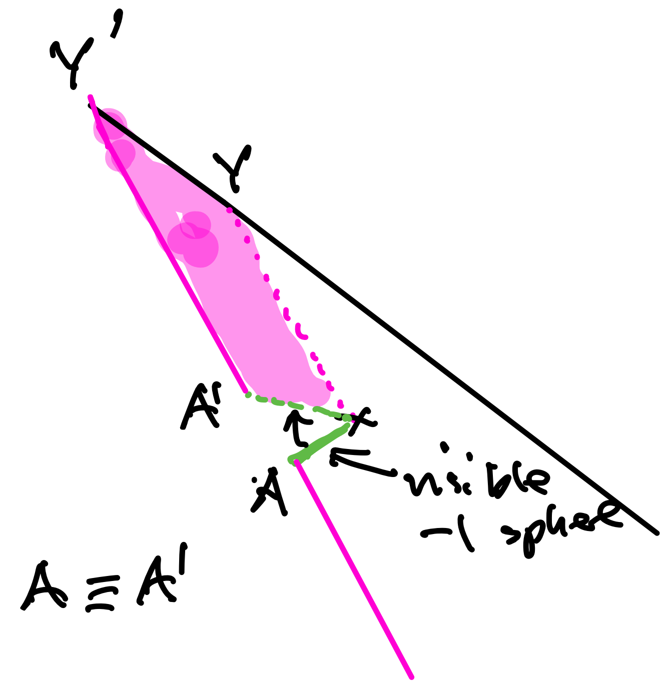
- More generally, any rational curve in X that is symplectically immersed with positive self-intersections and that locally projects to a line in direction (p,q) through 0 represents a (p,q)-cuspidal curve. If it is otherwise disjoint from the boundary divisor we call it a (p,q) well-placed sesqui-cuspidal curve. We can see these via scattering diagrams.

Holomorphic models from ATFs I.

Given an ATF with a nodal ray that is parallel to one edge of the ATF, we change its base diagram by rotating the branch cut by a right angle so that it meets that edge perpendicularly. The base diagram then has a triangle cut out of it, and the monodromy fixes the nodal direction and takes NA to NB.

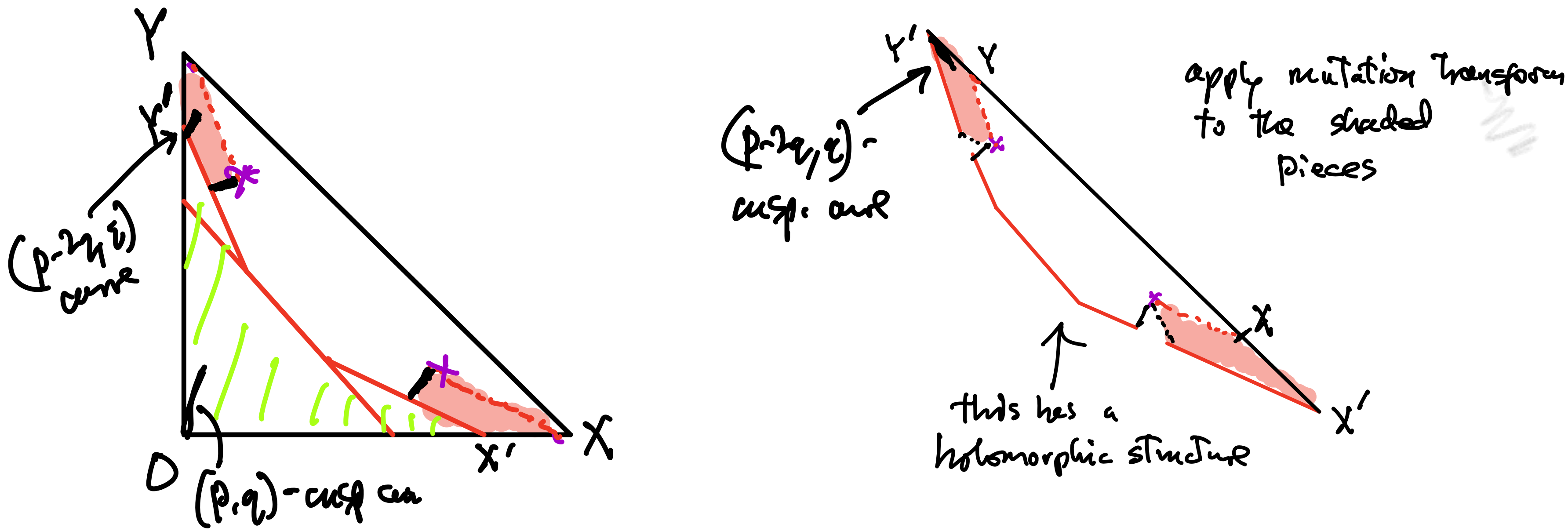
In this picture there is a visible symplectic -1 sphere lying over NA





Holomorphic models from ATFs II

- If we rotate the branch cut like this at both nodes of the standard ATF for $\mathbb{C}P^2$ we must first blow up at the smooth vertex 0 three times to get sides parallel to the nodal rays, and then we get a holomorphic model of $(Bl^3(\mathbb{C}P^2), \mathcal{N})$ (called a Looijenga pair).

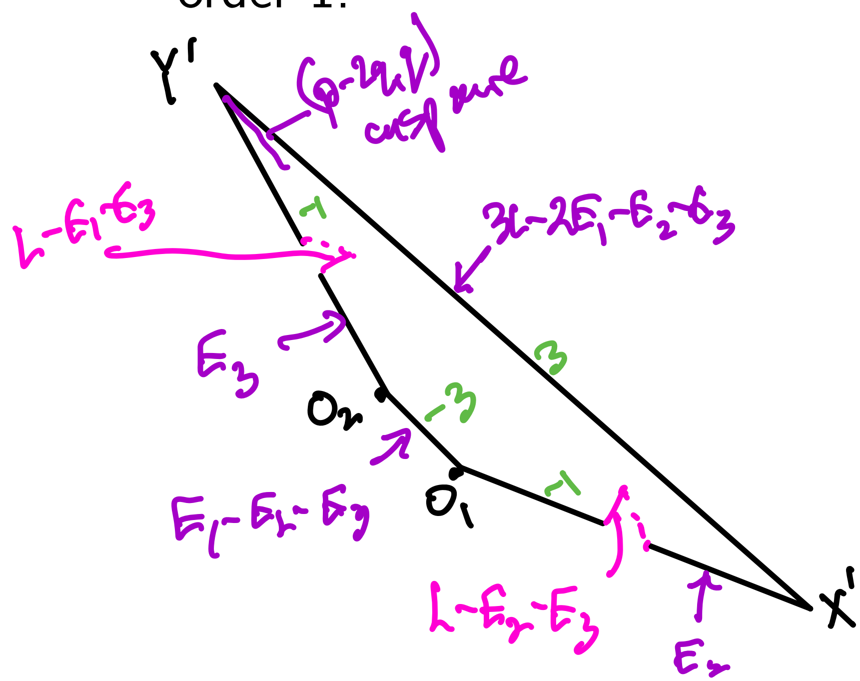


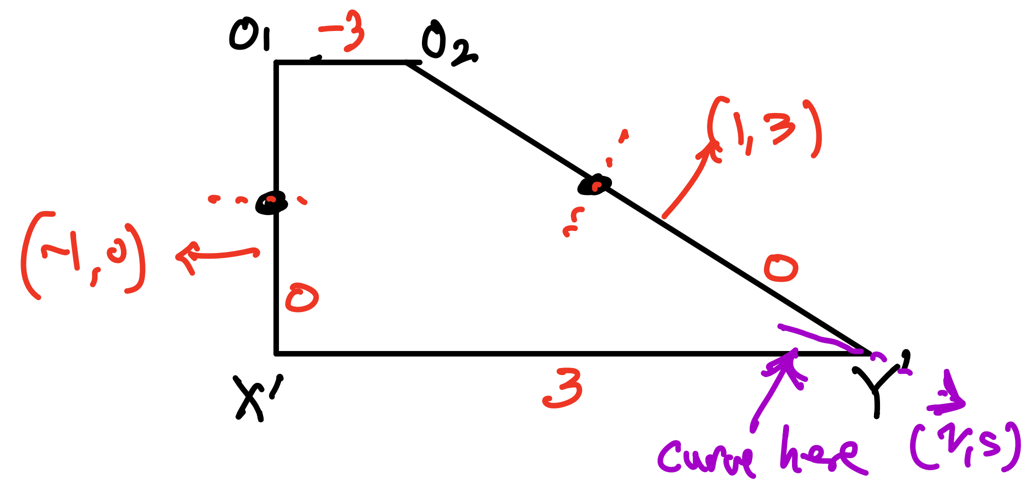
- Under this operation a (p, q)-cuspidal curve which is p-fold tangent to the y axis and where $p > 2q$ becomes a $(p - 2q, q)$-cuspidal curve $(p - 2q)$-fold tangent to ~~the y axis~~. This curve will intersect the two exceptional divisors but only intersects the boundary at its cuspidal point.

Y'Y

The scattering diagram picture I

- The boundary of the diagram consists of four curves of self-intersections $3, -1, -3, -1$. If we blow down the -1 curves, this boundary is the same as that of the toric manifold F_3 (the third Hirzebruch surface.)
  So our curve C sits in an edge blowup X of F_3; it intersects the two exceptional divisors that lie over points on ∂F_3, and is well-placed with respect to the anticanonical divisor of X. It is an (r, s) cuspidal curve and its normal crossing resolution intersects the edge with normal (r, s) to order 1.

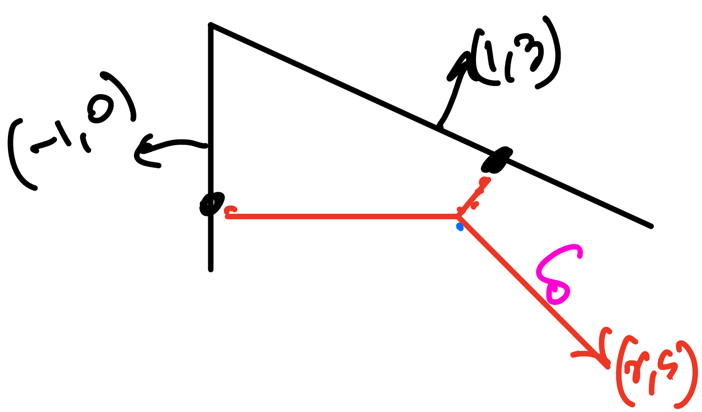


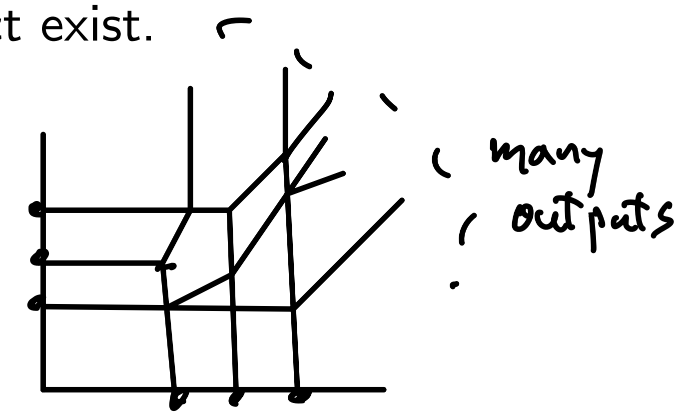


$$(r,s) = q(1,3) + (p-2q)(0,-1)$$
$$= (q, -p+5q)$$

The scattering diagram picture II

- In the scattering diagram we ignore all parts of the previous diagram except the two edges that contain the blown up points and the output ray that represents the normal crossing resolution of C. These two edges have outward normals $(-1,0)$ and $(1,3)$ and in these coordinates the output is $q(1,3)+(p-2q)(0,-1)=(q,5q-p)$. The scattering diagram assigns a function f_δ (a power series in t) to each ray δ in the direction $k(1,0)+\ell(-1,-3)=(r,s)$ for $k,\ell \geq 0$. If $\gcd(r,s)=1$, the first nonzero coefficient of this function is a count of the number of curves that meet the edge with normal (r,s) exactly once. So to show that C exists we just need to see that this coefficient is nonzero.
- The three-fold cover of our scattering diagram is of the standard type analyzed by Gross, Pandharipande, Siebert and it follows from their work that the curves we want to find do in fact exist.





The 3-fold cover is standard scattering diagram