Introduction

Let \( M \) be a locally compact Hausdorff topological space and \( \mathcal{H}(M) \) the group of homeomorphisms of \( M \). We introduce on \( \mathcal{H}(M) \) the compact-open topology, which is the topology with subbase given by all sets of the form \( \{ f \in \mathcal{H}(M) : f(K) \subset U \} \), where \( K \subset M \) is compact and \( U \subset M \) is open. \( \mathcal{H}(M) \) is a topological group in this topology, if, in addition, \( M \) is locally connected.

A topological group \( G \) is said to act continuously on \( M \) by homeomorphisms, if a continuous homomorphism \( \Phi : G \rightarrow \mathcal{H}(M) \) is specified. The continuity of \( \Phi \) is equivalent to the continuity of the action map

\[
\hat{\Phi} : G \times M \rightarrow M, \quad (g, p) \mapsto \Phi(g)(p) =: gp.
\]

By passing to a quotient of \( G \) if necessary, we assume that the action is effective (the kernel of \( \Phi \) is trivial).
The action of $G$ on $M$ is proper, if the map

$$\psi : G \times M \rightarrow M \times M, \quad (g, p) \mapsto (gp, p),$$

is proper, i.e. for every compact subset $C \subset M \times M$ its inverse image $\psi^{-1}(C) \subset G \times M$ is compact as well. Actions of compact groups are always proper.

**Theorem 1** [van Dantzig, van der Wärden (1928)] *If $(M, d)$ is a connected locally compact metric space, then the group $\text{Isom}_d(M)$ of all isometries of $M$ with respect to $d$ (considered with the compact-open topology) is a topological group acting properly on $M$ by homeomorphisms.*
We will be interested in actions by diffeomorphisms on smooth manifolds, in which case $M$ is a connected $C^\infty$-smooth manifold and $\Phi$ maps $G$ into the group $\text{Diff}(M) \subset \mathcal{H}(M)$ of all $C^\infty$-smooth diffeomorphisms of $M$.

The properness of the action implies:

- $G$ is locally compact, hence it carries the structure of a Lie group and the action map $\hat{\Phi}$ is smooth;
- $G$ and $\Phi(G)$ are isomorphic as topological groups;
- $\Phi(G)$ is a closed subgroup of $\text{Diff}(M)$.

Thus, we can assume that $G$ is a Lie group acting smoothly and properly on the manifold $M$, and that it is realized as a closed subgroup of $\text{Diff}(M)$ (hence $\Phi = \text{id}$).
Let \((M, g)\) be a Riemannian manifold. One can define a distance associated to the metric \(g\):

\[
d_g(p, q) := \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_g \, dt, \quad p, q \in M,
\]

where the inf is taken over all piecewise \(C^1\)-smooth curves \(\gamma : [0, 1] \to M\), such that \(\gamma(0) = p, \gamma(1) = q\). The distance \(d_g\) is continuous and inner, and hence induces the topology of \(M\). Thus, \(d_g\) turns \(M\) into a connected locally compact topological space. Therefore, by Theorem 1, the group \(\text{Isom}_{d_g}(M)\) (considered with the compact-open topology) acts properly on \(M\) by homeomorphisms.
In 1939 Myers and Steenrod showed that the group $\text{Isom}_{d_g}(M)$ coincides with the group $\text{Isom}(M, g) \subset \text{Diff}(M)$ of all isometries of $M$ with respect to $g$. Hence they obtained:

**Theorem 2 [Myers, Steenrod (1939)]** If $(M, g)$ is a Riemannian manifold, then the group $\text{Isom}(M, g)$ (considered with the compact-open topology) acts properly on $M$.

Conversely, the following holds:

**Theorem 3 [Palais (1961)]** If $G$ acts properly on a smooth manifold $M$, then $M$ admits a smooth $G$-invariant Riemannian metric.

Thus, Lie groups acting properly and effectively on a manifold $M$ by diffeomorphisms are precisely closed subgroups of $\text{Isom}(M, g)$ for all possible smooth Riemannian metrics $g$ on $M$. 
If $G$ acts properly on $M$, then for every $p \in M$ its isotropy subgroup

$$G_p := \{ g \in G : gp = p \}$$

is compact in $G$. Then the isotropy representation

$$\alpha_p : G_p \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg_p$$

is continuous and faithful. In particular, the linear isotropy group

$$LG_p := \alpha_p(G_p)$$

is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to $G_p$. In some coordinates in $T_p(M)$ the group $LG_p$ becomes a subgroup of the orthogonal group $O_m(\mathbb{R})$, where $m := \dim M$. Hence

$$\dim G_p \leq \dim O_m(\mathbb{R}) = m(m - 1)/2.$$
Furthermore, the orbit of $p$

$$G(p) := \{gp : g \in G\}$$

is a closed submanifold of $M$, and $\dim G(p) \leq m$. Thus, we obtain

$$\dim G = \dim G_p + \dim G(p) \leq m(m + 1)/2.$$ 

**Theorem 4** [Fubini (1903); E. Cartan (1928); Eisenhart (1933)]  
If $G$ acts properly on a smooth manifold $M$ of dimension $m$ and $\dim G = m(m + 1)/2$, then $M$ is isometric (with respect to a $G$-invariant metric) either to one of the standard complete simply-connected spaces of constant sectional curvature ($\mathbb{R}^m$, $S^m$, $\mathbb{H}^m$), or to $\mathbb{R}P^m$. 
Theorem 5 [H.-C. Wang (1947); I. P. Egorov (1949); Yano (1953)]
A group $G$ with

$$m(m - 1)/2 + 1 < \dim G < m(m + 1)/2$$

cannot act properly on a smooth manifold $M$ of dimension $m \neq 4$.

Theorem 6 [Ishihara (1955)] Let $M$ be a smooth manifold of dimension 4. Then a group of dimension 9 cannot act properly on $M$. If $M$ admits a proper action of an 8-dimensional group $G$, then $M$ has a $G$-invariant complex structure.
There exists also an explicit classification of pairs \((M, G)\), where \(m \geq 4\), \(G\) is connected, and \(\dim G = \frac{m(m-1)}{2} + 1\), due to Yano (1953) (a local classification), Ishihara (1955) (the case \(m = 4\)), Obata (1955), Kuiper (1956).

Further, Kobayashi and Nagano obtained a (reasonably explicit) classification of pairs \((M, G)\), where \(m \geq 6\), \(G\) is connected, and \((m-1)(m-2)/2 + 2 \leq \dim G \leq \frac{m(m-1)}{2}\) (1972).

There are many other results (Wu-Yi Hsiang, I. P. Egorov, Ishihara, Mann, Jänich, Wakakuwa, ...), especially for compact subgroups, but no complete classifications exist beyond dimension \((m-1)(m-2)/2 + 2\).
We will now assume that $M$ is a complex manifold of dimension $n \geq 2$ (hence $m = 2n \geq 4$) and that a (real) Lie group $G$ acts on $M$ by holomorphic transformations, that is,

$$G \subset \text{Aut}(M) \subset \text{Diff}(M),$$

where $\text{Aut}(M)$ is the group of holomorphic automorphisms of $M$.

**Theorem 7 [Kaup (1967)]** If $G \subset \text{Aut}(M)$ is closed and preserves a continuous distance on $M$, then $G$ acts properly on $M$.

Thus, Lie groups acting properly and effectively on a manifold $M$ by holomorphic transformations are precisely closed subgroups of $\text{Aut}(M)$ preserving a continuous distance on $M$. 
In some coordinates in $T_p(M)$ the group $LG_p$ becomes a subgroup of the unitary group $U_n$. Hence $\dim G_p \leq \dim U_n = n^2$, and therefore

$$\dim G \leq n^2 + 2n.$$ 

We note that $n^2 + 2n < (m - 1)(m - 2)/2 + 2$ for $m = 2n$ and $n \geq 5$. Thus, the group dimension range that arises in the complex case, for $n \geq 5$ lies strictly below the dimension range considered in the classical real case and thus is not covered by the existing results. Furthermore, overlaps with these results for $n = 3, 4$ occur only in relatively easy situations (described in Theorem 8 and statement (i) of Theorem 9 below) and do not lead to any significant simplifications in the complex case.

The only interesting overlap with the real case occurs for $n = 2$, $\dim G = 5$. 

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Proper Group Actions in Complex Geometry
Theorem 8 [Kaup (1967)] If $G \subset \text{Aut}(M)$ acts properly on $M$ and $\dim G = n^2 + 2n$, then $M$ is holomorphically isometric (with respect to a $G$-invariant Hermitian metric) to one of the standard complete simply-connected Kähler spaces of constant holomorphic sectional curvature:

(i) $\mathbb{B}^n := \{ z \in \mathbb{C}^n : \| z \| < 1 \}$,

(ii) $\mathbb{C}^n$,

(iii) $\mathbb{CP}^n$.

In particular, the manifold from Theorem 6 is equivalent to one of $\mathbb{B}^2$, $\mathbb{C}^2$, $\mathbb{CP}^2$. 
Let $F$ be an equivalence map. Then in case (i) $F$ transforms $G$ into $\text{Aut}(\mathbb{B}^n)$ which is the group of all transformations

$$z \mapsto \frac{Az + b}{cz + d},$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n,1}.$$

We have $\text{Aut}(\mathbb{B}^n) \simeq \text{PSU}_{n,1} := SU_{n,1}/(\text{center})$. 
In case (ii) $F$ can be chosen to transform $G$ into the group $G(\mathbb{C}^n)$ of all holomorphic automorphisms of $\mathbb{C}^n$ of the form

$$z \mapsto Uz + a,$$

where $U \in U_n$, $a \in \mathbb{C}^n$. We have $G(\mathbb{C}^n) \simeq U_n \ltimes \mathbb{C}^n$. The group $G(\mathbb{C}^n)$ is the full group of holomorphic isometries of the flat metric on $\mathbb{C}^n$. 
In case (iii) \( F \) can be chosen to transform \( G \) into the group \( G(\mathbb{CP}^n) \) of all holomorphic automorphisms of \( \mathbb{CP}^n \) of the form

\[
\zeta \mapsto U\zeta,
\]

where \( \zeta \) is a point in \( \mathbb{CP}^n \) given in homogeneous coordinates, and \( U \in SU_{n+1} \). We have \( G(\mathbb{CP}^n) \simeq PSU_{n+1} := SU_{n+1}/(\text{center}) \). The group \( G(\mathbb{CP}^n) \) is the full group of holomorphic isometries of the Fubini-Study metric on \( \mathbb{CP}^n \), and is a maximal compact subgroup of the full group of holomorphic automorphisms \( \text{Aut}(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/(\text{center}) \).
Theorem 9 [Isaev (2006)] Let $M$ be a complex manifold of dimension $n \geq 2$ and $G \subset \text{Aut}(M)$ a connected Lie group that acts properly on $M$ and has dimension satisfying $n^2 + 3 \leq \dim G < n^2 + 2n$. Then one of the following holds:

(i) $M$ is holomorphically equivalent to $\mathbb{C}^n$ by means of a map that transforms $G$ into the group $G_1(\mathbb{C}^n)$ which consists of all maps from $G(\mathbb{C}^n)$ with $U \in SU_n$ (here $\dim G = n^2 + 2n - 1$);

(ii) $n = 4$ and $M$ is holomorphically equivalent to $\mathbb{C}^4$ by means of a map that transforms $G$ into the group $G_2(\mathbb{C}^4)$ which consists of all maps from $G(\mathbb{C}^4)$ with $U \in e^{i\mathbb{R}} Sp_2$ (here $\dim G = n^2 + 3 = 19$).
Proposition 10 [Kaup (1967)] Let $M$ be a connected complex manifold of dimension $n$ and $G \subset \text{Aut}(M)$ a Lie group with $\dim G > n^2$ that acts properly on $M$. Then the action of $G$ is transitive.

Thus, the group $G$ from Theorem 9 acts transitively (i.e. for every $p \in M$ we have $\dim G(p) = 2n$) which implies:

$$n^2 - 2n + 3 < \dim LG_p < n^2.$$ 

It turns out that a connected closed subgroup of $U_n$ satisfying these inequalities either is $SU_n$ or is conjugate to $e^{i\mathbb{R}} Sp_2$ for $n = 4$. 

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Theorem 11 [Isaev (2006)] Let $M$ be a complex manifold of dimension $n \geq 2$ and $G \subset \text{Aut}(M)$ a connected Lie group that acts properly on $M$ and has dimension $n^2 + 2$. Then one of the following holds:

(i) $M$ is holomorphically equivalent to $M' \times M''$, where $M'$ is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and $M''$ is one of $\mathbb{B}^1$, $\mathbb{C}^1$, $\mathbb{CP}^1$; an equivalence map can be chosen to transforms $G$ into $G' \times G''$, where $G'$ is one of $\text{Aut}(\mathbb{B}^{n-1})$, $\text{G}(\mathbb{C}^{n-1})$, $\text{G}(\mathbb{CP}^{n-1})$, and $G''$ is one of $\text{Aut}(\mathbb{B}^1)$, $\text{G}(\mathbb{C}^1)$, $\text{G}(\mathbb{CP}^1)$, respectively;

(ii) $n = 4$ and $M$ is holomorphically equivalent to $\mathbb{C}^4$ by means of a map that transforms $G$ into the group $G_3(\mathbb{C}^4)$ which consists of all maps from $G(\mathbb{C}^4)$ with $U \in \text{Sp}_2$.

Remark. For $n = 2$ Case (i) is contained in the classification due to Ishihara of 4-dimensional real manifolds with transitive actions of 6-dimensional groups.
The group $G$ from Theorem 11 also acts transitively, and therefore for every $p \in M$ we have

$$\dim LG_p = n^2 - 2n + 2.$$ 

It turns out that the only connected closed subgroups of $U_n$ satisfying this condition are (up to conjugation) $U_{n-1} \times U_1$ and $Sp_2$ for $n = 4$. 
The Case \( \dim G = n^2 + 1 \)

Again, the action of \( G \) is transitive, and therefore for every \( p \in M \) we have

\[
\dim LG_p = (n - 1)^2.
\]

Lemma 12 [Isaev, Kruzhilin (2002)] Let \( H \) be a connected closed subgroup of \( U_n \) of dimension \((n - 1)^2\), \( n \geq 2 \). Then \( H \) is conjugate in \( U_n \) to one of the following subgroups:

(i) \( e^{iR}SO_3(\mathbb{R}) \) (here \( n = 3 \));
(ii) \( SU_{n-1} \times U_1 \) for \( n \geq 3 \);
(iii) the subgroup \( H_{k_1,k_2}^n \) of all matrices

\[
\begin{pmatrix}
A & 0 \\
0 & a
\end{pmatrix},
\]

for some \( k_1, k_2 \in \mathbb{Z}, (k_1, k_2) = 1, k_2 > 0 \), where \( A \in U_{n-1} \) and \( a \in (\det A)^{k_1/k_2} \).
Theorem 13 [Isaev (2006)] Let $M$ be a complex manifold of dimension 3 and $G \subset \text{Aut}(M)$ a connected Lie group that acts properly on $M$ and has dimension $10 = 3^2 + 1$. If for $p \in M$ the group $LG_p^0$ is conjugate to $e^{i\mathbb{R}}SO_3(\mathbb{R})$, then one of the following holds:

(i) $M$ is holomorphically equivalent to the Siegel space (the symmetric classical domain of type $(\text{III}_2)$)

$$\mathcal{S} := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : Z\bar{Z} \ll id\},$$

where

$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix};$$

in this case any equivalence map transforms $G$ into $\text{Aut}(\mathcal{S}) \simeq \text{Sp}_4(\mathbb{R})/\mathbb{Z}_2$.
(ii) $M$ is holomorphically equivalent to the complex quadric $Q_3 \subset \mathbb{CP}^4$ by means of a map that transforms $G$ into the group $SO_5(\mathbb{R})$ (which is a maximal compact subgroup of $\text{Aut}(Q_3) \simeq SO_5(\mathbb{C})$);

(iii) $M$ is holomorphically equivalent to $\mathbb{C}^3$ by means of a map that transforms $G$ into the group $G_2(\mathbb{C}^3)$ which consists of all maps from $G(\mathbb{C}^3)$ with $U \in e^{i\mathbb{R}} SO_3(\mathbb{R})$. 
Theorem 14 [Isaev (2006)] Let $M$ be a complex manifold of dimension $n \geq 3$ and $G \subset \text{Aut}(M)$ a connected Lie group that acts properly on $M$ and has dimension $n^2 + 1$. If for $p \in M$ the group $L G_p^0$ is conjugate to $SU_{n-1} \times U_1$, then $M$ is holomorphically equivalent to $\mathbb{C}^{n-1} \times M'$, where $M'$ is one of $\mathbb{B}^1$, $\mathbb{C}^1$, $\mathbb{C}P^1$, and an equivalence map can be chosen to transform $G$ into $G_1(\mathbb{C}^{n-1}) \times G'$, where $G'$ is one of the groups $\text{Aut}(\mathbb{B}^1)$, $G(\mathbb{C}^1)$, $G(\mathbb{C}P^1)$, respectively.
The case $LG^0_p$ is conjugate to $H^{n}_{k_1,k_2}$

A complete classification was obtained in a joint paper with N. Kruzhilin (Israel J. Math. 172(2009), 193–252). The classification is rather lengthy, and I will only give a few examples.
Example 1

$M$ is one of the domains

$$\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Re} \, z_n > |z'|^2\} \simeq \mathbb{B}^n,$$

$$\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Re} \, z_n < |z'|^2\} \simeq \mathbb{CP}^n \setminus (\mathbb{B}^n \cup L),$$

where $L$ is a complex hyperplane tangent to $S^{2n-1} = \partial \mathbb{B}^n$, and $G$ is the group of maps

$$z' \mapsto \lambda U z' + a,$$

$$z_n \mapsto \lambda^2 z_n + 2\lambda \langle U z', a \rangle + |a|^2 + i\alpha,$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\lambda > 0$, $\alpha \in \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^{n-1}$. 
Example 2

\[ M = \mathbb{CP}^3 \text{ and } G \text{ consists of all maps from } G(\mathbb{CP}^3) \text{ with } U \in Sp_2. \]
Example 3

$M$ is the Hopf manifold $\mathbb{C}^n^*/\{z \sim dz\}$, for $d \in \mathbb{C}^*$, $|d| \neq 1$, and $G$ consists of all maps of the form

$$[z] \mapsto [\lambda Uz],$$

where $U \in U_n$, $\lambda > 0$, and $[z] \in M_d$ denotes the equivalence class of a point $z \in \mathbb{C}^n^*$. 
Example 4

Let $n = 3$ and $(z : w)$ be homogeneous coordinates in $\mathbb{CP}^3$ with $z = (z_1 : z_2)$, $w = (w_1 : w_2)$. Set $M = \mathbb{CP}^3 \setminus \{w = 0\}$ and let $G$ be the group of all maps of the form

$$
\begin{align*}
z & \mapsto Uz + Aw, \\
w & \mapsto Vw,
\end{align*}
$$

where $U, V \in SU_2$, and

$$
A = \begin{pmatrix}
a & ib \\
b & -ia
\end{pmatrix},
$$

for some $a, b \in \mathbb{C}$. 
Example 5

$M = \mathbb{C}^3$, and $G$ consists of all maps of the form

\[
\begin{align*}
    z' & \mapsto Uz' + a, \\
    z_3 & \mapsto \det U z_3 + \left[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Uz' \right] \cdot a + b,
\end{align*}
\]

where $z' := (z_1, z_2)$, $U \in U_2$, $a \in \mathbb{C}^2$, $b \in \mathbb{C}$, and $\cdot$ is the dot product in $\mathbb{C}^2$. 
Example 6

\[ M = B^1 \times \mathbb{C}, \text{ and } G \text{ consists of all maps of the form} \]

\[ z_1 \mapsto \frac{az_1 + b}{bz_1 + \overline{a}}, \]

\[ z_2 \mapsto \frac{z_2 + cz_1 + \overline{c}}{bz_1 + \overline{a}}, \]

where \( a, b \in \mathbb{C}, \ |a|^2 - |b|^2 = 1, \ c \in \mathbb{C}. \]
Example 7

\[ M = \mathbb{C}^{n-1} \times \{ \text{Re} \, z_n > 0 \} , \] and for a fixed \( T \in \mathbb{R}^* \) the group \( G \) consists of all maps of the form

\[
\begin{align*}
z' & \mapsto \lambda^T Uz' + a, \\
z_n & \mapsto \lambda z_n + ib,
\end{align*}
\]

where \( z' := (z_1, \ldots, z_{n-1}) \), \( U \in U_{n-1} \), \( a \in \mathbb{C}^{n-1} \), \( b \in \mathbb{R} \), \( \lambda > 0 \).
Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. The Poincaré distance is defined as

$$
\rho(p, q) := \frac{1}{2} \ln \left( 1 + \frac{|p - q|}{1 - \overline{p}q} \right),
$$

where $p, q \in \Delta$. The Kobayashi pseudodistance on $M$ can now be introduced as follows:
\[ K_M(p, q) = \inf \sum_{j=1}^{m} \rho(s_j, t_j), \]

for all \( p, q \in M \), where the inf is taken over all \( m \in \mathbb{N} \), all pairs of points \( \{s_j, t_j\}_{j=1}^{m} \) in \( \Delta \) and all collections of holomorphic maps \( \{f_j\}_{j=1}^{m} \) from \( \Delta \) into \( M \) such that \( f_1(s_1) = p \), \( f_m(t_m) = q \), and \( f_j(t_j) = f_{j+1}(s_{j+1}) \) for \( j = 1, \ldots, m - 1 \). Then \( K_M \) is a pseudodistance on \( M \), which does not increase under holomorphic maps, i.e. for any holomorphic map \( f \) between two complex manifolds \( M_1 \) and \( M_2 \) we have

\[ K_{M_2}(f(p), f(q)) \leq K_{M_1}(p, q), \]

for all \( p, q \in M_1 \). In particular, \( K_M \) is \( \text{Aut}(M) \)-invariant.
A complex manifold $M$ for which the pseudodistance $K_M$ is a distance is called **Kobayashi-hyperbolic** or simply **hyperbolic**. The Kobayashi pseudodistance is always continuous, hence by Theorem 7, the group $\text{Aut}(M)$ acts properly on $M$, provided $M$ is hyperbolic.

From now on we assume that $M$ is hyperbolic and $G = \text{Aut}(M)^0$. In this situation, in addition to the results for general proper actions, there are complete explicit classifications for $\dim G = n^2$, and $\dim G = n^2 - 1$. 
Theorem 15 [Kim, Verdiani (2004)] Let $M$ be a simply-connected complete hyperbolic manifold of dimension $n \geq 2$, and $\dim G = n^2$. Then $M$ is holomorphically equivalent to one of the following domains:

(i) a Thullen domain

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'||^2 + |z_n|^{\alpha} < 1\right\},$$

with $\alpha > 0$, $\alpha \neq 2$;

(ii) $\mathbb{B}^1 \times \mathbb{B}^1 \times \mathbb{B}^1$ (here $n = 3$);

(iii) $\mathbb{B}^2 \times \mathbb{B}^2$ (here $n = 4$).

Observe that in cases (ii) and (iii) the action of $G$ is transitive, whereas in case (i) it is not.
Theorem 16 [Isaev (2005)] Let $M$ be a hyperbolic manifold of dimension $n \geq 2$, and $\dim G = n^2$. Then $M$ is holomorphically equivalent either to one of the domains listed in Theorem 15 or to one of:

(i) $\{z \in \mathbb{C}^n : r < ||z|| < 1\}/\mathbb{Z}_k$, $0 \leq r < 1$, $k \in \mathbb{N}$;

(ii) $\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'|| < 1, |z_n| < (1 - ||z'||^2)^{\alpha}\}$, $\alpha < 0$;

(iii) $\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : ||z'|| < 1, r (1 - ||z'||^2)^{\alpha} < |z_n| < (1 - ||z'||^2)^{\alpha}\}$, with either $\alpha \geq 0$, $0 \leq r < 1$, or $\alpha < 0$, $r = 0$. 

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(iv) \( \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|z'\| < 1, \ r \ (1 - \|z'\|^2)^\alpha < \exp(\text{Re} \ z_n) < (1 - \|z'\|^2)^\alpha \} \), with either \( \alpha = 1, 0 \leq r < 1 \), or \( \alpha = -1, r = 0 \) (the universal covers of domains (iii) for \( \alpha \neq 0 \));

(v) \( \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r \exp(\alpha \|z'\|^2) < |z_n| < \exp(\alpha \|z'\|^2) \} \), with either \( \alpha = 1, 0 < r < 1 \), or \( \alpha = -1, r = 0 \);

(vi) \( \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : -1 + \|z'\|^2 < \text{Re} \ z_n < \|z'\|^2 \} \)
(\text{the universal cover of domains (v) for } \alpha = 1).$

In all these cases the action of \( G \) is not transitive.
Theorem 17 [Isaev (2005)] Let $M$ be a hyperbolic manifold of dimension $n \geq 3$, and $\dim G = n^2 - 1$. Then $M$ is holomorphically equivalent to one of the following:

(i) $\mathbb{B}^{n-1} \times S$, where $S$ is a hyperbolic Riemann surface with discrete automorphism group;

(ii) $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\text{Re } z_1)^2 + (\text{Re } z_2)^2 + (\text{Re } z_3)^2 - (\text{Re } z_4)^2 < 0, \text{Re } z_4 > 0\}$. 

In case (ii) the action of $G$ is transitive, whereas in case (i) it is not. The domain in case (ii) is the symmetric classical domain of type $(I_{2,2})$ with $G = SU_{2,2}/\mathbb{Z}_4$. 
The classification for $n = 2, \dim G = 3$ (Isaev (2005-06)) includes the following domains:

(i) $\{(z : w : \zeta) \in \mathbb{CP}^2 : s|z^2 + w^2 + \zeta^2| < |z|^2 + |w|^2 + |\zeta|^2 < t|z^2 + w^2 + \zeta^2| \}, \ 1 \leq s < t < \infty$;

(ii) $\{(z, w) \in \mathbb{C}^2 : s|z^2 + w^2 - 1| < |z|^2 + |w|^2 - 1 < t|z^2 + w^2 - 1| \}, -1 \leq s < t \leq 1$;

(iii) $\{(z, w) \in \mathbb{C}^2 : s|1 + z^2 - w^2| < 1 + |z|^2 - |w|^2 < t|1 + z^2 - w^2|, \text{Im}(z(1 + \overline{w})) > 0 \}, \ 1 \leq s < t \leq \infty$. 