Real algebraic and real pseudoholomorphic curves in $\mathbb{RP}^2$

S. Yu. Orevkov

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First Part of Hilbert's 16th Problem (most common interpretation): Find isotopy types of non-singular real algebraic curves in $\mathbb{RP}^2$ of (each?) given degree.

Easy for degree ≤ 5
Solved by D.A. Gudkov for degree 6
Solved by O.Ya. Viro for degree 7

Harnack bound for the number of connected components:

$$b_0(\mathbb{R}A) \le g(A) + 1 = \tfrac{1}{2}(m-1)(m-2) + 1$$

where $m = \deg(A)$. If "$=$", then A is called *M-curve*

Oleg Viro observed that many restrictions have topological nature. He gave a precise meaning to this informal assertion.

Definition. (Viro) A *flexible* curve of degree m in $\mathbb{RP}^2$ is a smooth oriented 2-submanifold in $\mathbb{CP}^2$ such that:

- $[A] = m[\mathbb{CP}^1]$ in $H_2(\mathbb{CP}^2)$ (homological degree m)
- $\mathrm{genus}(A) = (m-1)(m-2)/2$
- $\mathrm{conj}(A) = A$ (invariance under conjugation)
- $\mathbb{C}T(\mathbb{R}A) \sim T(A)|_{\mathbb{R}A}$ ($\mathbb{R}$- and $\mathbb{C}$-tangent bundles)

Open question. Do there exist a flexible curve A such that $\mathbb{R}A$ is not isotopic to the real locus of any real algebraic curve of the same degree?

Notation:

$p = \#$(even ovals) =encercled by even nb of other ovals
$n = \#$(odd ovals) =encercled by odd nb of other ovals

Examples of topological restrictions for degree $m = 2k$

- Petrovsky Inequality: $|p - n| \leq \frac{3}{8}m^2 + \dots$
- Gudkov-Rohlin congr.: M-curve $\Rightarrow p - n \equiv k^2 \mod 8$

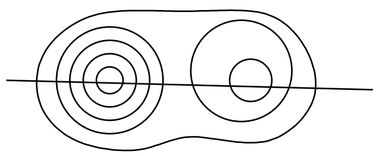
For any degree:

- (by definition) Harnack bound: $p + n \leq \frac{1}{2}m^2 + \dots$
- Arnold Inequality: $\#$(non-empty ovals) $\leq \frac{1}{4}m^2 + \dots$
- Various improvments of the linear terms in all the above

Non-topological restrictions.

(Formally speaking: not (yet?) known to be topological)
Bezout Theorem for auxiliary curves.

Examples:
A curve of degree m cannot have:
2 nests of depths d_1, d_2 if $2(d_2 + d_2) > m$ (auxiliary line)
5 nests of depths $d_1, ..., d_5$ if $2\sum d_i > m$ (auxiliary conic)



(Impossible for $m < 14$)

Def. A *nest of depth* d is a union of d nested ovals.

Middle class (between flexible and real algebraic) is

Real pseudoholomorphic curves.

These are Conj-invariant J-holomorphic curves in $\mathbb{CP}^2$ where J is a conj-anti-invariant almost complex structure tamed by the Fubini-Studi symplectic form.

Being flexible curves, they satisfy all topological restrictions.

Due to Gromov's theory they satisfy all Bezout restrictions with rational auxiliary curves.

(I do not know if there are Bezout restrictions with non-rational curves which cannot be proven with rational ones.)

Open question (Top vs. Symp) Do there exist flexible curve A such that $\mathbb{R}A$ is not isotopic to the real locus of any real pseudoholomorphic curve of the same degree?

Open question (Symp vs. Alg) Do there exist a **smooth** real pseudoholomorphic curve A such that $\mathbb{R}A$ is not isotopic to the real locus of any real algebraic curve of the same degree?

Weak "Symp vs. Alg" (but it was open a month ago)
Do there exist a **smooth** real pseudoholomorphic curve A such that $(\mathbb{CP}^2, A)$ does not admit any conj-equivariant diffeomorhism with $(\mathbb{CP}^2, B)$ with a real algebraic B?

In this talk I give an affirmative answer.

Remark 1. The weak version of "Top vs. Symp" does not make much sense because one can always spoil a flexible curve by gluing a conjugate pair of knotted 2-spheres.

Remark 2. The question "Symp vs. Alg" without the smoothness condition has a positive answer. The simplest examples are pseudoline arrangements not satisfying Pappus or Desargues Theorem.

Remark 3. I classified up to isotopy real ps.-hol. M-curves of degree 8 (subject of my talk in TAU about 20 years ago).

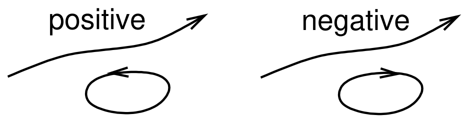
Remark 4. Real ps.-holo. curves serve not only to mark the limits of restrictions. I used them to prove algebraic unrealizability of two oval arrangements by 8th degree M-curves.

End of the introduction

Definition. A *real alg. curve* A is a complex curve invariant by (endowed with) a complex conjugation. $\mathbb{R}A = \mathrm{Fix}(\mathrm{conj})$.

Definition. A is *separating* or *Type I* if $A \setminus \mathbb{R}A = A_+ \sqcup A_-$ is not connected. A *complex orientation* of $\mathbb{R}A$ is the boundary orientation from A_+ or from A_-.

Definition. An oval of a separating odd degree curve A is *positive* (*negative*) if it is oriented w.r.t. the pseudoline of $\mathbb{R}A$ like this:



Notation.

Λ^{p}_{+} = number of positive even ovals,
Λ^{p}_{-} = number of negative even ovals,
Λ^{n}_{+} = number of positive odd ovals,
Λ^{n}_{-} = number of negative odd ovals.

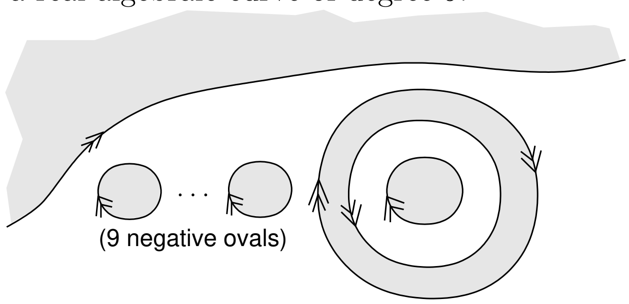
Main Theorem. Let A be a smooth real algebraic separating curve in $\mathbb{P}^2$ of degree $m = 2k+1$ with l ovals. Then

$$\Lambda^{\mathrm{p}}_{+} + \Lambda^{\mathrm{n}}_{-} + 1 \geq \frac{l - k^2 + 2k}{2} \quad \text{and} \quad \Lambda^{\mathrm{n}}_{+} + \Lambda^{\mathrm{p}}_{-} \geq \frac{l - k^2 + 2k}{2}.$$

Setting $l = g - 2s$ (i.e., saying that A is an $(M-2s)$-curve):

$$\Lambda^{\mathrm{p}}_{+} + \Lambda^{\mathrm{n}}_{-} + 1 \geq \frac{k^2 + k}{2} - s \quad \text{and} \quad \Lambda^{\mathrm{n}}_{+} + \Lambda^{\mathrm{p}}_{-} \geq \frac{k^2 + k}{2} - s.$$

Corollary. The following complex orientations are unrealizable by a real algebraic curve of degree 9.



Indeed, if $k = 4$, $l = 12$ and $\Lambda^{\mathrm{p}}_{+} = \Lambda^{\mathrm{n}}_{-} = 0$, then
$\Lambda^{\mathrm{p}}_{+} + \Lambda^{\mathrm{n}}_{-} + 1 = 1 \not\geq \frac{1}{2}(l - k^2 + 2k) = \frac{1}{2}(12 - 16 + 8) = 2$

These complex orientations are realizable by a real pseudoholomorphic curve of degree 9.

There are similar examples for any degree $m \equiv 9 \mod 12$.

Proof of Main Theorem

Definition. $f : A \to \mathbb{P}^1$ is *separating* if $f^{-1}(\mathbb{RP}^2) = \mathbb{R}A$.

Theorem. (Alexandre Gabard, 2006) Let A be a separating real algebraic curve. If genus$(A) = g$ and $b_0(\mathbb{R}A) = r$, then there exists a separating morphism of degree $\leq (r + g + 1)/2$ (the average of the actual number of components and the maximal possible number of components of $\mathbb{R}A$).

Remark 1. Thm. with "$\leq g+1$" was proven by Alfors (1950), but this is not enough for "Symp vs. Alg".

Remark 2. The bound $\frac{1}{2}(r+g+1)$ is sharp (Coppens, 2013).

Theorem. (Abel-Jacobi + Poincaré residue). Let:
S be a smooth real alg. surface,
A be a smooth irreducible real separating curve on S,
$D = 2D_0 + D_1 \in |A + K_S|$ s.th. $A \not\subset \operatorname{supp}(D)$, D_1 is reduced.

We introduce a chess-board orientation of $\mathbb{R}S \setminus (\mathbb{R}A \cup \mathbb{R}D_1)$ and its boundary orientation on $\mathbb{R}A \setminus \operatorname{supp}(D)$. Let $f : A \to \mathbb{P}^1$ be a separating morphism and $p_0 \in \mathbb{RP}^1$. Then it is impossible that the chosen orientation coincides with the complex orientation at each point of $f^{-1}(p_0) \cap \mathbb{R}A \setminus \operatorname{supp}(D)$.

Proof. We have $D - A \sim K_S$. Let Ω be a 2-form with divisor $D - A$ and let ω be its Poincaré resudue on A.

Locally, if $A = \{F(x, y) = 0\}$ and $\Omega = g(x, y)\, dx \wedge dy$, then

$$\omega = (g\, dx/F'_y)|_A.$$

Then ω is a holomorphic 1-form on A and it defines the chosen orientation on $\mathbb{R}A \setminus \operatorname{supp}(D)$. Let t be a parameter on $\mathbb{RP}^1$ at p_0, and let $\{p_1(t), \dots, p_n(t)\} = f^{-1}(t)$. By Abel-Jacobi,

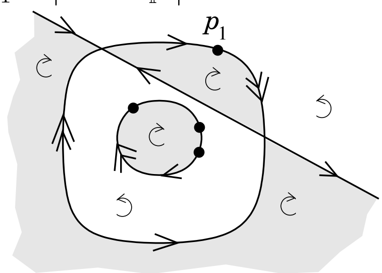
$$\sum_{k=0}^{n} \int_{p_k([0,t])} \omega = 0.$$

$\frac{d}{dt}$ at $t = 0$ yeilds $\sum \omega(v_k) = 0$ for $v_k = p'_k(0) \in T_{p_k(0)}A$.
The orientations coincide $\Rightarrow$ $\omega(v_k) > 0$ when $p_k(0) \notin \operatorname{supp}(D)$.
Q.E.D.

Example 1. Let A be a real hyperbolic quartic curve in $\mathbb{RP}^2$ (hyperbolic means: some central projection is a separating morphism). Then $\mathbb{R}A$ has two nested ovals.

Then the theorem implies that a separating morphism $f : A \to \mathbb{P}^1$ cannot have covering degree 1 over the outer oval.

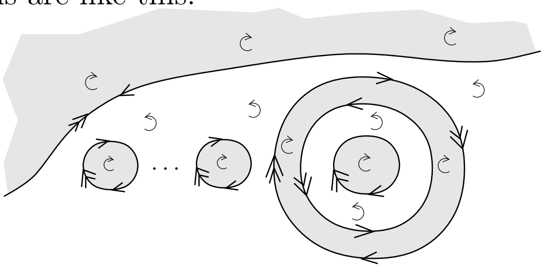
Indeed, if such f exist, we get a contradiction with the thm. by choosing $D = D_1 \in |A + K_{\mathbb{P}^2}|$ to be a line like this:



Example 2. (Proof of Main Theorem in case of the above ps.-holo curve of degree 9). Suppose it is algebraic.
Let $p_0 \in \mathbb{RP}^1$ and f be a separating morphism of degree

$$\tfrac{1}{2}(r+g+1) = \tfrac{1}{2}(13+\tfrac{7\times 8}{2}+1) = \tfrac{1}{2}(13+28+1) = 21$$

We choose $D = 2D_0 \in |A + K_{\mathbb{P}^2}|$ (a double cubic). Then the orientations are like this:



Let J be the pseudoline of $\mathbb{R}A$. We have at least 1 pt of $f^{-1}(p_0)$ on each oval, hence $|J \cap f^{-1}(p_0)| \le 21 - 12 = 9$. Thus we may choose D passing through $J \cap f^{-1}(p_0)$. Even if we have less points in this set, we trace D through 9 point of J. Then $f^{-1}(p_0) \not\subset D$. Otherwise D cuts each oval twice, then

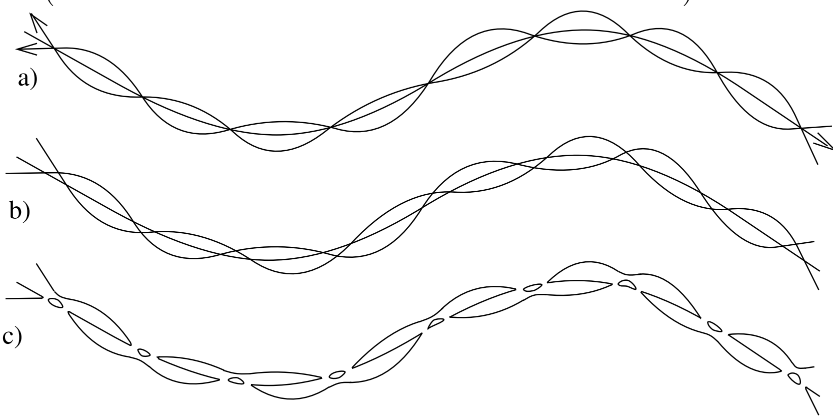
$9 \times 3 = A.D_0 \ge A.J + A.(\text{ovals}) \ge 9 + 2 \times 12$ (contradiction)

Q.E.D.

The proof of Main Theorem in the general case is the same but the computations are with polynomials in k and l rather than with small integers.

Construction of the 9 degree ps.-holo curve

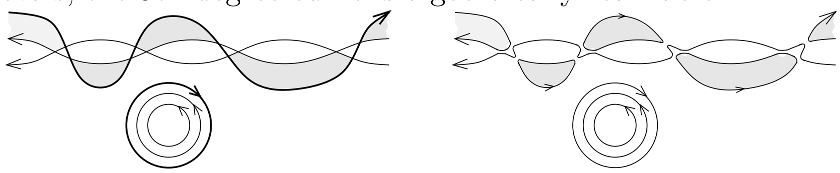
(the 3 nested ovals are not shown but assumed)



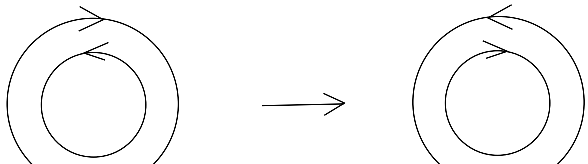
Note that the 3 cubics in Fig.(b) contradict Abel-Jacobi.

This was the initial hint that the obtained 9 degree ps-holo curve might be algebraically unrealizable.

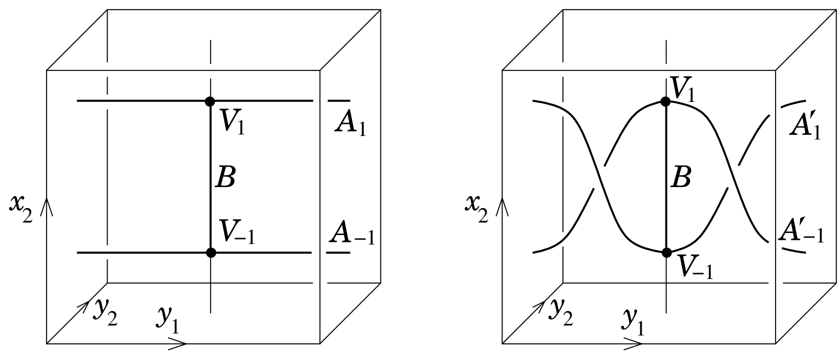
If we swap the orientations of any two consecutive nested ovals, the 9th degree curve is algebraically realizable:



The *braiding construction* (Auroux-Donaldson-Katzarkov, 2003) always allows us to swap the complex orientations of any two nested ovals of a separating real ps.-holo curve provided there are no other ovals between them:



Here is the ADK braiding construction transforming A to A':

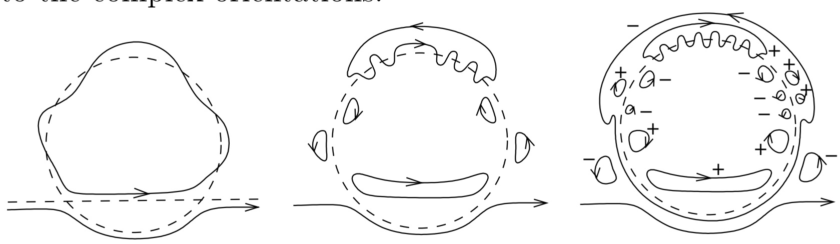


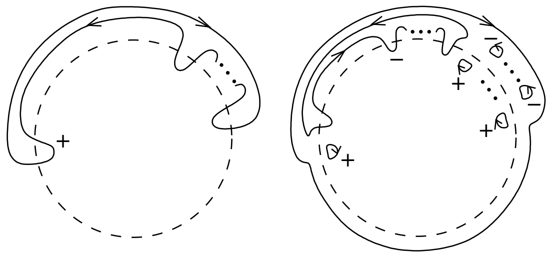
Sections $x_1 = \text{const}$ of $A \cup B$ and $A' \cup B$

By Weinstein Neighbourhood Theorem we identify the annulus B between the ovals $V_{\pm 1}$ with the annulus $-1 \leq x_2 \leq 1$ in the real locus of $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$ with coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (then x_1 is defined mod $\mathbb{Z}$).

Construction for any degree $m \equiv 9 \mod 12$

The 1st stage is a particular case of the classical Hilbert's construction of M-curves with a deep nest, but we pay attention to the complex orientations.





Then we apply the same tripling construction as for deg 9.