A quadratically enriched Abramovich-Bertram formula

Joint work with Kirsten Wickelgren

Erwan Brugallé
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Laboratoire de Mathématiques Jean Leray
Nantes Université
A quick overview

- How many lines through 2 points?
A quick overview

- How many lines through 2 points? 1

Theorem (Kass, Levine, Solomon, Wickelgren)
One can count rational curves in del Pezzo surfaces with quadratic forms instead of integers.

In general generality:
$L_1, \ldots, L_m$ extensions of $k$, such that
\[ \sum_i [L_i : k] = 8 \]

\[ 2^{-1} + 2^{-1} + \sum_i \text{Tr} L_i / k(\langle 1 \rangle) \]

\[ 2 \]
A quick overview

- How many lines through 2 points? \(1\)
- How many conics through 5 points?
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- How many rational cubics through 8 points?
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- How many conics through 5 points? 1
- How many rational cubics through 8 points?
  depends on the *(nice)* base field $k$

In general: $10 < 1^2 + 8^2 + \sum_{i} \text{Tr} L_i/k(<1)$
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**Goal**

*Relate these quadratic invariants for different k-forms on the same underlying surface over \( \overline{k} \).*
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Generalizes previous results over $\mathbb{R} (B)$.
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Exemple

• $\mathbb{P}^2(\mathbb{R})$ blown-up at two real points – $\mathbb{P}^2(\mathbb{R})$ blown-up at two complex conjugated points.
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\[ x^2 - y^2 + z^2 - \delta w^2 = 0 \]

$\delta \in k^*/(k^*)^2$
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- Quadrics $Q_\delta$ in $\mathbb{P}^3$ with equation

$$x^2 - y^2 + z^2 - \delta w^2 = 0 \quad \delta \in k^*/(k^*)^2$$
Enumerative invariants over $\mathbb{R}$, as well as several of their properties, generalize to arbitrary nice fields $k$. 
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Message to take home

Enumerative invariants over $\mathbb{R}$, as well as several of their properties, generalize to arbitrary nice fields $k$.

Proofs over $\mathbb{R}$ generalize to arbitrary nice fields $k$.

If you proved something in real enumerative geometry, you may proved it in enumerative geometry over any nice field!
Examples

- Blow-up of $\mathbb{P}^2$ at two points

**Theorem (B-Wickelgren)**

$$N_{\mathbb{P}^2_{k(\sqrt{\delta})}, \sigma}(dL - aE_1 - aE_2) = N_{\mathbb{P}^2_{k,k}, \sigma}(dL - aE_1 - aE_2)$$
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\[ N_{\mathbb{P}^2_{k(\sqrt{\delta})},\sigma}(dL - aE_1 - aE_2) = N_{\mathbb{P}^2_{k,k'},\sigma}(dL - aE_1 - aE_2) \]
\[ - (\langle 2 \rangle - \langle 2\delta \rangle) \sum_{j \geq 1} (-1)^j N_{\mathbb{P}^2_{k,k'},\sigma}(dL - (a + j)E_1 - (a - j)E_2) \]
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- $Q_\delta : \ x^2 - y^2 + z^2 - \delta w^2 = 0 \quad \delta \in k^*/(k^*)^2$

$Q_1 = \mathbb{P}^1 \times \mathbb{P}^1, \quad \text{Pic}(Q_1) = \mathbb{Z}^2$
Examples

• Blow-up of \( \mathbb{P}^2 \) at two points

Theorem (B-Wickelgren)

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N_{\mathbb{P}^2, k(\sqrt{\delta}), \sigma}(dL - aE_1 - aE_2) = N_{\mathbb{P}^2, k, \sigma}(dL - aE_1 - aE_2)
- (\langle 2 \rangle - \langle 2\delta \rangle) \sum_{j \geq 1} (-1)^j N_{\mathbb{P}^2, k, \sigma}(dL - (a + j)E_1 - (a - j)E_2)
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• \( Q_\delta : x^2 - y^2 + z^2 - \delta w^2 = 0 \quad \delta \in k^*/(k^*)^2 \)

\( Q_1 = \mathbb{P}^1 \times \mathbb{P}^1 \), \( \text{Pic}(Q_1) = \mathbb{Z}^2 \supset \mathbb{Z}(1, 1) = \text{Pic}(Q_\delta) \quad \delta \neq 1 \)
Examples

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**Theorem (B-Wickelgren)**

\[
N_{Q_\delta,\sigma}(a,a) = N_{Q_1,\sigma}(a,a)
\]
Examples

- Blow-up of $\mathbb{P}^2$ at two points

**Theorem (B-Wickelgren)**

$$N_{\mathbb{P}^2_{k(\sqrt{\delta})},\sigma}(dL - aE_1 - aE_2) = N_{\mathbb{P}^2_{k,k},\sigma}(dL - aE_1 - aE_2)$$

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- $Q_\delta: \quad x^2 - y^2 + z^2 - \delta w^2 = 0 \quad \delta \in k^*/(k^*)^2$

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**Theorem (B-Wickelgren)**

$$N_{Q_\delta,\sigma}(a,a) = N_{Q_1,\sigma}(a,a) + (<2> - <2\delta>) \sum_{j \geq 1} (-1)^j N_{Q_1,\sigma}(a+j, a-j)$$
Quadratically enriched enumeration

\[ x^2 - y^2 = 0 \]
\[ \langle 1 \rangle \]

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\[ \langle 1 \rangle + \langle 1 \rangle + \langle 1 \rangle = 3 \]

\[ \langle 1 \rangle \]
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\[ x^2 - y^2 = 0 \quad <1> \]

\[ x^2 - y^2 = 0 \quad + \quad <1> \]

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\[ <1> + <1> + <1> = 3 <1> \]
Quadratically enriched enumeration

\[ \text{Gal}(\mathbb{C} : \mathbb{R}) = \{ \text{Id}, \text{conj} \} \]

\[ x^2 - y^2 = 0 \quad x^2 - y^2 = 0 \quad x^2 - y^2 = 0 \]

\[ <1> \quad <1> \quad <1> \]
Quadratically enriched enumeration

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< 1 >

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< 1 >
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\[ x^2 - y^2 = 0 \]
\[ x^2 + y^2 = 0 \]
\[ x^2 - y^2 = 0 \]

\(<1> \]
\(<-1> \]
Quadratically enriched enumeration

\[ \text{Gal}(\mathbb{C} : \mathbb{R}) = \{ \text{id}, \text{conj} \} \]

\[
\begin{align*}
& x^2 - y^2 = 0 \\
& \langle 1 \rangle \\
& x^2 + y^2 = 0 \\
& \langle -1 \rangle \\
& x^2 - y^2 = 0 \\
& \langle 1 \rangle 
\end{align*}
\]
Quadratically enriched enumeration

\[ Gal(\mathbb{C} : \mathbb{R}) = \{ id, \text{conj} \} \]

\[
\begin{align*}
  x^2 - y^2 &= 0 & x^2 + y^2 &= 0 & x^2 + y^2 &= 0 \\
  <1> & & <1> & & <1> \\
  <1> & & <1> & & <1>
\end{align*}
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\begin{align*}
    x^2 - y^2 &= 0 \\
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\end{align*}
\]

\[
\begin{align*}
    < 1 > &+ < -1 > + < -1 > = < 1 > + 2 < -1 >
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\[ x^2 - y^2 = 0 \quad x^2 + y^2 = 0 \quad x^2 + y^2 = 0 \]

\[ <1> + <-1> + <-1> = <1> + 2 < -1 > \]
Quadratically enriched enumeration

\[ \text{Gal}(\mathbb{R}(i) : \mathbb{R}) = \{ \text{Id}, \text{conj} \} \]

\[ x^2 - y^2 = 0 \quad \text{and} \quad x^2 + y^2 = 0 \quad \text{and} \quad x^2 + y^2 = 0 \]

\[ <1> + <-1> + <-1> = <1> + 2 < -1 > \]
Quadratically enriched enumeration

\[ Gal(k(\sqrt{\delta}) : k) = \{ Id, \sigma \} \]

\[
\begin{align*}
x^2 - y^2 &= 0 \\
<1> &+ <\delta> + <\delta> = <1> + 2 <\delta>
\end{align*}
\]
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\[ 1^\sigma = 1 + 2\delta \]
Quadratically enriched enumeration

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\[ \langle 1 \rangle \quad + \quad \langle 1 \rangle + \langle 1 \rangle \quad = \langle 1 \rangle + \langle 2 \rangle + \langle 2\delta \rangle \]
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\[ Gal(k(\sqrt{\delta}) : k) = \{ \text{id}, \sigma \} \]

\[ x^2 - y^2 = 0 \]

\[ <1> \]

\[ <\text{Tr}_{k(\sqrt{\delta})/k}(<1>)) = <1> + <2> + <2\delta> \]
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\[ \langle 1 \rangle \]

\[ x^2 - y^2 = 0 \]

\[ \langle 2 \rangle + \langle 2\delta \rangle \]
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\[ x^2 - y^2 = 0 \]

\[ \langle 1 \rangle + \langle 2 \rangle + \langle 2\delta \rangle = \langle 1 \rangle + \langle 2 \rangle + \langle 2\delta \rangle \]
Quadratically enriched enumeration

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3 \langle 1 \rangle \quad \langle 1 \rangle + 2 \langle \delta \rangle \quad \langle 1 \rangle + \langle 2 \rangle + \langle 2\delta \rangle
\{\text{non-degenerate quadratic forms on } k\}/\text{equivalence} \text{ is a commutative monoid for the direct sum.}

It can be embedded in the Grothendieck-Witt group $GW(k)$, obtained by adding a formal substraction.
{non-degenerate quadratic forms on \( k \}) /\text{equivalence} \text{ is a commutative monoid for the direct sum.}

It can be embedded in the Grothendieck-Witt group \( GW(k) \), obtained by adding a formal substraction.

\( GW(k) \) is generated by rank 1 quadratic forms \( < a > \), with \( a \in k^*/(k^*)^2 \).
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It can be embedded in the Grothendieck-Witt group \( GW(k) \), obtained by adding a formal substraction.

\( GW(k) \) is generated by rank 1 quadratic forms \( <a> \), with \( a \in k^*/(k^*)^2 \).

\( GW(k) \) is a ring for the product generated by

\[ <a><b> = <ab> \].\]
Trace form

Given $[L : k] = m$, there exists an additive map

$$Tr_{L/k} : GW(L) \longrightarrow GW(k)$$

defined by the composition

$$L \xrightarrow{q} L \xrightarrow{tr_{L/k}} k$$

Note: $rk(\text{Tr}_{L/k}(q)) = m \times rk(q)$
Given $[L : k] = m$, there exists an additive map 

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$$k^m = L \xrightarrow{q} L \xrightarrow{\text{tr}_{L/k}} k$$
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k^m &= L \xrightarrow{q} L \xrightarrow{\text{tr}_{L/k}} k
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**Exemple \((\mathbb{C}/\mathbb{R})\)**

\[
\begin{align*}
\text{Tr}_{\mathbb{C}/\mathbb{R}}(\langle 1 \rangle) &= \langle 1 \rangle + \langle -1 \rangle
\end{align*}
\]
Trace form

Given \([L : k] = m\), there exists an additive map

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Tr_{L/\overline{k}} : \ GW(L) \rightarrow \ GW(k)
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k^m = L \xrightarrow{q} L \xrightarrow{tr_{L/\overline{k}}} k
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**Exemple \((\mathbb{C}/\mathbb{R})\)**

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Tr_{\mathbb{C}/\mathbb{R}}(\langle 1 \rangle) = \langle 1 \rangle + \langle -1 \rangle
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\[
(x + iy)^2 + (x - iy)^2 = 2x^2 - 2y^2
\]
Given \([L : k] = m\), there exists an additive map

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**Exemple (\(\mathbb{C}/\mathbb{R}\))**

\[ Tr_{\mathbb{C}/\mathbb{R}}(<1>) = <1> + <-1> \]

\[ (x + iy)^2 + (x - iy)^2 = 2x^2 - 2y^2 \]

Note: \(rk(Tr_{L/k}(q)) = m \times rk(q)\)
Quadratic multiplicity of a rational curve

- $C$ is a rational nodal curve in $\mathbb{P}^2(k)$ defined by a polynomial $f \in k[x, y]:$

\[
\mu_k(C) = \prod_{\text{nodes } p \text{ of } C \otimes \overline{k}} < -\det(\text{Hess}(f)_p) > \in GW(k)
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- $C = \{C_1, \ldots, C_m\}$ is the $Gal(\overline{k} : k)$-orbit of a rational nodal curve $C_1$ with residue field $L$

\[
\mu_k(C) = Tr_{L/k}(\mu_L(C_1))
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Quadratic multiplicity of a rational curve

- $C$ is a rational nodal curve in $\mathbb{P}^2(k)$ defined by a polynomial $f \in k[x, y]$

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- $C = \{C_1, \ldots, C_m\}$ is the $Gal(\overline{k} : k)$-orbit of a rational nodal curve $C_1$ with residue field $L$

$$\mu_k(C) = Tr_{L/k}(\mu_L(C_1))$$

Note: $rk(\mu_k(C)) = m$
\[ \text{car}(K) = 0 \]

Let \( d \geq 1 \), and \( \sigma = \{L_1, \ldots, L_n\} \) be \( k \)-extensions such that

\[ \sum [L_i : k] = 3d - 1 \]
Quadratically enriched invariants – $\mathbb{P}^2$

$\text{car}(K) = 0$

Let $d \geq 1$, and $\sigma = \{L_1, \ldots, L_n\}$ be $k$-extensions such that

$$\sum [L_i : k] = 3d - 1$$

Let $p_1, \ldots, p_n$ be points in $\mathbb{P}^2$ such that $p_i$ has residue field $L_i$. 
Quadratically enriched invariants – $\mathbb{P}^2$

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$$\sum [L_i : k] = 3d - 1$$

Let $p_1, \ldots, p_n$ be points in $\mathbb{P}^2$ such that $p_i$ has residue field $L_i$.

**Theorem (Kass, Levine, Solomon, Wickelgren)**

$$N_{\mathbb{P}^2, \sigma}(d) = \sum_C \mu_k(C)$$

does not depend on a generic choice of the points $p_i$, where the sum is taken over all $\text{Gal}(\overline{k} : k)$-orbits of rational curves of degree $d$ passing through the points $p_1, \ldots, p_n$. 
Replace $\mathbb{P}^2$ with a $k$-rational del Pezzo surface $S$. Replace $d$ by $D \in \text{Pic}_k(S)$, and let $\sigma = \{L_1, \ldots, L_n\}$ be $k$-extensions such that

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Remark

When $k = \mathbb{R}$, this is a reformulation of a weak version of Welschinger invariants.
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When $k = \mathbb{R}$, this is a reformulation of a weak version of Welschinger invariants.
Invariants of quadrics (C–Abramovich-Bertram, R–B-Puignau)

\[ Q_\delta : \quad x^2 - y^2 + z^2 - \delta w^2 = 0 \quad \delta \in k^*/(k^*)^2 \]
\[ Q_1 = \mathbb{P}^1 \times \mathbb{P}^1, \quad \text{Pic}(Q_1) = \mathbb{Z}^2 \supset \mathbb{Z}(1, 1) = \text{Pic}(Q_\delta) \quad \delta \neq 1 \]

**Theorem (B-Wickelgren)**

\[ N_{Q_\delta, \sigma}(a, a) = N_{Q_1, \sigma}(a, a) + (\langle 2 \rangle - \langle 2\delta \rangle) \sum_{j \geq 1} (-1)^j N_{Q_1, \sigma}(a+j, a-j) \]
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Diagram:

- \( Q_1 \): \( x^2 - y^2 + z^2 - w^2 \)
- \( Q_0 \): \( x^2 - y^2 + z^2 \)
- \( Q_{-1} \): \( x^2 - y^2 + z^2 + w^2 \)
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**Strategy** : deduce a curve count in \( Q_\delta \) by counting curves in \( Q_0 \).
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Express an invariant quantity in terms of non-invariant quantities.
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Express an invariant quantity in terms of non-invariant quantities.

**Spectacular fact**: a suitable combination of these expressions makes all non-invariant terms disappear!
Consider $x^2 - y^2 + z^2 - \delta w^2 = 0$ as a nodal quadric $\mathcal{Q}$ in $\mathbb{A}^4$. 

$x^2 - y^2 + z^2 - \delta w^2$

$\mathcal{Q}_w = \mathcal{Q}_\delta \ w \neq 0$

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From $Q_0$ to $Q_\delta$

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$\widetilde{Q} :$ blow up $Q$ at the point 0.
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In $\tilde{Q}_0$ : choose $p_1, \ldots, p_n$ in $\mathbb{F}_2$, which doesn’t know about $\delta$. 
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**Exemple** \( N_{\mathbb{P}^2,\{k^5,\mathbb{S}_6\}}(4) = 218 < 1 > + 190 < -1 > + 17 \text{Tr}(L^5) \)

(B-Rau-Wickelgren) \( + 4 \text{Tr}(L^{4,2}) + \text{Tr}(L^4) + \text{Tr}(L^{3,3}) \)