Joint work with T. Pelka (U. Warsaw)

① Kontsevich-Soibelman version of SYZ
② Construction of symplectic A'Campo space
③ How to use it towards SYZ.

① Def: $f : X \to \mathbb{D}$ $\quad X^\circ = f^{-1}(\mathbb{D}^\circ)$ $\quad \mathbb{D}^\circ = \mathbb{D} \setminus \{0\}$

(CY Calabi-Yau degeneration). $\quad X^\circ \longrightarrow \mathbb{D}^\circ$ $\quad X_t$ is CY. $\exists\, \Omega \in H^0(X_t, K_{X_t})$ s.t. $\operatorname{div}(\Omega) = 0$

$n = \dim(X_t)$

Size of Jordan blocks of monodromy $\leq n$.

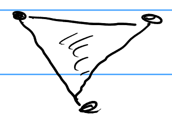
f is maximal CY if the bound is achieved.

Essential skeleton of a CY degeneration.

$X_0 = f^{-1}(0) = \sum m_i D_i$ $\quad X_0$ is simple normal crossings.

$\Omega \in H^0(X, K_X)$ By adjunction + CY

$\operatorname{div}(\Omega) = \sum a_i D_i$ $\quad K_X = \sum a_i D_i$ $\quad K_X + (X_0)_{red} = \sum \nu_i D_i$



Δ_X dual complex of X_0.

$\Delta^{ess} \subset \Delta_X$ subcomplex.

$S = \{ i : \frac{\nu_i}{m_i} \text{ is minimal} \}$ $\quad \Delta^{ess} :=$ subcomplex generated by the the vertexes $\{v_i : i \in S\}$

Prop: f is maximal $\Leftrightarrow$ $\dim(\Delta^{ess}) = n$

Recall Calabi-Yau theorem

X_t is CY and $[\omega]$ is any Kähler class in X

$\exists!\ \omega^{CY} \in [\omega]$ such that $\omega^{CY}(-, J-)$ is Ricci-flat.

$\Omega \in H^0(X, K_X)$ $\quad \omega^{CY}$ is Ricci flat $\Leftrightarrow$ it satisfie the complex MA equation $\quad \frac{1}{n!}(\omega_{cy})^n = \mathrm{vol}^{\Omega}$

$$\mathrm{vol}^{\Omega} = \left(\frac{i}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega \wedge \bar{\Omega} \leftarrow \text{real volume form.}$$

dd^c Lemma. $\quad \omega^{CY} = \omega + dd^c \phi$ $\quad \phi : X_t \to \mathbb{R}$ smooth.

Kontsevich - Soibelmann conjectures.

Gromov-Hausdorff limit conjecture.

$f: X^\circ \to \mathbb{D}^\circ$

Maximal CY degeneration.

Choose a scaling ω_t^{CY} Calabi-Yau metric in X_z

$\mathrm{diam}(X_z, \omega_z^{CY}(-, J-))$ is finite and non zero in the limit as $z \to 0$.

Let Δ be the Gromov-Hausdorff limit of X_z

There exists $\Delta^{sing} \subset \Delta$ of codim $= 2$ (Hausdorff dimension).

such that $\Delta^{sm} := \Delta \setminus \Delta^{sing}$ is an affine manifold of dim $= n$ (real). ($\underline{\{y_1, \dots, y_n\}}$ affine coordinate locally)

The metric restricted to Δ^{sm} is given by a Riemannian metric that has a local potential.

$$g_{ij} = \frac{\partial^2 K}{\partial y_i \partial y_j}$$

K local function.

Real MA equation. $\boxed{\det(g_{ij}) = \text{constant}}$

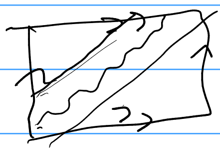
Conjecture 2 (Lagrangian fibration)

X° $\exists\, U \subset X^\circ$ open subset $Z_z^{sm} = U \cap X_z$.

$\downarrow$ $Z_z^{sm} \longrightarrow \Delta^{sm}$ is a fibration by special
$\mathbb{D}^\circ$. Lagrangian tori (Lagrangian w.r.t ω_{CY}) X_z

$\cdot\ \mathrm{im}(e^{i\theta}\Omega)|_L = 0$ $\theta \in S^1$

Ω dz



$$\lim (Z_z, Z_z \setminus Z_z^{sm}) = (\Delta, \Delta^{sing}).$$

Ω_n

$$\mathrm{vol}(Z_z^{sm}) / \mathrm{vol}(Z_z) \longrightarrow 1 \quad |z| \to 0.$$

$\Omega \wedge \bar{\Omega}$

ω_{CY}^n

$$\lim_{z \to 0} \frac{\int_{Z_z^{sm}} (\omega^{CY})^n_t}{\int_{X_z} (\omega^{CY})^n_t} \to 1. \quad \text{(Yang Li)}$$

The A'Campo space and its symplectic form

$f: X \to \mathbb{D}$ any snc. degeneration. ↝ Kato Nakayama space.

Log-analytic spaces
↓ Functor
Topological spaces.

$\mathbb{C}^2, \{xy=0\}$

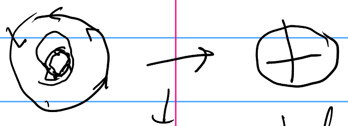
$$\begin{array}{ccc} X_{\log} & \to & (X, D) \\ {\scriptstyle f_{\log}}\downarrow & \circ & \downarrow {\scriptstyle f} \\ \mathbb{D}_{\log} & \to & (\mathbb{D}, 0) \end{array}$$

$f^{-1}(0) = D$.

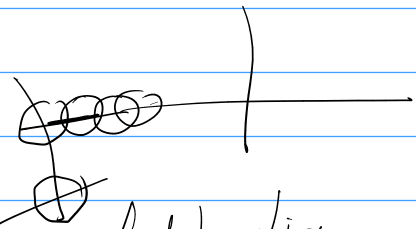
$(\mathbb{C},0)\times(\mathbb{C},0)$.

$(\mathbb{C}^2, \{xy\}=0) \xrightarrow{\log} (r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$

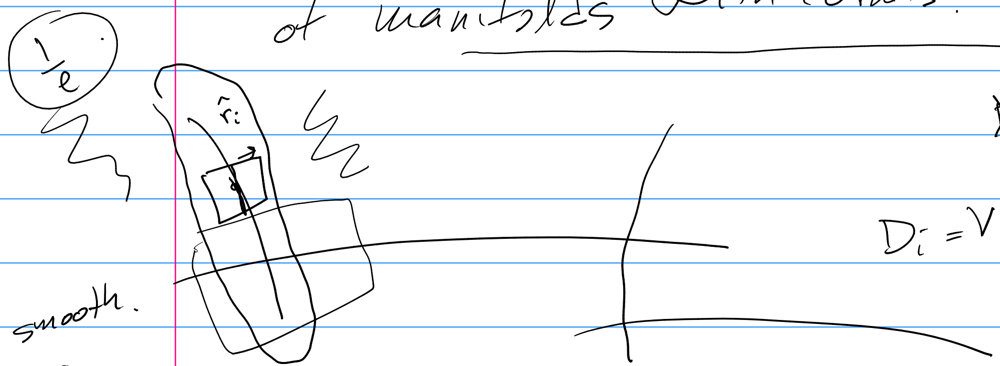
$r_1, \theta_1, r_2, \theta_2 \to$



real oriented blow up.



$f_{\log}$ becomes a locally trivial fibration of manifolds with corners.



$X_0 \subset X$

$D_i = V(z_i)$ $r_i := |z_i|$

$r_i = \lambda_i \hat r_i$

$\lambda_i \neq 0$ everywhere in the chart.

$\hat r_i : X \to \mathbb{R}_{\geq 0}$. "distance to D_i function"

It is comparable with a standard distance function.

$\hat r_i^{-1}(0) = D_i$

$\max r_i < \frac{1}{e}$

Our main set of functions:

$\hat s_i := \log \hat r_i$ $t = \dfrac{-1}{\log|f|}$

$\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$\eta(s) = \dfrac{1}{1-\log s}$

$\hat\alpha_i = -d^c \hat s_i$

$f^{-1}(0) = \sum m_i D_i$

$\hat t_i = \dfrac{-1}{m \log \hat r_i}$ → "Natural function"

$\hat w_i = \dfrac{t}{\hat t_i}$ → tropical function.

$\sum_{i=1}^{N} \hat w_i \to 1$ as $z \to X_0$

$f = z_1 \cdot z_1$ $\quad t_i = \frac{-1}{\log|z_i|}$ $\quad w_1 + w_2 = I.$

Hybrid functions: $\hat{v}_i = \hat{t}_i - y(\hat{w}_i)$ $\quad \hat{u}_i = y(\hat{w}_i).$

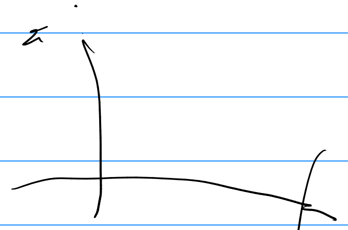
$\to g = f \lambda(t)$ $\quad t = -\frac{1}{\log|t|}.$

$$X_0 = \sum_{i=1}^{N} m_i D_i$$

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d] \arrow[dd, bend right] & \overline{\Gamma}_\mu \arrow[r] \arrow[d] & \Gamma_\mu \subset X \times \mathbb{R}^N \arrow[d] \\
Y_{\log} \arrow[r] \arrow[d] & X \arrow[r, dashed, "\mu"] \arrow[d] & \mathbb{R}^N \\
D_{\log} \arrow[r] & D &
\end{tikzcd}$$

$\mu := (\hat{w}_1 \ldots \hat{w}_n).$

$\hat{w}_i = \frac{\log|z_i|}{\log|f|}$ asymptotic.



Intuition. $f: \mathbb{C}^2 \to \mathbb{C}$

$z_1, z_2 \longmapsto z_1 \cdot z_2.$

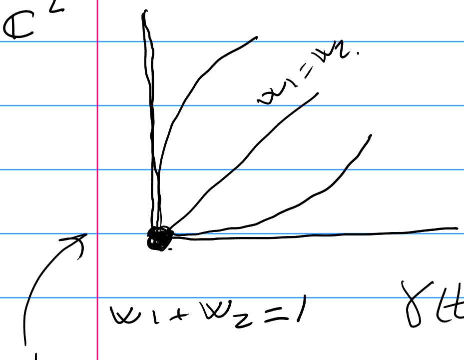
$\mathbb{R}^2_{\geq 0} \times (S^1)^2 \to \mathbb{R}_{\geq 0} \times S^1$

$r_1, r_2, \quad \theta_1, \theta_2 \longmapsto r_1 r_2, \theta_1 + \theta_2.$

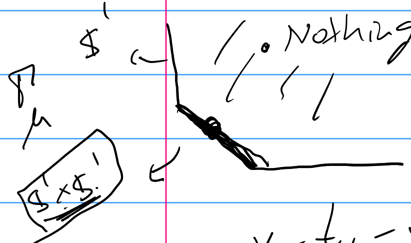
$\mathbb{C}^2 \longrightarrow \mathbb{C}$

$z_1, z_2 \longrightarrow z_1 z_2.$

$\mathbb{C}^2$



$w_1 + w_2 = 1$ $\quad \gamma(t) = (t^{w_1}, t^{w_2})$ $\quad w_i(\gamma(t)) = w_i$



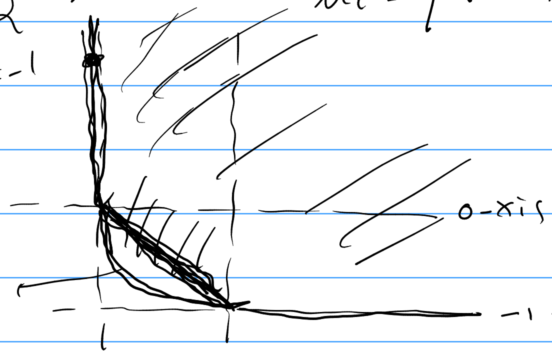
Nothing.

$\bar{v}_i = t_i - w_i$ $\quad$ Role of $u_i = y(w_i)$

$\Gamma_\mu \xrightarrow{(\bar{v}_1, v_2)} \mathbb{R}^2.$

$\hat{v}_1, \hat{v}_2$ $\quad v_1 = -1$

$v_1 = t_1 - w_1$

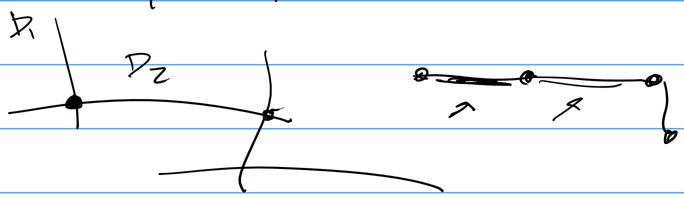


$w_1 = 1$

$\boxed{w_i}$ are moves

$w_1 + w_2 = -1.$ $\quad t_1 = t_2 = 0.$

$v_1 + v_2 = w_1 + w_2.$



Theorem:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, "\pi"] & \Gamma_\mu \arrow[d] & \\
Y_{\log} \arrow[r] \arrow[d] & X \arrow[r, dashed] \arrow[d] & \mathbb{R}^N. \\
D_{\log} & D &
\end{tikzcd}$$

Dual complex $\}$ $\quad w_1 + \cdots + w_N = 1 \subset \mathbb{R}^N_{\geq 0}.$

There exists a C^∞ atlas in A such that.

f_A is a C^∞ loc. trivial fibration. $D_{\log} = (g, \theta)$.
$|z|$

- The function $\hat{v}_i$ are smooth.

θ_i are smooth. the map π is smooth.

$\hat{\alpha}_i$ is smooth. $\hat{\alpha}_i = d\hat{\theta}_i + \pi^*\beta$

($\beta \in \Omega^1(X)$).

The symplectic forms in the A'Campo space.

The A'Campo form $\lambda^A := \sum_{i=1}^{N} \hat{v}_i \hat{\alpha}_i$

Lemma $\lambda^A = d^c \phi^A$ fibrewise $\qquad \omega^A := d\lambda^A = dd^c \phi^A$ fibrewise.

The semiflat form $\phi^{sf} := \sum_{i \in S} \frac{1}{t} \hat{\omega}_i^2$

Maximal cy degeneration

$\lambda^{sf} = d^c \phi^{sf}$
$\omega^{sf} = dd^c \phi^{sf}$ → Both extend to forms in the whole A

Def (A modification of the Fubini Studi form)

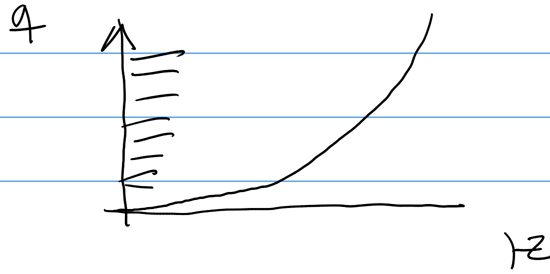
Family of forms depending on a parameter q.

$$\omega_q = \pi^* \omega^{FS} + \varepsilon\left(q\omega^A + (1-q)\omega^{sf}\right)$$

$\omega_q = \pi^*\omega^{FS} + dd^c$ potential.

ω_q is non degenerate in A for any $q > 0$

$\omega_{q(|z|)}$ ~ gives a form in X_z



Thus $(X_z, \omega_{q(|z|)})$ satisfies the Gromov Hausdorff

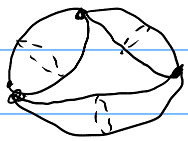
Limit conjecture. $w_1 = w_2 =$ constant. $S^1 \times S^1$ Lagrangian.

We produce a torus Lagrangian fibration at radius 0

Base is the (expanded) essential skeleton

minus codim $= 2$ faces

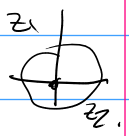
$\{xyz + w(x^3 + y^3 + z^3) = 0\} \subset \mathbb{P}^2 \times \Delta$ $w \in \Delta$.

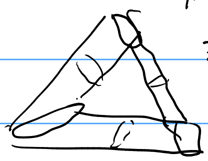
X_0  $\leftarrow$ $(X_0)_{log,(0,0)}$ $\theta_1 + \theta_1 =$ const. X_{log}

$\mathbb{P}^1, \{0, \infty\}$

z div$(z) = \{0, \infty\}$

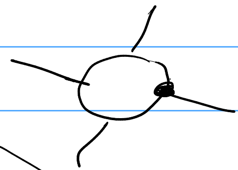
$|z| =$ const.

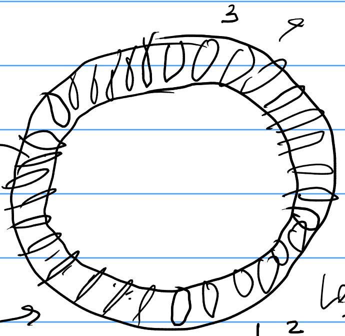




$\downarrow$

Natural piece. D_{log}

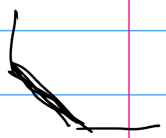




$(X_{log,0,0})$ $\mathbb{P}^1_{2\,pts}$

f_A $(0,0)$

foliations glue



Exceptional piece is

$w_i \in [0,1] \times S^1$ Tori $:= w_i =$ constant.

$$\begin{array}{c} A \\ f_A \downarrow \\ D_{log} \end{array}$$

ω_q nondegenerate.