Proper Group Actions in Complex Geometry

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Introduction

Let M be a locally compact Hausdorff topological space and $\mathcal{H}(M)$ the group of homeomorphisms of M. We introduce on $\mathcal{H}(M)$ the compact-open topology, which is the topology with subbase given by all sets of the form $\{f \in \mathcal{H}(M) : f(K) \subset U\}$, where $K \subset M$ is compact and $U \subset M$ is open. $\mathcal{H}(M)$ is a topological group in this topology, if, in addition, M is locally connected.

A topological group G is said to act continuously on M by homeomorphisms, if a continuous homomorphism $\Phi : G \to \mathcal{H}(M)$ is specified. The continuity of Φ is equivalent to the continuity of the action map

$$\hat{\Phi} : \ G \times M \to M, \quad (g, p) \mapsto \Phi(g)(p) =: gp.$$

By passing to a quotient of G if necessary, we assume that the action is effective (the kernel of Φ is trivial).

The action of G on M is proper, if the map

$$\Psi: \; G \times M \to M \times M, \quad (g, p) \mapsto (gp, p),$$

is proper, i.e. for every compact subset $C \subset M \times M$ its inverse image $\Psi^{-1}(C) \subset G \times M$ is compact as well. Actions of compact groups are always proper.

Theorem 1 [van Dantzig, van der Wærden (1928)] *If* (M, d) *is a connected locally compact metric space, then the group* $\mathrm{Isom}_d(M)$ *of all isometries of* M *with respect to* d *(considered with the compact-open topology) is a topological group acting properly on* M *by homeomorphisms.*

We will be interested in actions by diffeomorphisms on smooth manifolds, in which case M is a connected C^∞-smooth manifold and Φ maps G into the group $\text{Diff}(M) \subset \mathcal{H}(M)$ of all C^∞-smooth diffeomorphisms of M.

The properness of the action implies:

- G is locally compact, hence it carries the structure of a Lie group and the action map $\hat{\Phi}$ is smooth;
- G and $\Phi(G)$ are isomorphic as topological groups;
- $\Phi(G)$ is a closed subgroup of $\text{Diff}(M)$.

Thus, we can assume that G is a Lie group acting smoothly and properly on the manifold M, and that it is realized as a closed subgroup of $\text{Diff}(M)$ (hence $\Phi = \text{id}$).

Riemannian Manifolds

Let (M, g) be a Riemannian manifold. One can define a distance associated to the metric g:

$$d_g(p, q) := \inf_{\gamma} \int_0^1 ||\gamma'(t)||_g \, dt, \quad p, q \in M,$$

where the inf is taken over all piecewise C^1-smooth curves $\gamma : [0, 1] \to M$, such that $\gamma(0) = p$, $\gamma(1) = q$. The distance d_g is continuous and inner, and hence induces the topology of M. Thus, d_g turns M into a connected locally compact topological space. Therefore, by Theorem 1, the group $\mathrm{Isom}_{d_g}(M)$ (considered with the compact-open topology) acts properly on M by homeomorphisms.

In 1939 Myers and Steenrod showed that the group $\text{Isom}_{d_g}(M)$ coincides with the group $\text{Isom}(M,g) \subset \text{Diff}(M)$ of all isometries of M with respect to g. Hence they obtained:

Theorem 2 [Myers, Steenrod (1939)] *If (M,g) is a Riemannian manifold, then the group $\text{Isom}(M,g)$(considered with the compact-open topology) acts properly on M.*

Conversely, the following holds:

Theorem 3 [Palais (1961)] *If G acts properly on a smooth manifold M, then M admits a smooth G-invariant Riemannian metric.*

Thus, Lie groups acting properly and effectively on a manifold M by diffeomorphisms are precisely closed subgroups of $\text{Isom}(M,g)$ for all possible smooth Riemannian metrics g on M.

If G acts properly on M, then for every $p \in M$ its isotropy subgroup

$$G_p := \{g \in G : gp = p\}$$

is compact in G. Then the isotropy representation

$$\alpha_p : G_p \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg_p$$

is continuous and faithful. In particular, the linear isotropy group

$$LG_p := \alpha_p(G_p)$$

is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to G_p. In some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the orthogonal group $O_m(\mathbb{R})$, where $m := \dim M$. Hence $\dim G_p \leq \dim O_m(\mathbb{R}) = m(m-1)/2$.

Furthermore, the orbit of p

$$G(p) := \{gp : g \in G\}$$

is a closed submanifold of M, and $\dim G(p) \leq m$. Thus, we obtain

$$\dim G = \dim G_p + \dim G(p) \leq m(m+1)/2.$$

Theorem 4 [Fubini (1903); E. Cartan (1928); Eisenhart (1933)]
If G acts properly on a smooth manifold M of dimension m and $\dim G = m(m+1)/2$, then M is isometric (with respect to a G-invariant metric) either to one of the standard complete simply-connected spaces of constant sectional curvature ($\mathbb{R}^m$, S^m, $\mathbb{H}^m$), or to $\mathbb{RP}^m$.

Theorem 5 [H.-C. Wang (1947); I. P. Egorov (1949); Yano (1953)] *A group G with*

$$m(m-1)/2+1 < dim\, G < m(m+1)/2$$

cannot act properly on a smooth manifold M of dimension $m \neq 4$.

Theorem 6 [Ishihara (1955)] *Let M be a smooth manifold of dimension 4. Then a group of dimension 9 cannot act properly on M. If M admits a proper action of an 8-dimensional group G, then M has a G-invariant complex structure.*

There exists also an explicit classification of pairs (M, G), where $m \geq 4$, G is connected, and $\dim G = m(m-1)/2 + 1$, due to Yano (1953) (a local classification), Ishihara (1955) (the case $m = 4$), Obata (1955), Kuiper (1956).

Further, Kobayashi and Nagano obtained a (reasonably explicit) classification of pairs (M, G), where $m \geq 6$, G is connected, and $(m-1)(m-2)/2 + 2 \leq \dim G \leq m(m-1)/2$ (1972).

There are many other results (Wu-Yi Hsiang, I. P. Egorov, Ishihara, Mann, Jänich, Wakakuwa, ...), especially for compact subgroups, but no complete classifications exist beyond dimension $(m-1)(m-2)/2 + 2$.

The Complex Case

We will now assume that M is a complex manifold of dimension $n \geq 2$ (hence $m = 2n \geq 4$) and that a (real) Lie group G acts on M by holomorphic transformations, that is,

$$G \subset \mathrm{Aut}(M) \subset \mathrm{Diff}(M),$$

where $\mathrm{Aut}(M)$ is the group of holomorphic automorphisms of M.

Theorem 7 [Kaup (1967)] *If $G \subset Aut(M)$ is closed and preserves a continuous distance on M, then G acts properly on M.*

Thus, Lie groups acting properly and effectively on a manifold M by holomorphic transformations are precisely closed subgroups of $\mathrm{Aut}(M)$ preserving a continuous distance on M.

In some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the unitary group U_n. Hence $\dim G_p \leq \dim U_n = n^2$, and therefore

$$\dim G \leq n^2 + 2n.$$

We note that $n^2 + 2n < (m-1)(m-2)/2 + 2$ for $m = 2n$ and $n \geq 5$. Thus, the group dimension range that arises in the complex case, for $n \geq 5$ lies strictly below the dimension range considered in the classical real case and thus is not covered by the existing results. Furthermore, overlaps with these results for $n = 3, 4$ occur only in relatively easy situations (described in Theorem 8 and statement (i) of Theorem 9 below) and do not lead to any significant simplifications in the complex case.

The only interesting overlap with the real case occurs for $n = 2$, $\dim G = 5$.

Theorem 8 [Kaup (1967)] *If* $G \subset Aut(M)$ *acts properly on* M *and* $\dim G = n^2 + 2n$, *then* M *is holomorphically isometric (with respect to a* G*-invariant Hermitian metric) to one of the standard complete simply-connected Kähler spaces of constant holomorphic sectional curvature:*

(i) $\mathbb{B}^n := \{z \in \mathbb{C}^n : ||z|| < 1\}$,

(ii) $\mathbb{C}^n$,

(iii) $\mathbb{CP}^n$.

In particular, the manifold from Theorem 6 is equivalent to one of $\mathbb{B}^2$, $\mathbb{C}^2$, $\mathbb{CP}^2$.

Let F be an equivalence map. Then in case (i) F transforms G into $Aut(\mathbb{B}^n)$ which is the group of all transformations

$$z \mapsto \frac{Az + b}{cz + d},$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n,1}.$$

We have $Aut(\mathbb{B}^n) \simeq PSU_{n,1} := SU_{n,1}/(center)$.

In case (ii) F can be chosen to transform G into the group $G(\mathbb{C}^n)$ of all holomorphic automorphisms of $\mathbb{C}^n$ of the form

$$z \mapsto Uz + a,$$

where $U \in U_n$, $a \in \mathbb{C}^n$. We have $G(\mathbb{C}^n) \simeq U_n \ltimes \mathbb{C}^n$. The group $G(\mathbb{C}^n)$ is the full group of holomorphic isometries of the flat metric on $\mathbb{C}^n$.

In case (iii) F can be chosen to transform G into the group $G(\mathbb{CP}^n)$ of all holomorphic automorphisms of $\mathbb{CP}^n$ of the form

$$\zeta \mapsto U\zeta,$$

where ζ is a point in $\mathbb{CP}^n$ given in homogeneous coordinates, and $U \in SU_{n+1}$. We have $G(\mathbb{CP}^n) \simeq PSU_{n+1} := SU_{n+1}/(center)$. The group $G(\mathbb{CP}^n)$ is the full group of holomorphic isometries of the Fubini-Study metric on $\mathbb{CP}^n$, and is a maximal compact subgroup of the full group of holomorphic automorphisms $Aut(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/(center)$.

Theorem 9 [Isaev (2006)] *Let M be a complex manifold of dimension $n \geq 2$ and $G \subset Aut(M)$ a connected Lie group that acts properly on M and has dimension satisfying $n^2 + 3 \leq \dim G < n^2 + 2n$. Then one of the following holds:*

(i) M is holomorphically equivalent to $\mathbb{C}^n$ by means of a map that transforms G into the group $G_1(\mathbb{C}^n)$ which consists of all maps from $G(\mathbb{C}^n)$ with $U \in SU_n$ (here $\dim G = n^2 + 2n - 1$);

(ii) $n = 4$ and M is holomorphically equivalent to $\mathbb{C}^4$ by means of a map that transforms G into the group $G_2(\mathbb{C}^4)$ which consists of all maps from $G(\mathbb{C}^4)$ with $U \in e^{i\mathbb{R}} Sp_2$ (here $\dim G = n^2 + 3 = 19$).

Proposition 10 [Kaup (1967)] *Let M be a connected complex manifold of dimension n and $G \subset Aut(M)$ a Lie group with $\dim G > n^2$ that acts properly on M. Then the action of G is transitive.*

Thus, the group G from Theorem 9 acts transitively (i.e. for every $p \in M$ we have $\dim G(p) = 2n$) which implies:

$$n^2 - 2n + 3 < \dim LG_p < n^2.$$

It turns out that a connected closed subgroup of U_n satisfying these inequalities either is SU_n or is conjugate to $e^{i\mathbb{R}} Sp_2$ for $n = 4$.

Theorem 11 [Isaev (2006)] *Let M be a complex manifold of dimension $n \geq 2$ and $G \subset Aut(M)$ a connected Lie group that acts properly on M and has dimension $n^2 + 2$. Then one of the following holds:*

(i) M is holomorphically equivalent to $M' \times M''$, where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and M'' is one of $\mathbb{B}^1$, $\mathbb{C}^1$, $\mathbb{CP}^1$; an equivalence map can be chosen to transforms G into $G' \times G''$, where G' is one of $Aut(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, and G'' is one of $Aut(\mathbb{B}^1)$, $G(\mathbb{C}^1)$, $G(\mathbb{CP}^1)$, respectively;

(ii) $n = 4$ and M is holomorphically equivalent to $\mathbb{C}^4$ by means of a map that transforms G into the group $G_3(\mathbb{C}^4)$ which consists of all maps from $G(\mathbb{C}^4)$ with $U \in Sp_2$.

Remark. For $n = 2$ Case (i) is contained in the classification due to Ishihara of 4-dimensional real manifolds with transitive actions of 6-dimensional groups.

The group G from Theorem 11 also acts transitively, and therefore for every $p \in M$ we have

$$\dim LG_p = n^2 - 2n + 2.$$

It turns out that the only connected closed subgroups of U_n satisfying this condition are (up to conjugation) $U_{n-1} \times U_1$ and Sp_2 for $n = 4$.

The Case $\dim G = n^2 + 1$

Again, the action of G is transitive, and therefore for every $p \in M$ we have

$$\dim LG_p = (n-1)^2.$$

Lemma 12 [Isaev, Kruzhilin (2002)] *Let H be a connected closed subgroup of U_n of dimension $(n-1)^2$, $n \geq 2$. Then H is conjugate in U_n to one of the following subgroups:*

(i) $e^{i\mathbb{R}} SO_3(\mathbb{R})$ (here $n = 3$);

(ii) $SU_{n-1} \times U_1$ for $n \geq 3$;

(iii) the subgroup $H^n_{k_1,k_2}$ of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix},$$

for some $k_1, k_2 \in \mathbb{Z}$, $(k_1, k_2) = 1$, $k_2 > 0$, where $A \in U_{n-1}$ and $a \in (\det A)^{k_1/k_2}$.

Theorem 13 [Isaev (2006)] *Let M be a complex manifold of dimension 3 and $G \subset Aut(M)$ a connected Lie group that acts properly on M and has dimension $10 = 3^2 + 1$. If for $p \in M$ the group LG_p^0 is conjugate to $e^{i\mathbb{R}} SO_3(\mathbb{R})$, then one of the following holds:*

(i) M is holomorphically equivalent to the Siegel space (the symmetric classical domain of type (III_2))

$$\mathscr{S} := \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : Z\overline{Z} \ll id \right\},$$

where

$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix};$$

in this case any equivalence map transforms G into $Aut(\mathscr{S}) \simeq Sp_4(\mathbb{R})/\mathbb{Z}_2$;

(ii) M is holomorphically equivalent to the complex quadric $\mathcal{Q}_3 \subset \mathbb{CP}^4$ by means of a map that transforms G into the group $SO_5(\mathbb{R})$ (which is a maximal compact subgroup of $Aut(\mathcal{Q}_3) \simeq SO_5(\mathbb{C})$);

(iii) M is holomorphically equivalent to $\mathbb{C}^3$ by means of a map that transforms G into the group $G_2(\mathbb{C}^3)$ which consists of all maps from $G(\mathbb{C}^3)$ with $U \in e^{i\mathbb{R}} SO_3(\mathbb{R})$.

Theorem 14 [Isaev (2006)] *Let M be a complex manifold of dimension $n \geq 3$ and $G \subset Aut(M)$ a connected Lie group that acts properly on M and has dimension n^2+1. If for $p \in M$ the group LG_p^0 is conjugate to $SU_{n-1} \times U_1$, then M is holomorphically equivalent to $\mathbb{C}^{n-1} \times M'$, where M' is one of $\mathbb{B}^1$, $\mathbb{C}^1$, $\mathbb{CP}^1$, and an equivalence map can be chosen to transform G into $G_1(\mathbb{C}^{n-1}) \times G'$, where G' is one of the groups $Aut(\mathbb{B}^1)$, $G(\mathbb{C}^1)$, $G(\mathbb{CP}^1)$, respectively.*

The case LG_p^0 is conjugate to $H^n_{k_1,k_2}$

A complete classification was obtained in a joint paper with N. Kruzhilin (Israel J. Math. 172(2009), 193–252). The classification is rather lengthy, and I will only give a few examples.

Example 1

M is one of the domains

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z_n > |z'|^2\right\} \simeq \mathbb{B}^n,$$

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z_n < |z'|^2\right\} \simeq \mathbb{CP}^n \setminus (\overline{\mathbb{B}^n} \cup L),$$

where L is a complex hyperplane tangent to $S^{2n-1} = \partial \mathbb{B}^n$, and G is the group of maps

$$\begin{aligned} z' &\mapsto \lambda U z' + a, \\ z_n &\mapsto \lambda^2 z_n + 2\lambda \langle U z', a \rangle + |a|^2 + i\alpha, \end{aligned}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\lambda > 0$, $\alpha \in \mathbb{R}$, and $\langle \cdot\,, \cdot \rangle$ is the inner product in $\mathbb{C}^{n-1}$.

Example 2

$M = \mathbb{CP}^3$ and G consists of all maps from $G(\mathbb{CP}^3)$ with $U \in Sp_2$.

Example 3

M is the Hopf manifold $\mathbb{C}^{n*}/\{z \sim dz\}$, for $d \in \mathbb{C}^*$, $|d| \neq 1$, and G consists of all maps of the form

$$[z] \mapsto [\lambda U z]\,,$$

where $U \in U_n$, $\lambda > 0$, and $[z] \in M_d$ denotes the equivalence class of a point $z \in \mathbb{C}^{n*}$.

Example 4

Let $n = 3$ and $(z : w)$ be homogeneous coordinates in $\mathbb{CP}^3$ with $z = (z_1 : z_2)$, $w = (w_1 : w_2)$. Set $M = \mathbb{CP}^3 \setminus \{w = 0\}$ and let G be the group of all maps of the form

$$\begin{array}{rcl} z & \mapsto & Uz + Aw, \\ w & \mapsto & Vw, \end{array}$$

where $U, V \in SU_2$, and

$$A = \begin{pmatrix} a & i\bar{b} \\ b & -i\bar{a} \end{pmatrix},$$

for some $a, b \in \mathbb{C}$.

Example 5

$M = \mathbb{C}^3$, and G consists of all maps of the form

$$\begin{aligned} z' &\mapsto Uz' + a, \\ z_3 &\mapsto \det U\, z_3 + \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Uz' \right] \cdot a + b, \end{aligned}$$

where $z' := (z_1, z_2)$, $U \in U_2$, $a \in \mathbb{C}^2$, $b \in \mathbb{C}$, and $\cdot$ is the dot product in $\mathbb{C}^2$.

Example 6

$M = \mathbb{B}^1 \times \mathbb{C}$, and G consists of all maps of the form

$$z_1 \mapsto \frac{az_1 + b}{\overline{b}z_1 + \overline{a}},$$

$$z_2 \mapsto \frac{z_2 + cz_1 + \overline{c}}{\overline{b}z_1 + \overline{a}},$$

where $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$, $c \in \mathbb{C}$.

Example 7

$M = \mathbb{C}^{n-1} \times \{\operatorname{Re} z_n > 0\}$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \lambda^T U z' + a, \\ z_n &\mapsto \lambda z_n + ib, \end{aligned}$$

where $z' := (z_1, \ldots, z_{n-1})$, $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$.

Kobayashi-Hyperbolic Manifolds

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. The Poincaré distance is defined as

$$\rho(p, q) := \frac{1}{2} \ln \frac{1 + \left| \dfrac{p-q}{1-\overline{p}q} \right|}{1 - \left| \dfrac{p-q}{1-\overline{p}q} \right|},$$

where $p, q \in \Delta$. The Kobayashi pseudodistance on M can now be introduced as follows:

$$K_M(p,q) = \inf \sum_{j=1}^{m} \rho(s_j, t_j),$$

for all $p, q \in M$, where the inf is taken over all $m \in \mathbb{N}$, all pairs of points $\{s_j, t_j\}_{j=1,\dots,m}$ in Δ and all collections of holomorphic maps $\{f_j\}_{j=1,\dots,m}$ from Δ into M such that $f_1(s_1) = p$, $f_m(t_m) = q$, and $f_j(t_j) = f_{j+1}(s_{j+1})$ for $j = 1, \dots, m-1$. Then K_M is a pseudodistance on M, which does not increase under holomorphic maps, i.e. for any holomorphic map f between two complex manifolds M_1 and M_2 we have

$$K_{M_2}(f(p), f(q)) \le K_{M_1}(p,q),$$

for all $p, q \in M_1$. In particular, K_M is $\mathrm{Aut}(M)$-invariant.

A complex manifold M for which the pseudodistance K_M is a distance is called Kobayashi-hyperbolic or simply hyperbolic. The Kobayashi pseudodistance is always continuous, hence by Theorem 7, the group $\mathrm{Aut}(M)$ acts properly on M, provided M is hyperbolic.

From now on we assume that M is hyperbolic and $G = \mathrm{Aut}(M)^0$. In this situation, in addition to the results for general proper actions, there are complete explicit classifications for $\dim G = n^2$, and $\dim G = n^2 - 1$.

Theorem 15 [Kim, Verdiani (2004)] *Let M be a simply-connected complete hyperbolic manifold of dimension $n \geq 2$, and $\dim G = n^2$. Then M is holomorphically equivalent to one of the following domains:*

(i) a Thullen domain

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'||^2 + |z_n|^\alpha < 1\right\},$$

with $\alpha > 0$, $\alpha \neq 2$;

(ii) $\mathbb{B}^1 \times \mathbb{B}^1 \times \mathbb{B}^1$ *(here $n = 3$);*

(iii) $\mathbb{B}^2 \times \mathbb{B}^2$ *(here $n = 4$).*

Observe that in cases (ii) and (iii) the action of G is transitive, whereas in case (i) it is not.

Theorem 16 [Isaev (2005)] *Let M be a hyperbolic manifold of dimension $n \geq 2$, and $\dim G = n^2$. Then M is holomorphically equivalent either to one of the domains listed in Theorem 15 or to one of:*

(i) $\{z \in \mathbb{C}^n : r < ||z|| < 1\}/\mathbb{Z}_k$, $0 \leq r < 1$, $k \in \mathbb{N}$;

(ii) $\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'|| < 1, \, |z_n| < \left(1 - ||z'||^2\right)^{\alpha}\right\}$, $\alpha < 0$;

(iii) $\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'|| < 1, \, r\left(1 - ||z'||^2\right)^{\alpha} < |z_n| < \left(1 - ||z'||^2\right)^{\alpha}\right\}$, *with either* $\alpha \geq 0$, $0 \leq r < 1$, *or* $\alpha < 0$, $r = 0$;

(iv) $\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'|| < 1,\ r\left(1 - ||z'||^2\right)^{\alpha} < \exp\left(Re\, z_n\right) < \left(1 - ||z'||^2\right)^{\alpha}\right\}$, *with either* $\alpha = 1$, $0 \le r < 1$, *or* $\alpha = -1$, $r = 0$ *(the universal covers of domains (iii) for* $\alpha \neq 0$*);*

(v) $\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r \exp\left(\alpha ||z'||^2\right) < |z_n| < \exp\left(\alpha ||z'||^2\right)\right\}$, *with either* $\alpha = 1$, $0 < r < 1$, *or* $\alpha = -1$, $r = 0$*;*

(vi) $\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : -1 + ||z'||^2 < Re\, z_n < ||z'||^2\right\}$ *(the universal cover of domains (v) for* $\alpha = 1$*).*

In all these cases the action of G is not transitive.

Theorem 17 [Isaev (2005)] *Let M be a hyperbolic manifold of dimension $n \geq 3$, and $\dim G = n^2 - 1$. Then M is holomorphically equivalent to one of the following:*

(i) $\mathbb{B}^{n-1} \times S$, *where S is a hyperbolic Riemann surface with discrete automorphism group;*

(ii) $$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (Re\, z_1)^2 + (Re\, z_2)^2 + (Re\, z_3)^2 - (Re\, z_4)^2 < 0,\ Re\, z_4 > 0\}.$$

In case (ii) the action of G is transitive, whereas in case (i) it is not. The domain in case (ii) is the symmetric classical domain of type $(I_{2,2})$ with $G = SU_{2,2}/\mathbb{Z}_4$.

The classification for $n = 2$, $\dim G = 3$ (Isaev (2005-06)) includes the following domains:

(i) $\left\{(z : w : \zeta) \in \mathbb{CP}^2 : s|z^2 + w^2 + \zeta^2| < |z|^2 + |w|^2 + |\zeta|^2 < t|z^2 + w^2 + \zeta^2|\right\}$, $1 \leq s < t < \infty$;

(ii) $\left\{(z, w) \in \mathbb{C}^2 : s|z^2 + w^2 - 1| < |z|^2 + |w|^2 - 1 < t|z^2 + w^2 - 1|\right\}$, $-1 \leq s < t \leq 1$;

(iii) $\left\{(z, w) \in \mathbb{C}^2 : s|1 + z^2 - w^2| < 1 + |z|^2 - |w|^2 < t|1 + z^2 - w^2|, Im\,(z(1 + \overline{w})) > 0\right\}$, $1 \leq s < t \leq \infty$.