Intersection Theory of Compactified Jacobians

1. The problem

$$\mathrm{Jac}(\mathcal{C}_{g,n}) = \{(C, x_1, \ldots, x_n, L)\}$$

abelian scheme

dim $4g-3+n$

$$\big\downarrow$$

$$\mathcal{M}_{g,n}$$

dim $3g-3+n$

1. The problem

For discrete data $A = (a_1, \dots, a_n)$ $\quad \sum a_i = 0$

$$\mathrm{Jac}(C_{g,n}) = \{(C, x_1, \dots, x_n, L)\}$$

abelian scheme

dim $4g-3+n$

$$\begin{tikzcd}
\mathrm{Jac}(C_{g,n}) \arrow[d] \\
M_{g,n} \arrow[u, bend left=50, "0"] \arrow[u, bend right=50, "\sigma_A"']
\end{tikzcd}$$

$\sigma_A(C, x_1, \dots, x_n)$

$= \mathcal{O}_C(\sum a_i x_i)$

dim $3g-3+n$

Setup introduces several cycles

(1) "Double Ramification"

locus $= a_{J_A} \cap [0] = \{ (C, x_1, \dots, x_n) \mid \mathcal{O}(\sum a_i x_i) \simeq \mathcal{O} \}$

cycle $= a_{J_A}^* ([0]) \in CH^g(M_{g,n})$

Setup introduces several cycles

(1) "Double Ramification"

$$\text{locus} = a_{J_A} \cap [0] = \{(C, x_1, \dots, x_n) \mid \mathcal{O}(\textstyle\sum a_i x_i) \simeq \mathcal{O}\}$$

$$\text{cycle} = a_{J_A}^*([0]) \in CH^g(M_{g,n})$$

(2) "Higher ramification" $A_1, \dots, A_\ell$

$$\{(C, x_1, \dots, x_n) \mid \mathcal{O}(\textstyle\sum a_{i1} x_i) \simeq \mathcal{O}(\sum a_{i2} x_i) \simeq \dots \simeq \mathcal{O}(\sum a_{i\ell} x_i) \simeq \mathcal{O}\}$$

$$\prod_{i=1}^{\ell} a_{J_{A_i}}^*([0]) \in CH^{\ell g}(M_{g,n})$$

(3) Brill-Noether cycles

Can change $a_{J_A} = \omega^k(\sum a_i x_i)$, $k(2g-2) + \sum a_i = d$

locus $W^d_{g,r,A} = \{(C, x_1, \dots, x_n) \mid h^0(\omega^k(\sum a_i x_i)) \geq r+1\}$

cycle $w^d_{g,r,A} = \Delta^{r+1}_{g+r-d}\, c(-R\pi_* \omega^k(\sum a_i x_i))$

"determinantal" formula

$r=0$: $= c_{g-d}(-R\pi_* \omega^k(\sum a_i x_i))$

Three Problems ($CH(\overline{\mathcal{M}}_{g,n})$ membership card)

(A) Meaningfully extend classes to $\overline{\mathcal{M}}_{g,n}$.

(B) Are extensions in $R^*(\overline{\mathcal{M}}_{g,n})$

(C) Calculate class of extensions

(in terms of standard generators of $R^*(\overline{\mathcal{M}}_{g,n})$)

Of these problems, A has been solved:

(1) + (2) via Gromov-Witten theory

(For experts: (1) $DR_{g,A} = p_*\left(\left[K^{\sim}_{g,A}(\mathbb{P}^1, 0+\infty)\right]^{vir} \right)$

"rubber" relative stable maps

(2) $DR_{g,A_1,\ldots,A_\ell} = p_*\left(\left[K^{\sim}_{g,A_1,\ldots,A_\ell}(\mathbb{P}^1 \times \ldots \times \mathbb{P}^1)\right]^{vir} \right)$)

(3) Via compactified Jacobians

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Graber-Vakil 2001

"rubber" relative stable maps

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Abramovich-Chen-Gross-Siebert 2011

(3) Via compactified Jacobians (Pagani-Ricolfi-van Zelm 2018)

$$\boxed{\text{Caution}}$$

(2) $\mathrm{DR}_{g, A_1, \dots, A_\ell} \neq \prod_{i=1}^{\ell} \mathrm{DR}_{g, A_i}$ on $\overline{\mathcal{M}}_{g,n}$

(3) $w^d_{g,r,A}$ <u>not</u> determinantal locus on $\overline{\mathcal{M}}_{g,n}$

B : (1) was solved by Faber-Pandharipande (2002)

(2)-(3) were conjectured

C : (1) solved by Janda-Pandharipande-Pixton-Zvonkine (2013)

(2)-(3) out of reach

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Goal for today : - Describe framework in which we can approach all problems on same footing and solve A, B, C for all (1)-(3).

Why do we care about these classes?

1. The $DR_{g,A_1,\dots,A_n}$ completely determine the log GW theory of toric varieties.

   (Thm of Ranganathan - Urundolil Kumaran)

2. The $w^d_{g,r}$ determine relations in $R^*(\overline{M}_{g,n})$

   For example, $w^{d<0}_{g,0,A} = 0$

Point of view: problems are hard, because extending Jacobians to $\overline{M}_{g,n}$ is subtle:

$$\boxed{\begin{tikzcd} \mathrm{Jac}(C_{g,n}) \arrow[r, phantom, "\subset"] \arrow[d] & ? \arrow[d] \\ M_{g,n} \arrow[r, phantom, "\subset"] & \overline{M}_{g,n} \end{tikzcd}}$$

Obvious candidates fail. Not compact, Hausdorff.

Less naive canditates don't have a_J.

Our problem requires compactification with Abel-Jacobi Section

Main Results

1. **Thm**: (M-Wise, 18?) An optimal compactification exists:

$$\begin{tikzcd}
\mathrm{Jac}(C_{g,n}) \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{LogJac}(\overline{C}_{g,n}) \arrow[d] \\
M_{g,n} \arrow[r, phantom, "\subset"] & \overline{M}_{g,n} \arrow[u, bend right, "a_J"']
\end{tikzcd}$$

2. **Thm**: (Holmes-M-Pandharipande-Pixton-Schmitt '22, Abreu-M-Pagani) soon

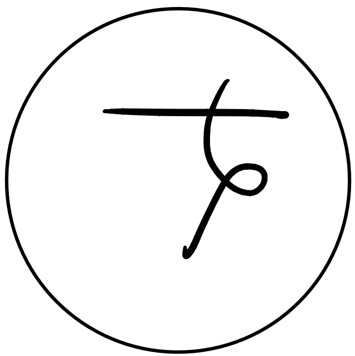
There are explicit formulas for all classes (1)-(3)

(Some of) The Techniques

Log and Tropical Geometry

Basic Object: (X, D) normal crossings pair

(or toroidal embedding)



Structures:

1. Log Structure $\quad \mathcal{O}_X^* \subset M_X \subset \mathcal{O}_X$

$$M_X = \{ f \in \mathcal{O}_X \mid f \text{ is a unit on } X - D \}$$

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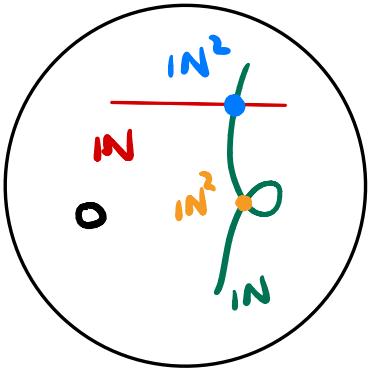
2. Stratification

$$\{ \overline{M}_X = M_X / \mathcal{O}_X^* \text{ constant} \}$$



3. Tropicalization

Around each $x \in X$, we have (etale) local chart

$$(X, D) \longrightarrow (V(\sigma_x), V(\sigma_x) - T) = \operatorname{Spec} \mathbb{Z}[\bar{M}_{X,x}]$$

toric variety

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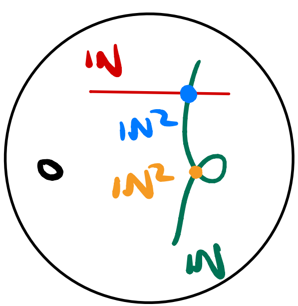
$$(X, D) \longrightarrow (V(\sigma_x), V(\sigma_x) - T) = \operatorname{Spec} \mathbb{Z}[\overline{M}_{X,x}]$$

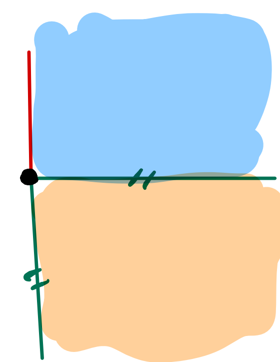
toric variety

$$(X, D) \rightsquigarrow \Sigma_{(X,D)} = \varinjlim_{x \in X} \sigma_x = \varinjlim_{x \in X} \operatorname{Hom}(\overline{M}_{X,x}, \mathbb{R}_{\geq 0})$$

tropicalization of (X, D)



$$\rightsquigarrow \Sigma_{(X,D)} =$$



Examples:

- X toric $\rightsquigarrow$ Strata $=$ T-orbits

$D = X - T$ $\qquad \Sigma_{(X,D)} =$ Fan

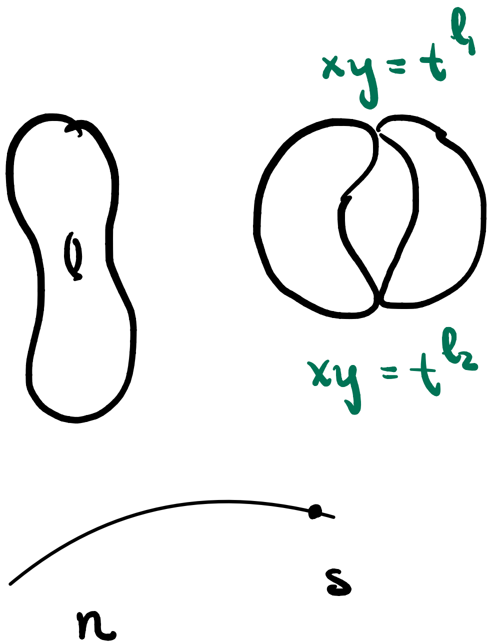
- $\overline{M}_{g,n}$, $D = \overline{M}_{g,n} - M_{g,n}$

Strata $\longleftrightarrow$ Stable graphs Γ

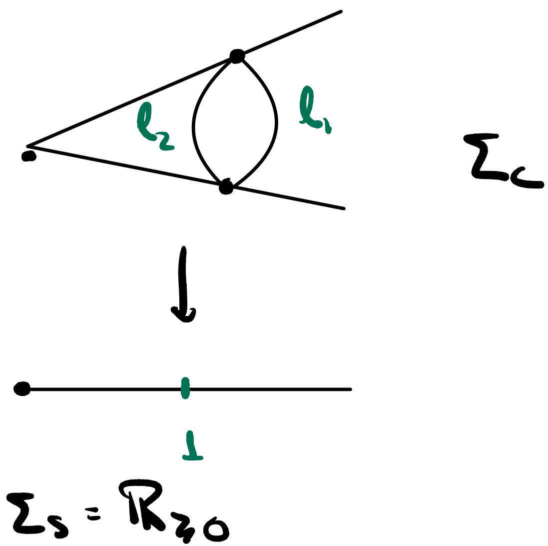
$\Sigma_{\overline{M}_{g,n}} = M_{g,n}^{trop}$ (moduli space of tropical curves)

$$= \varinjlim_{\Gamma} \mathbb{R}_{\geq 0}^{E(\Gamma)}$$

• $S = \operatorname{Spec} \mathbb{C}[[t]]$, $\quad C \longrightarrow S$ degeneration of curves



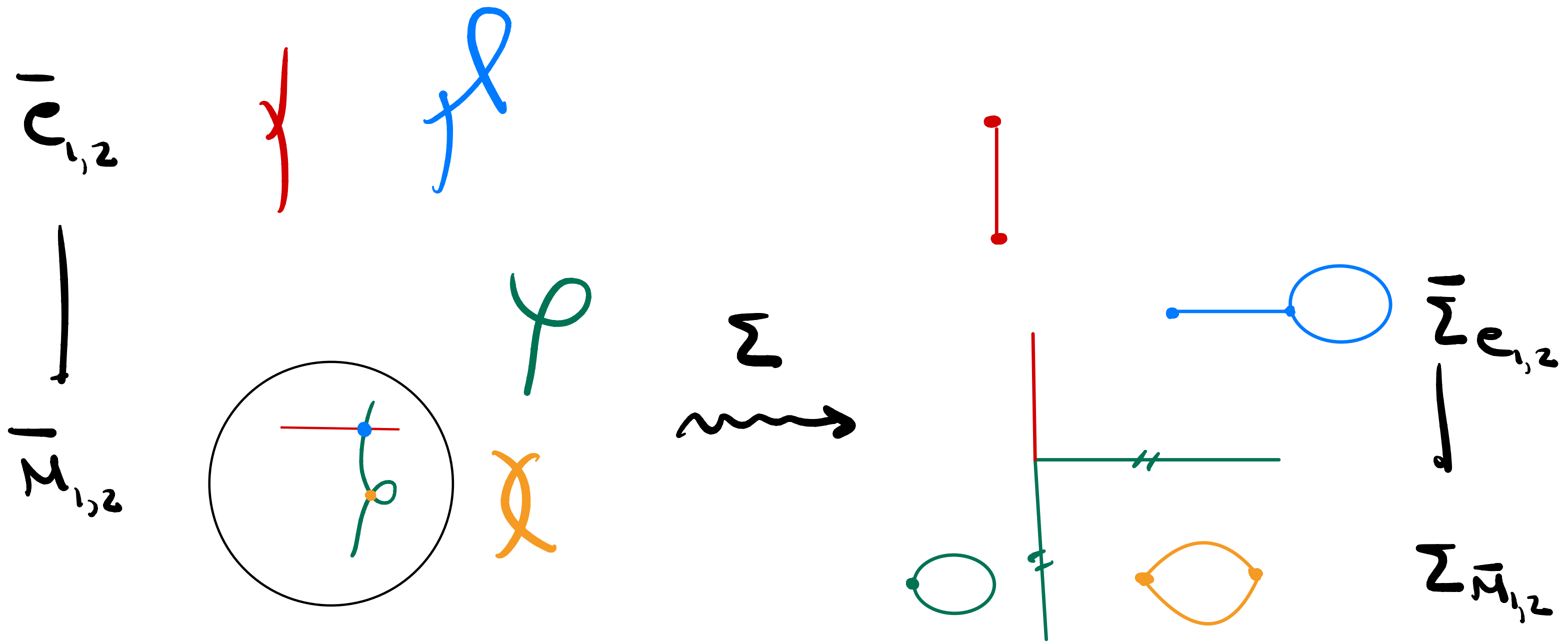
tropical curve = metric dual graph



More generally, S = toroidal, $C \longrightarrow S$ proper, flat, log smooth family of curves (a $\boxed{\text{log curve}}$)

(aka any nice family of nodal curves)

$$\begin{array}{ccc} C & & \Sigma_C \\ \downarrow & \rightsquigarrow & \downarrow \\ S & & \Sigma_S \end{array}$$ family of tropical curves

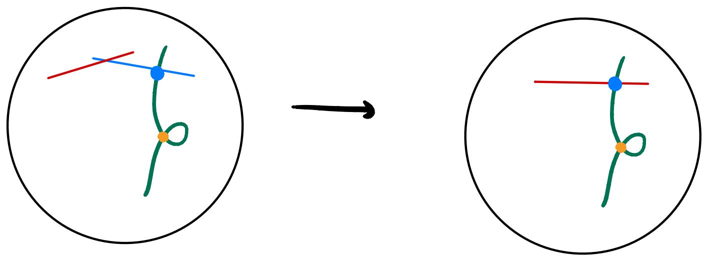


$\overline{\mathcal{C}}_{1,2}$
$\overline{\mathcal{M}}_{1,2}$
Σ
$\overline{\Sigma\mathcal{C}}_{1,2}$
$\Sigma\overline{\mathcal{M}}_{1,2}$

Operations : Log Blowups

Essentially : Pullbacks of toric blowups from local charts

Simplest Example : Blowup along smooth stratum



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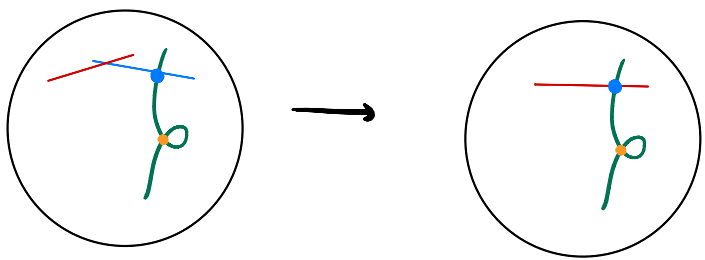
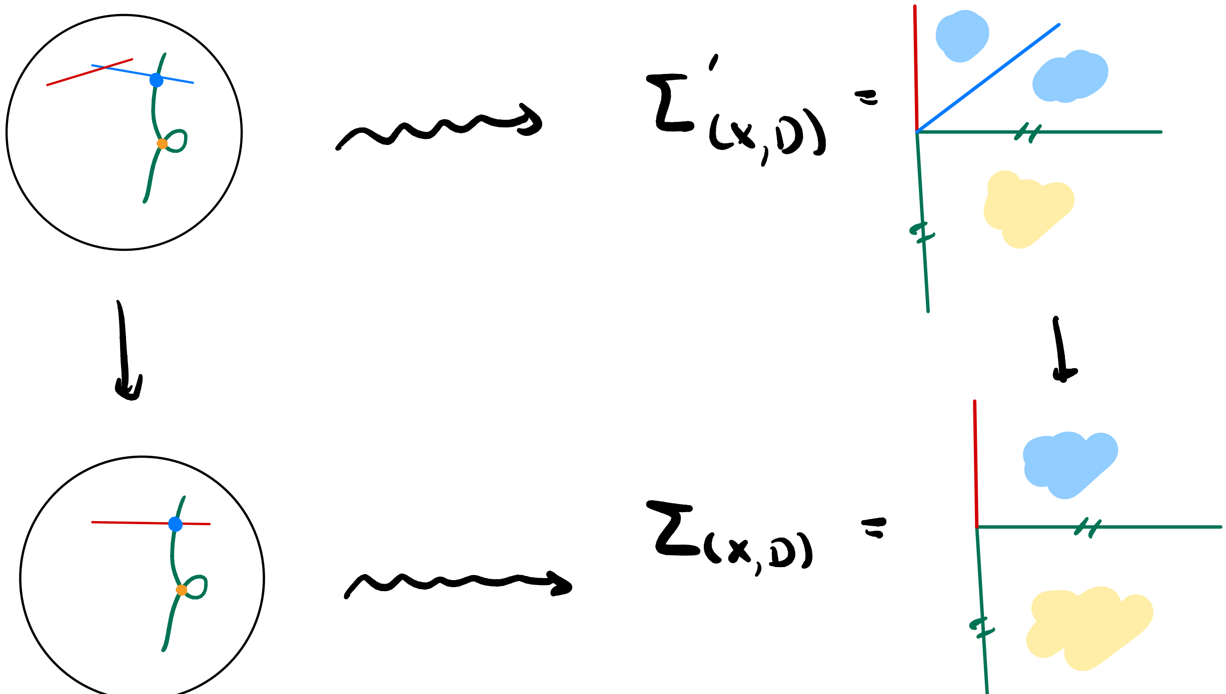


Log Blowup: Proper birational $X' \longrightarrow X$ that can be dominated by iterated strata blowups

Better: $\{\text{Log Blowups}\} \longleftrightarrow \left\{ \begin{array}{c} \text{Subdivisions} \\ \Sigma_{(X',D')} \to \Sigma_{(X,D)} \end{array} \right\}$



4. $\log CH^*(X, D) = \varinjlim_{X' \to X} CH^*(X')$

log blowups

4. $\log CH^*(X, D) = \varinjlim_{\substack{X' \to X \\ \text{log blowups}}} CH^*(X')$

- A class $\xi \in \log CH(X, D)$ is a collection of classes $\xi_Y \in CH(Y)$ for sufficiently fine $Y \to X$, s.t $\xi_Y|_W = \xi_Z|_W$ for all

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ Y \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

- There is an inclusion $CH(X) \subset \log CH(X,D)$

  and a retraction

$$ret_* : \log CH(X,D) \longrightarrow CH(X)$$

- X = toric, $\log CH(X,D)$ = McMullen's polytope algebra.

In general, full understanding way out of reach.

Nevertheless, we can work with it.

Large supply of classes come from combinatorics

<u>Thm</u> (M-Pandharipande-Schmitt, M-Ranganathan)

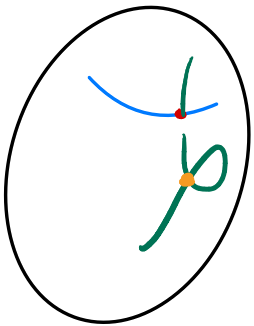
$\exists$ degree preserving homomorphism

$$\Phi : PP(\Sigma_{(X,D)}) \longrightarrow \log CH^*(X)$$

(analogue of non-equivariant limit $CH_T(V) \to CH(V)$)

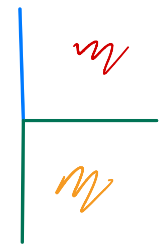
Example

(X, D)

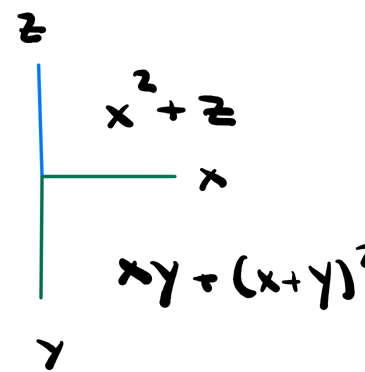


$\rightsquigarrow$

$\Sigma_{(X,D)}$



$[\smile] + [\,\cdot\,] + j_*[c_1(N(\ \xrightarrow{i} X))]$ $\xleftarrow{\phi}$



If $CH^*(X)$ has distinguished $R^*(X) \subset CH^*(X)$

$$\log R^*(X) = \langle R^*(X), PP(\Sigma_X) \rangle \subset \log CH^*(X)$$

often large enough for applications

Summary

(X,D) gives

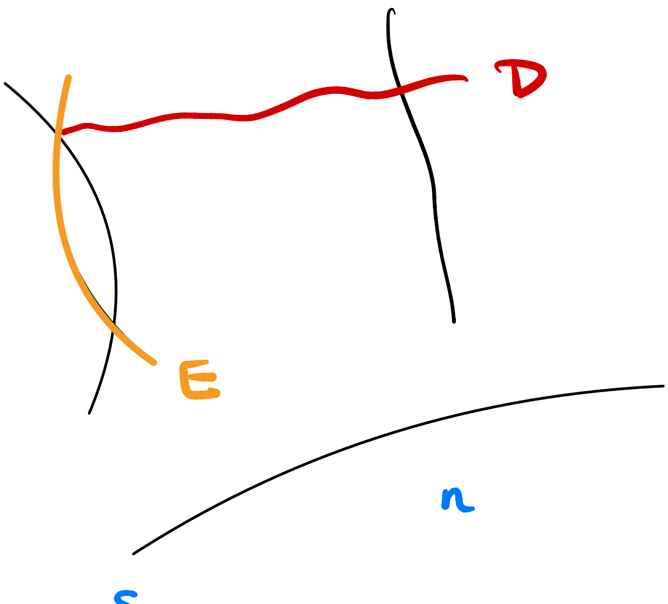
- log structure
- tropicalization
- cohomology theory $\log CH(X,D)$
- nodal curves $C \to S$ have log structures.

Controls log blowups via subdivisions

built out of $CH(X)$, $PP(\Sigma_X)$

Jacobians

Why is the Jacobian complicated for singular curves?



Spec $\mathbb{C}[[t]]$

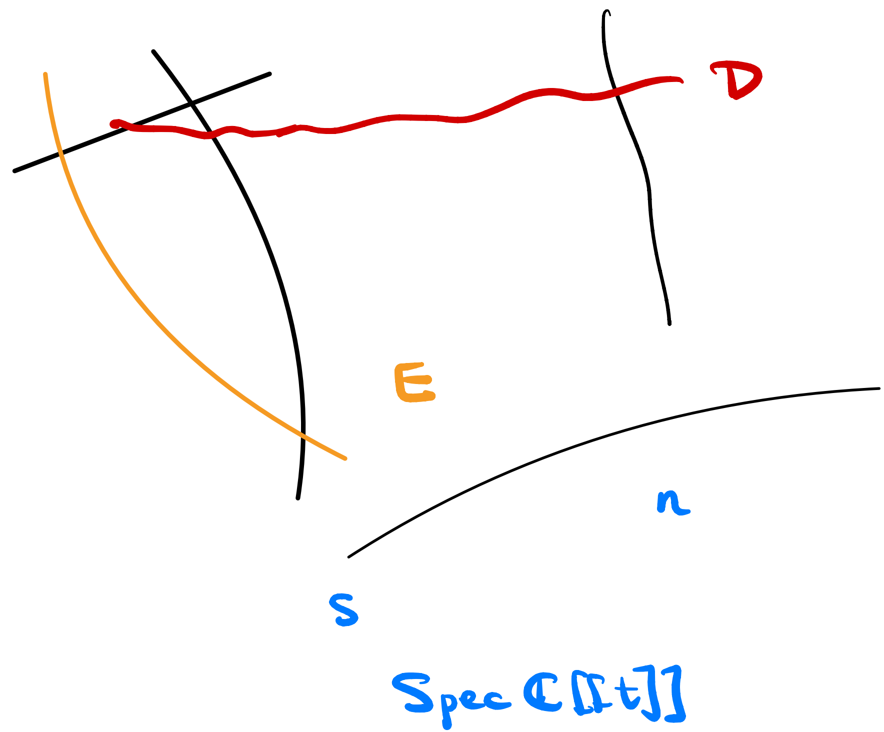
(1) $\mathcal{L}_\eta = \mathcal{O}(D_\eta)$

$\overline{D}_\eta$ meets singular locus

(2) $\mathcal{L}_\eta = \mathcal{O}(E_\eta) \simeq \mathcal{O}$

but $\mathcal{L} = \mathcal{O}(E) \neq \mathcal{O}$

Problem (1) can be solved by (log) blowup.



Problem (2) more delicate.

$$\left\{ \begin{array}{c} \text{Cartier divisors } E \\ \text{supported on boundary} \end{array} \right\} := \text{Twistors} \underset{\text{group}}{\swarrow}$$

$$\text{Twistors} \cong PL(\Sigma_c)$$

For tropical geometers

$$\text{Twistors} = PL(\Sigma_c) = \{ \text{Chip firing moves} \}$$

Established Solution

Solve (1) by allowing up to one blowup for each node (quasi-stable curve)

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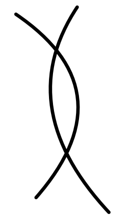
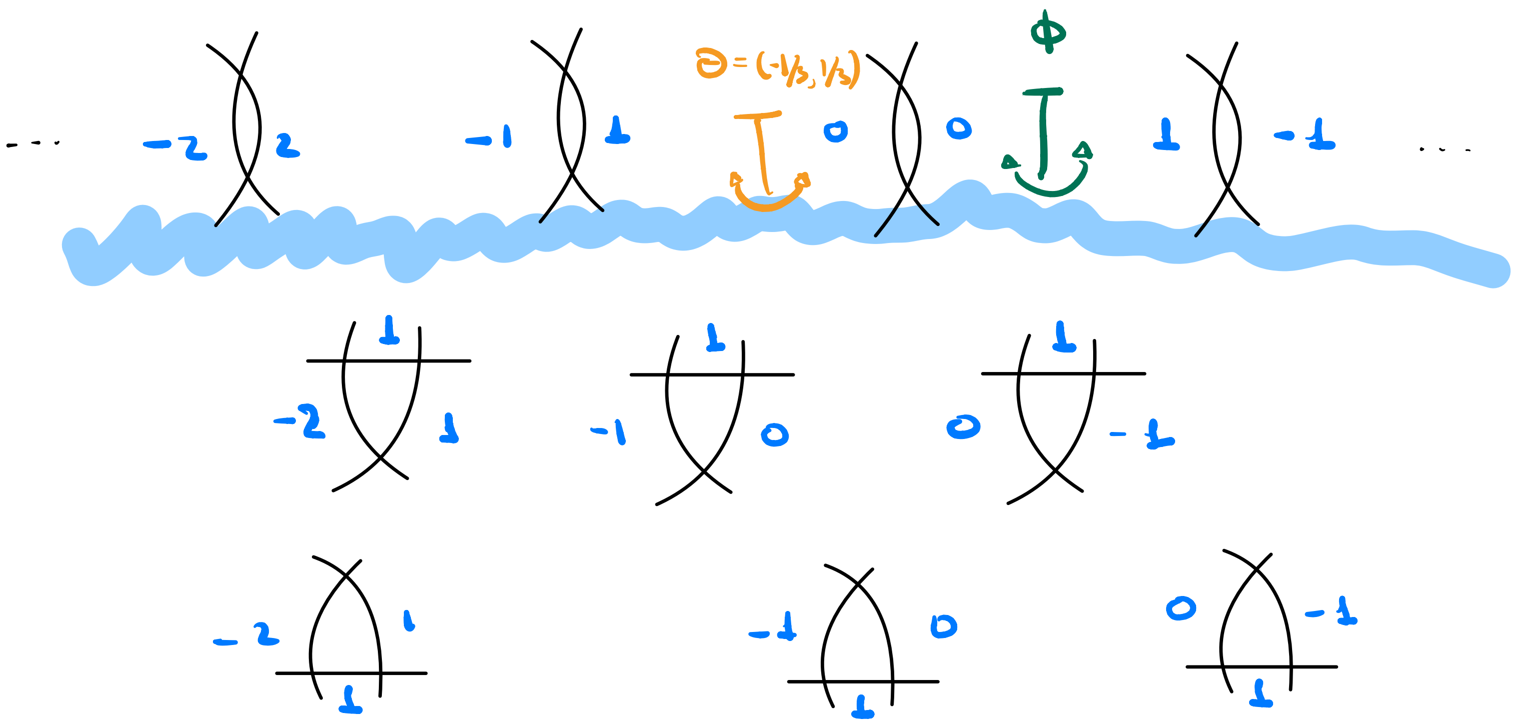
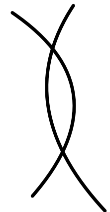
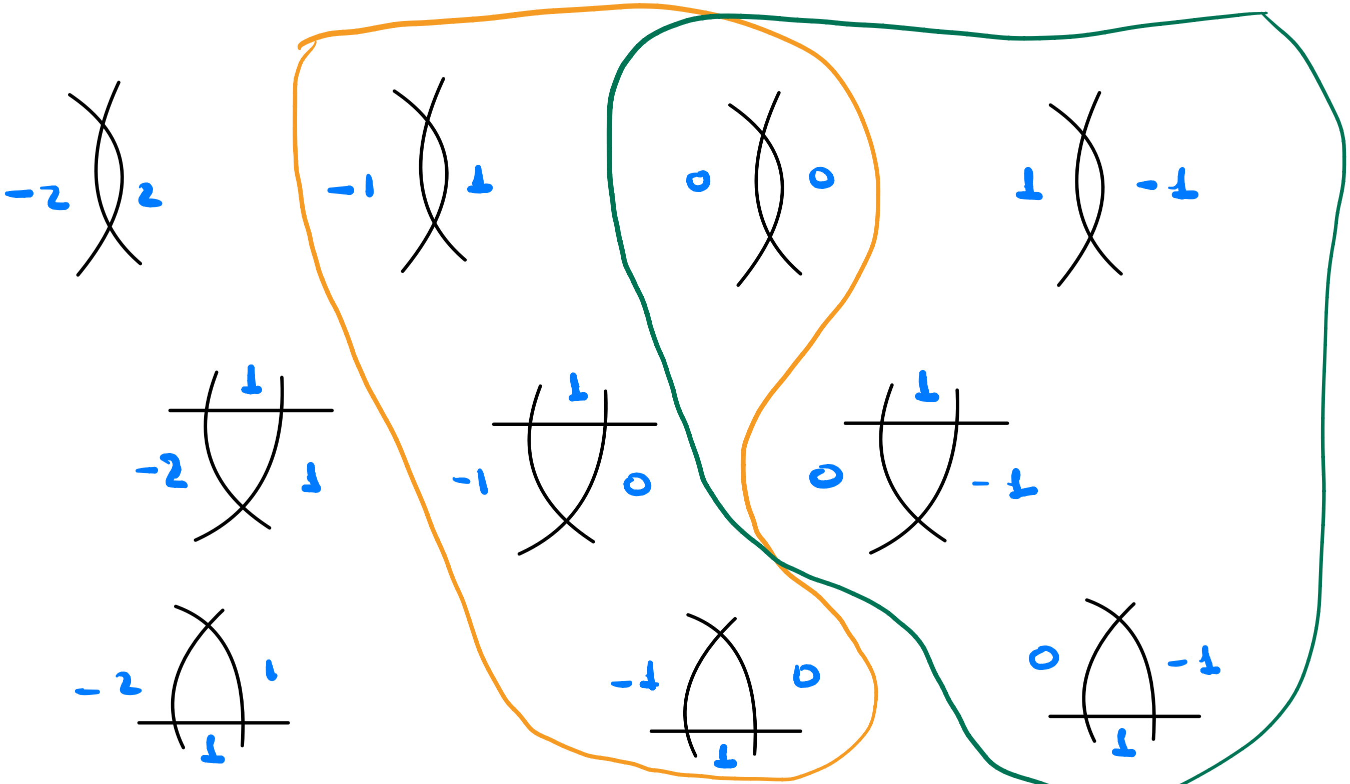
Solve (2) by stability condition: Fix small set of allowed multidegrees ("stable multidegrees") and twist limit to unique θ-stable one.

$$\mathcal{L}_n \simeq \mathcal{O}(D_n) \rightsquigarrow \underbrace{\mathcal{O}(\overline{D}_n) \otimes \alpha}_{\theta\text{-stable}}$$

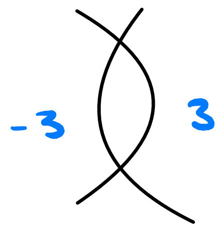
Example On  , stability condition may be



Example On  , stable multidegrees may be:

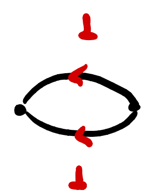


For example, if $\overline{D}_n$ is Cartier, of multidegree

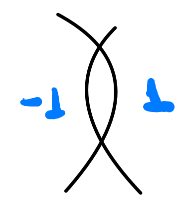


, stable limit is

$\mathcal{O}(\overline{D}_n) \otimes \underbrace{\mathcal{O}(\quad)}_{\mathcal{O}(-E)}$



$\simeq$



Thm (Oda-Seshadri, Caporaso, Kass-Pagani 70s - ...)

(Mumford, Faltings-Chai, Alexeev 60s - ...)

For every (generic) θ, there is a proper, smooth

$$
\begin{tikzcd}
\{C, L\} = \mathrm{Jac} \arrow[d] & \subset & \overline{\mathrm{Jac}}^{\theta} = \{C', L\} \arrow[d] \\
\mathcal{M}_{g,n} & \subset & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}
$$

quasi-stable

θ-stable mdeg

But: No aj section.

Log Solution

Morally, problem is as follows.

$\mathcal{L}_n$ determined by transition functions

$$u_{ij} \in \mathcal{O}^*_{\mathcal{C}_n}$$

Over s, u_{ij} may acquire zeroes and poles

But $u_{ij} \in j_* \mathcal{O}^*_{\mathcal{C}_n} = M_{\mathcal{C}}^{gp}$ canonically.

Def Let $C \to S$ be a log curve.

$$\mathrm{LogJac}(C/S) = \{ M_C^{gp}\text{-torsors on } C \}$$

Thm (M-Wise)

- $\mathrm{LogJac}(C/S)$ is proper, smooth, group compactification of $\mathrm{Pic}^{[0]}(C/S)$.

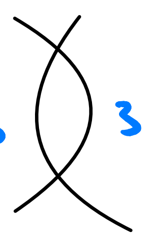
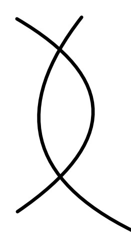
- Not representable: algebraic log stack

How to think about it?

1. Algebraically: $$\mathrm{Log}\,\mathrm{Jac}(C/S) = \varinjlim_{\substack{\text{log blowups} \\ C' \to C}} \mathrm{Jac}(C') \big/ PL(\Sigma_{C'})$$

Miracle Invariance: $H^1(C, M_C^{gp}) = H^1(C', M_{C'}^{gp})$

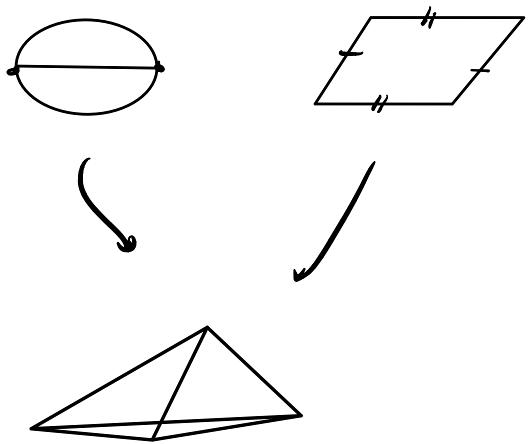
Twistor Vanishing:

-3  3 $=$ -1  1 in LogPic

2. Tropically: There is an exact sequence

$$0 \longrightarrow \mathrm{Pic}^{[0]}(C/S) \longrightarrow \mathrm{Log\,Jac}(C/S) \longrightarrow \mathrm{Tro\,Jac}(\Sigma_C/\Sigma_S) \longrightarrow 0$$

Σ_C



Σ_S

$$\mathrm{TroJac}(\Sigma_C/\Sigma_S) = \mathrm{Hom}(H_1(\Gamma),\mathbb{R})/_{H_1(\Gamma)}$$

family of real tori

Mikhalkin - Zharkov

(3) By descent

Every compactified Jacobian Jac^{∂} is

a log blowup of LogJac

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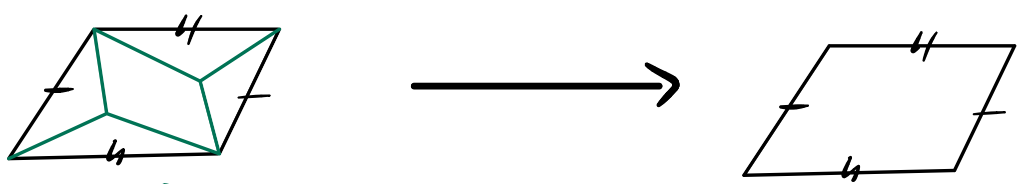
Every compactified Jacobian Jac^{∂} is a log blowup of LogJac

$$\left\{ \begin{array}{c} \text{Toroidal Compactifications} \\ \text{of } \mathrm{Pic}^{[0]} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Polyhedral} \\ \text{subdivisions} \\ \text{of TroJac} \end{array} \right\}$$



$$\mathrm{Jac}^{\partial} \longrightarrow \mathrm{TroJac}$$

Intersection Theory

Recall our diagram

$$\mathrm{Jac}^d(\mathcal{C}_{g,n}) = \{(C, x_1, \dots, x_n, L)\}$$

$$\begin{tikzcd}
\mathrm{Jac}^d(\mathcal{C}_{g,n}) \arrow[d] \\
\mathcal{M}_{g,n} \arrow[u, bend left=50, "o"] \arrow[u, bend right=50, "a_{JA}"']
\end{tikzcd}$$

$\rightsquigarrow$ Got DR, HR, BN on $\mathcal{M}_{g,n}$ out of this

1 2 3

Look at

$$\begin{tikzcd}
\mathrm{Jac}(C_{g,n}) \arrow[d] & \subset & \mathrm{LogJac}(\bar{C}_{g,n}) \arrow[d] \\
\mathcal{M}_{g,n} & \subset & \overline{\mathcal{M}}_{g,n} \arrow[u, bend left=40, "0"] \arrow[u, bend right=40, "a_{J_A}"']
\end{tikzcd}$$

For DR problem (1)

Want: $a_{J_A}^* : \mathrm{CH}(\mathrm{LogJac}) \longrightarrow \mathrm{CH}(\overline{\mathcal{M}}_{g,n})$

$$[0] \longrightarrow \mathrm{DR}_{g,A} = a_{J_A}^*([0])$$

but cannot.

Settle: $a|_A^* : \log CH(\mathcal{L}og\mathcal{J}ac) \longrightarrow \log CH(\overline{\mathcal{M}}_{g,n})$

$$[0] \longrightarrow a|_A^*([0]) := \log DR_{g,A}$$

Settle: $aj_A^* : \log CH(\mathrm{LogJac}) \longrightarrow \log CH(\overline{M}_{g,n})$

$$[0] \longmapsto aj_A^*([0]) := \log DR_{g,A}$$

In practice: For every ∂ (also Abreu-Pacini)

modular resolution of indeterminacies of aj_A by log blowup

$$\begin{tikzcd} M_{g,A}^{\partial} \arrow[r, "aj_A^{\partial}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Jac}^{\partial} \arrow[d] \\ \overline{M}_{g,n} \arrow[r, "aj_A"'] & \mathrm{LogJac} \end{tikzcd}$$

log blowup

The collection $(a_{J_A}^{\circledcirc})^*([0]) = \mathrm{DR}_{g,A}^{\circledcirc}$ is the logDR.

<u>Fact</u> ("A") $\mathrm{DR}_{g,A} = \mathrm{ret}_*(\log \mathrm{DR}_{g,A})$

<u>Thm</u> ("C") (Holmes - M - Pandharipande - Pixton - Schmitt)

There exists explicit formula (can implement on computer)

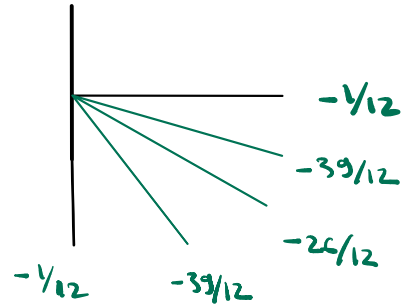
$$\log \mathrm{DR}_{g,A} = F(\psi_v, \kappa, Q) \in \log R^*(\overline{\mathcal{M}}_{g,n})$$

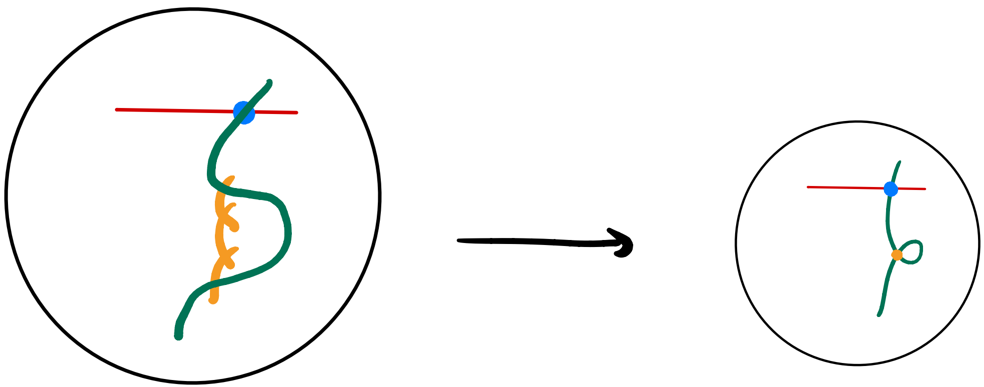
$\psi_v, \kappa \in R^*(\overline{\mathcal{M}}_{g,n})$

$Q \in PP(\Sigma \overline{\mathcal{M}}_{g,n})$

$g = 1, \quad A = (3,-3)$

$$\log DR_{1,(3,-3)} = \frac{9}{2}\psi_1 + \frac{9}{2}\psi_2 +$$





$$\frac{9}{2}\psi_1 + \frac{9}{2}\psi_2 + -\frac{1}{12}D - \frac{39}{12}E_t - \frac{26}{12}E_m - \frac{39}{12}E_b$$

What has been gained?

- Product Formula restored in $\log CH^*(\overline{\mathcal{M}}_{g,n})$

$$\text{(``A 2'')} \quad DR_{g,A_1,\dots,A_\ell} = (\mathrm{ret})_*\left(\prod_{i=1}^{\ell} \log DR_{g,A_i}\right) \in CH^{\ell g}(\overline{\mathcal{M}}_{g,n})$$

So formulas for logDR $\rightsquigarrow$ formulas for all higher ramification cycles (classes "2" above)

Examples

A, B, C 3 vectors of length 3

$\dim_{\mathbb{C}} = 3g$

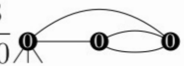
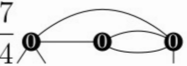
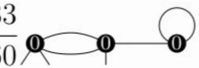
$$DR_{g,A,B,C} = \frac{\delta^{2g}}{2^{3g}\, g!\, (2g+1)!!} \in CH^{3g}(\overline{\mathcal{M}}_{g,3}) = \mathbb{Q}$$

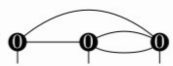
δ = gcd of 2×2 minors of $\begin{pmatrix} A & B & C \end{pmatrix}$

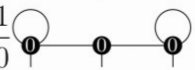
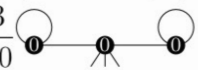
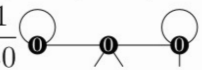
(Generalization of Buryak-Rossi)

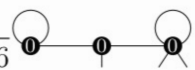
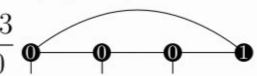
$A=(3,-3,0)$

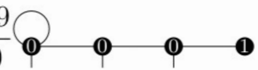
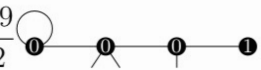
$B=(0,3,-3)$

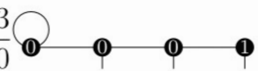
$\mathsf{DDR}_{2,A,B} = \frac{93}{640}$  $-\ \frac{87}{64}$  $+\ \frac{183}{160}$ 

$-\ \frac{49}{160}$  $+\ \frac{27}{320}$  $+\ \frac{213}{640}$ 

$+\ \frac{711}{640}$  $-\ \frac{93}{640}$  $+\ \frac{321}{1280}$ 

$+\ \frac{9}{256}$  $-\ \frac{549}{20}$  $+\ \frac{243}{20}$ 

$+\ \frac{7569}{160}$  $+\ \frac{639}{32}$  $+\ \frac{1251}{160}$ 

$-\ \frac{693}{160}$  $+\ \frac{6561}{20}$  $.$

$\in CH^4(\overline{M}_{2,3})$

The Brill-Noether loci can also be handled.

$$\begin{array}{ccc} & & \mathcal{M}_{g,A}^{\ominus} \\ & C & \downarrow \\ \mathcal{M}_{g,n} & \subset & \overline{\mathcal{M}}_{g,n} \end{array}$$

fails here

recall $w^{d}_{g,r,A} = \Delta^{r+1}_{g+r-d}\, c\left(-R\pi_* \omega^k\left(\sum a_i x_i\right)\right)$

Mumford's formula (GRR)

$\mathcal{M}_{g,A}^{\ominus}$ by construction has

$$C^{\ominus} \longrightarrow \mathcal{M}_{g,A}^{\ominus}, \quad \alpha \in PL(\Sigma_C^{\partial})$$

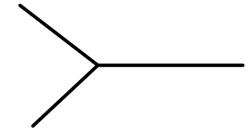
$$w^{\theta}_{g,r,A} = \Delta^{r+1}_{g+r-d}\left(c\left(-R\pi_*\omega^k\left(\Sigma a_i x_i\right) \otimes \alpha\right)\right)$$

$$\in \log CH(\overline{M}_{g,n})$$

is right extension: $ret_*(w^{\theta}_{g,r,A}) = w^{d}_{g,r,A}$

<u>Thm</u> (Abreu - M - Pagani)

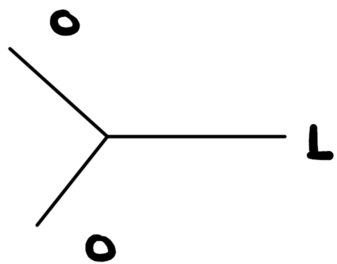
Let $C \to S$ be a log curve. There exists an explicit extension of Grothendieck-Riemann-Roch to $\log CH(C) \to \log CH(S)$

E.g.: On $\overline{M}_{0,4}$, $\Sigma_{\overline{M}_{0,4}} =$ 

theory

$$c_1\left(-R\pi_* \mathcal{O}(x_1 - x_2 - x_3) \otimes \alpha\right) = w^{-1}_{g,0,A} = 0$$

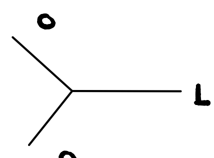
GRR gives

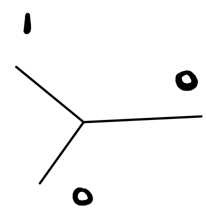
$\psi_1 -$ 

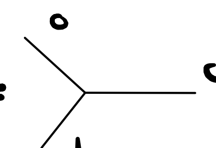
$= 0 \in R^*(\overline{M}_{0,4})$

WDVV

This way, one obtains

$\psi_1 = \psi_2 = \psi_3 = \psi_4 =$ 

$=$ 

$=$ 

Take-away:

- Passing from $CH(\overline{\mathcal{M}}_{g,n})$ to $\log CH(\overline{\mathcal{M}}_{g,n})$ restores essential properties of the classes.

Morally, because they are born on LogJac.

- Additional complications are handled tropically and theory is workable.

Where do we go from here?

- Properties of the classes (polynomiality, CohFT)
- Properties of the formulas (Invariance, Wall-crossing)
- Study of BN relations (structure, completeness)
- Study of $\log CH(\bar{A}_g)$

- $\log CH(\mathrm{LogPic})$ expected to have great deal of structure.

  - $\log CH(\mathrm{LogPic}) \xrightarrow{\int} \log CH(\overline{M}_{g,n})$

  - Theory of Fourier-Mukai transform and weight filtration

(Ongoing work with Younghan Bae)

Thank you for your attention!