Holomorphic Lagrangians, Gromov-Witten invariants, and Real structures*

* overpromised

ANU has 2 postdoctoral positions open, with applications closing January 3

1. Weinstein Symplectic 'Category'
2. Holomorphic symplectic category
3. Logarithmic symplectic category

In log Calabi-Yau 3-folds

4. GW invariants as Lagrangian correspondences
5. (speculative) Real structure + curves relative special Lagrangians

Weinstein symplectic "category"

Objects: symplectic manifolds

(M, ω) $\quad d\omega = 0$ $\quad \iota_\omega : TM \to T^*M$ iso, $v \mapsto \iota_v \omega$

Morphisms $M \longrightarrow N$ are Lagrangians $L \subset (M, -\omega_M) \times (N, \omega_N) := \bar{M} \times N$

Def $S \subset (M, \omega)$ is isotropic if $\omega|_S = 0$

S isotropic $\Rightarrow$ $\dim S \leq \frac{1}{2} \dim M$

$S \subset (M, \omega)$ is Lagrangian if it is isotropic and $\dim S = \frac{1}{2} \dim M$

Example Lagrangians:

Graph of a symplectomorphism

$L = \{(x, f(x))\} \subset \bar{M} \times N$ $\quad f : M^n \to N^n$ $\quad f^*\omega_N = \omega_M$

Any smooth map

$f : X \to Y$

$f^*T^*Y \hookrightarrow T^*\bar{X} \times T^*Y$

$$\begin{tikzcd} T^*X \arrow[d] & f^*T^*Y \arrow[l] \arrow[r] \arrow[dl] & T^*Y \arrow[d] \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

The wavefront set of a Fourier-integral operator (following Hormander.)

Fourier-integral operator $C^\infty(X) \to C^\infty(Y)$

$f \mapsto \int e^{i\frac{\phi(x,y)}{\hbar}} f(x)\, dx$

quantisation

semi classical limit

$\phi : X \times Y \to \mathbb{R}$

$\{(x, -d_x\phi, y, d_y\phi)\} \subset T^*\bar{X} \times T^*Y$

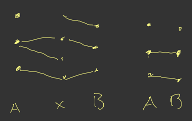
Lagrangian

Maslov, Hörmander:

Composition of operators $\rightsquigarrow$ geometric composition of transverse Lagrangian correspondences

Composition of relations

$L_1 \subset \bar{A} \times X$
$L_2 \subset \bar{X} \times B$



$$\begin{array}{c} L_1 \times_X L_2 \subset \bar{A} \times X \times B \\ \pi \big\downarrow \text{project} \\ L_1 \circ_X L_2 \subset \bar{A} \times B \end{array}$$

$= (a,b) \mid \exists x$ s.t. $(a,x) \in L_1, (x,b) \in L_2$

Composition is Lagrangian*

* (In nice cases)

on $L_1 \times_X L_2$, $\omega_A = \omega_X$ & $\omega_X = \omega_B$

$\Downarrow$

$\omega_A = \omega_B$ on $L_1 \circ_X L_2$

ie $L_1 \circ_X L_2$ is isotropic in $\bar{A} \times B$

Virtual dimension matches Lagrangian bound

$L_1 \circ_X L_2$ isotropic $\Longrightarrow$

$$\dim L_1 \circ_X L_2 \leq \text{vdim} = \dim L_1 \times L_2 - \dim X = \frac{1}{2} \dim(A \times B)$$

The problem:

Smooth intersections can be pathological

$L_1 \circ_{X} L_2$ might not be a manifold. So morphisms may not compose.

Solutions:

- Wehrheim-Woodward (holomorphic quilts, Fukaya categories).
- Algebraic/holomorphic intersections are less pathological.

Option: derived algebraic geometry following Pantev, Toen, Vaquie, Vezzosi.

We use homology supported on Lagrangians. Composition will correspond to gluing relative Gromov-Witten invariants.

Holomorphic Weinstein symplectic category

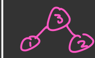
Objects: (M, ω) smooth, alg/$\mathbb{C}$ $\quad d\omega = 0 \quad$ $\iota_\omega : TM \to T^*M$, $v \mapsto \iota_v \omega$ iso

Morphisms $M \to N$ are $\mathrm{Lag}(M, N) := \mathbb{Z}\left\{\begin{array}{l}\text{Lagrangian subvarieties} \\ \text{of } M^- \times N, \\ \text{proper over } M^-\end{array}\right\}$

Examples

① $M = \prod$ CY surfaces

② $X^{[n]}$ Hilbert scheme of points on CY 2-fold (X, ω)



Lagrangian related to Nakajima basis

$$\begin{tikzcd}
 & \Delta^\mu \arrow[dl, hook] \arrow[dr, hook] & \\
(X^\mu)^- := \prod_k (X, -\mu_k \omega) \arrow[dr] & & X^{[n]} \arrow[dl] \\
 & S^n(X) &
\end{tikzcd}$$

μ = partition of n

$\mu = (\mu_1, \ldots, \mu_d)$

$\Delta^\mu \in \mathrm{Lag}\left(\prod_k (X, -\mu_k \omega), X^{[n]}\right)$

Composition of relations

$L_1 \subset A^- \times X$

$L_2 \subset X^- \times B$

$$\begin{array}{c} L_1 \times_X L_2 \subset A^- \times Y \times B \\ \pi \Big\downarrow \text{project} \\ L_1 \circ_X L_2 \subset A^- \times B \end{array}$$

$$\pi_*(L_1 \times_X L_2) \overset{\text{canonical}}{\leftrightarrows} L_1 \star_X L_2 \in \mathrm{Lag}(A^- \times B)$$

$\dim L_1 \circ_X L_2 \leq \mathrm{vdim} = \dim L_1 \times L_2 - \dim X$
$= \frac{1}{2} \dim(A \times B)$

Donaldson-Thomas invariants as Lagrangians

(Y, X) log C-Y 3fold, X smooth

R. Thomas: $\quad DT(Y, X) \in \mathrm{Lag}\left(\coprod_n X^{[n]}\right)$

Holomorphic Weinstein category properties

This is a strict symmetric monoidal category. $\quad X \times Y \xrightarrow{\cong} Y \times X$ is $\{(x, y, y, x)\} \subset X^- \times Y^- \times Y \times X$

Duals: $X^- = (X, -\omega)$ $\qquad \mathrm{Lag}(X \times Y, Z) \supseteq \mathrm{Lag}(X, Y^- \times Z)$ equality if Y compact

- $\Delta_X \subset X^- \times X$ is $\mathrm{Id}_X \in \mathrm{Lag}(X, X)$ and counit $\in \mathrm{Lag}(X \times X^-, \bullet)$ and unit $\in \mathrm{Lag}(\bullet, X \times X^-)$ if X compact

- Adjoints: $L \subset X^- \times Y$
  $\updownarrow$
  $L^\dagger \subset Y^- \times X$ $\qquad$ Eg graph of symplectomorphism is unitary

Symbolic calculus of dagger compact closed category
(restricted to the subcategory of compact objects)

$\star\Big($  $\Big) = (L_1 \star_X L_2 \star_Z \Delta_Z) \star_Y \Delta_Y \in \mathrm{Lag}(A, B)$

depends only on wiring diagram

Nakajima basis as unitary Lagrangian correspondence

$$\mathcal{L} := \sum_{\mu} \Big(\prod_k \sqrt{-1}^{\,\mu_k - 1}\Big) \Delta^{\mu} \in \mathrm{Lag}\Big(\coprod_{\mu} \mathcal{X}^{\mu}, X^{[n]}\Big) \otimes \mathbb{Z}[\sqrt{-1}]$$

$$X^{\mu} := \prod_k (X, -\mu_k \omega)$$

$$\begin{tikzcd} & \Delta^{\mu} \arrow[dl, hook] \arrow[dr, hook] & \\ X^{\mu} \arrow[dr] & & X^{[n]} \arrow[dl] \\ & S^n(X) & \end{tikzcd}$$

$$\mathcal{X}^{\mu} := X^{\mu} / \mathrm{Aut}_{\mu}$$

'evaluation stack for GW invariants'

$$\mathcal{L}^{-1} = \mathcal{L}^{\dagger}$$

$$GW \underset{\text{conjecture}}{=} \mathcal{L} \star DT$$

Logarithmic Weinstein symplectic category

Objects: (X, D, ω) $\quad d\omega = 0 \quad \iota_\omega : TM \to T^*M$ iso, $v \longmapsto \iota_v \omega$

log smooth$/\mathbb{C}$

Morphisms $M \to N$ are $\mathrm{Lag}(M, N)$

$$\mathrm{Lag}(M,N) = \mathbb{Z}\left\{\begin{array}{l}\text{Lagrangian subvarieties} \\ \text{of } M^- \times N, \\ \text{proper over } M\end{array}\right\}$$

Def $U \subset (X, D, \omega)$ is a Lagrangian subvariety if
- U is a subvariety of X
- $U \not\subset D$
- $\omega|_{U \setminus D} = 0$ & $\dim U = \frac{1}{2} \dim X$

Composition of relations

$L_1 \subset A^- \times X$

$L_2 \subset X^- \times B$

$$\begin{array}{c} L_1 \times_X L_2 \subset A^- \times X \times B \\ \pi \downarrow \text{project} \\ L_1 \circ_X L_2 \subset A^- \times B \end{array}$$

Now uses log Chow ring, or refined cohomology

$$\pi_*(L_1 \times_X L_2) \overset{\text{canonical}}{=} L_1 \star_X L_2 \in \mathrm{Lag}(A^- \times B)$$

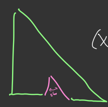
Examples

1) X toric $\quad (\mathbb{C}^*)^{2n}$ + str at ∞

$$\omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} + \frac{dz_3}{z_3} \wedge \frac{dz_4}{z_4} + \cdots$$

2) X = product of log Calabi–Yau surfaces

3) Non-toric blowup



$(X, D, \omega) = \left(\mathbb{CP}^2, \text{toric boundary}, \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}\right)$

Y = non toric blowup at $[1,1,0]$

$$\begin{tikzcd} & \left(\mathrm{Bl}_{[1,1,0]} X, \pi^{-1} D\right) \arrow[dl, "\pi"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$\pi^* \omega = \pi_2^* \omega_Y$

$\mathrm{Image}(\pi, \pi_2) \in \mathrm{Lag}(X, Y)$

Log Chow ring or refined cohomology

Captures 'intersection at infinity' of X\D

Eg $X = (\mathbb{CP}^1)^2$  

Has pullbacks/pushforwards compatible with fibre products

$$\begin{tikzcd} A \times_C B \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \qquad g^* \circ f_* = f'_* (g')^*$$

Is needed for tropical gluing formula for GW invariants

Is infinite dimensional

$$``H^*(\mathbb{CP}^2)" = \lim H^*(\text{Toric } 2\text{-folds})$$

GW invariants of log CY 3-folds are Lagrangian correspondences

$$(X, D, \Omega)$$

$$D = \bigcup_k D_k$$

Log CY 3-fold Examples

① (X, D) toric 3-fold $\Omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3}$

$(\mathbb{C}^*)^3$ + 'structure at ∞'

② non-toric blowup

③ 'Tropological vertex' (log version of Local CY)

Holomorphic symplectic evaluation space:

(D_k, Ω_{D_k}) is a log C-Y surface

eg $(\mathbb{C}^*)^2$ + structure at ∞

Locally, if $D = \{z=0\}$, $\Omega|_{D_k} = \frac{dz}{2\pi i z} \wedge \Omega_{D_k}$

Contact data $\upsilon \rightsquigarrow$ log Calabi-Yau surface $(X_\upsilon, \Omega_{X_\upsilon})$

$\upsilon \in \mathbb{N}[D_i]$

eg $X_\upsilon = (\mathbb{C}^*)^3 / \mathbb{C}^*_\upsilon$ + str at ∞

$\Omega_{X_\upsilon} = \pi_! \Omega$

$$\begin{tikzcd} (\mathbb{C}^*)^3 \arrow[dr] & H \arrow[l, hook'] \arrow[d, "\pi"] \\ & (\mathbb{C}^*)^3/\mathbb{C}^* = H/S^1 \end{tikzcd}$$

S^1-invariant hypersurface

to construct X_υ, take log modification $(X', D') \to (X, D)$ (ie blow up components of D)
s.t. $\upsilon \rightsquigarrow$ contact of order k with D'_i
$(X_\upsilon, \Omega_{X_\upsilon}) = (D'_i, k\,\Omega_{D'_i})$

Evaluation map:

$$ev : \mathcal{M}_{g,(\upsilon_1, \dots, \upsilon_n), E} \longrightarrow \prod (X_{\upsilon_i}, \Omega_{X_{\upsilon_i}})$$

Symplectic form

$$\omega = \sum \Omega_{X_{\upsilon_i}}$$

$$\boxed{\text{Lemma} \quad ev^* \omega = 0}$$

Proof idea

$$\begin{tikzcd} C \arrow[r, "\widetilde{ev}"] \arrow[d] & (X, \Omega) \\ F \arrow[r, "ev"] & \prod X_{\upsilon_i} \end{tikzcd}$$

$$ev^* \omega = \sum \text{residues of } \widetilde{ev}^* \Omega = 0$$

Cor 1) $\dim ev(\mathcal{M}) \leq \operatorname{vdim}[\mathcal{M}]$

2) $ev_* [\mathcal{M}_{g,(\upsilon_1, \dots, \upsilon_n), E}] \in \mathrm{Lag}(\prod X_{\upsilon_i}) \otimes \mathbb{Q}$

Canonical, and robust, independent of VFC construction method.

$$GW \underset{\text{conjecture}}{=} \mathcal{L} \star DT$$

$q^{\frac{1}{2}} = i e^{\frac{i\lambda}{2}}$ — string coupling constant

evaluation stack $\coprod_{p} \prod_{\upsilon} X_\upsilon^{p(\upsilon)} / S_{p(\upsilon)} \times \mathbb{Z}_M$

$GW(X) \in \mathrm{Lag}(\mathcal{X}) \otimes (\text{Novikov ring})$

$DT'(X) \in \mathrm{Lag}(\text{Hilbert scheme}) \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$

Real log CY 3-fold

Special Lagrangian * (using some kähler form on X)

$$(X, D, \Omega) \circlearrowleft \iota \qquad \iota^*\Omega = \overline{\Omega} \quad \rightsquigarrow \quad \begin{array}{l} L = X^{\iota} \\ \mathrm{Im}[\Omega]|_L = 0 \end{array}$$

Equivariant evaluation map

$$\hat{\iota} \circlearrowright \mathcal{M}(X) \xrightarrow{eu} (\mathcal{X}, \omega) \circlearrowleft \iota_0$$

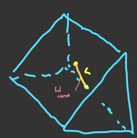
$$\begin{tikzcd} (\Sigma, j) \arrow[r, "f"] \arrow[d] & X \arrow[d, "\iota"] \\ (\Sigma, -j) \arrow[r, "\iota f"'] & X \end{tikzcd} \qquad \iota_0^* \omega = \overline{\omega}$$

$$\Longrightarrow \boxed{\begin{array}{l} eu(\mathcal{M}(X)^{\hat{\iota}}) \subset (eu(\mathcal{M}(X)))^{\iota_0} \leftarrow \text{hyper isotropic } \dim \leq \tfrac{1}{4}\dim(\mathcal{X}) \\ \dim eu(\mathcal{M}(X)^{\hat{\iota}}) \leq v\dim \end{array}}$$

I don't know how to define invariants in this situation,because of orientation issues

Curves relative a special Lagrangian

$$(X, D, \Omega, L) \qquad \mathrm{Im}[\Omega]|_L = 0$$

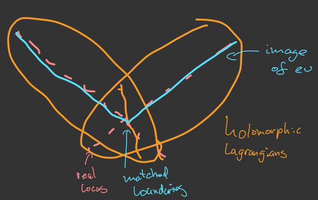


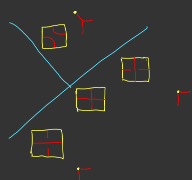
$$eu : \mathcal{M}(X, L) \longrightarrow (\mathcal{X}_L, \omega_L)$$

Real symplectic form constructed using residues of $\mathrm{Im}[\Omega]$

$\boxed{\text{Lemma: } eu^* \omega_L = 0}$ $\boxed{\text{Cor: } \dim eu(\mathcal{M}(X,L)) \leq v\dim \mathcal{M}(X,L)}$

Hope: an invariant based on 'boundary matching', like Ekholm and Shende's "skeins on branes"





Speculation: these invariants are more like "real DT invariants" than open GW invariants

★ $\rightsquigarrow$ Gluing of GW invariants

eg X CY 3-fold $\xrightarrow[\text{degenerate}]{}$ $X_+ \,|_D\, X_-$

$GW(X_+) \in \mathrm{Lag}\left(\mathrm{Exp}\,\mathbb{H}_\hbar(D, k\Omega_D)\right)$

$GW(X_-) \in \mathrm{Lag}\left(\mathrm{Exp}\,\mathbb{H}_\hbar(D, -k\Omega_D)\right)$

$GW(X) = GW(X_+) \star GW(X_-)$

$\Big\downarrow$ SYZ degeneration

Exploded CY

$GW(X) = GW(X') = \sum\limits_{\substack{\text{rigid}\\ \text{tropical}\\ \text{curves } \gamma \text{ in } \mathrm{Trop}(X')}} GW_\gamma(X')$