A quadratically enriched Abramovich-Bertram formula

Joint work with Kirsten Wickelgren

Erwan Brugallé

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Laboratoire de Mathématiques Jean Leray
Nantes Université

A quick overview

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- *Quadrics* Q_δ *in* $\mathbb{P}^3$ *with equation*

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If you proved something in real enumerative geometry, you may proved it in enumerative geometry over any nice field !

Examples

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Theorem (B-Wickelgren)

$$N_{\mathbb{P}^2_{k(\sqrt{\delta})},\sigma}(dL - aE_1 - aE_2) = N_{\mathbb{P}^2_{k,k},\sigma}(dL - aE_1 - aE_2)$$

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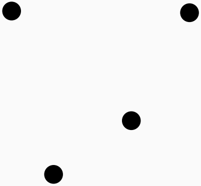
Quadratically enriched enumeration



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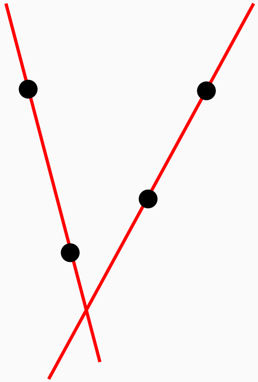

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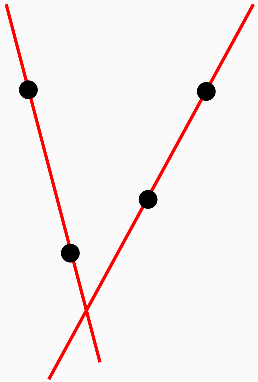

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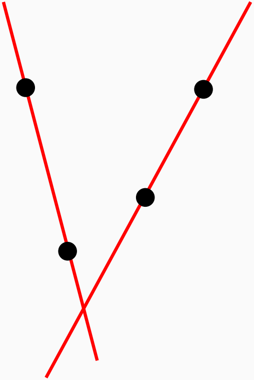

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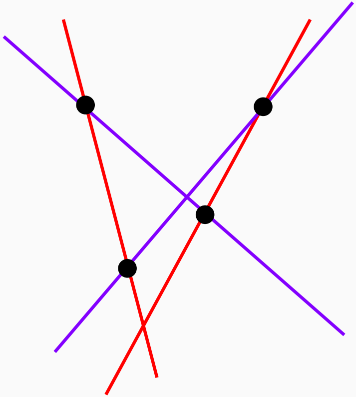

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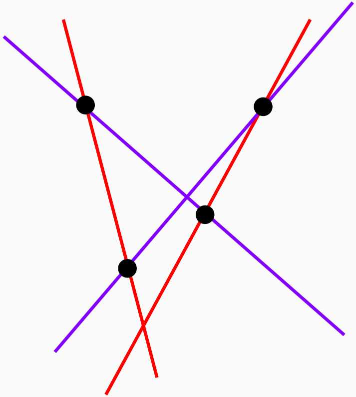

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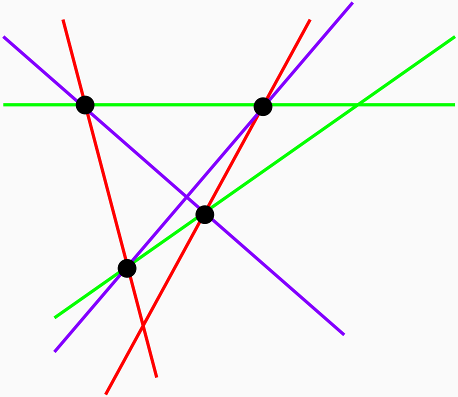

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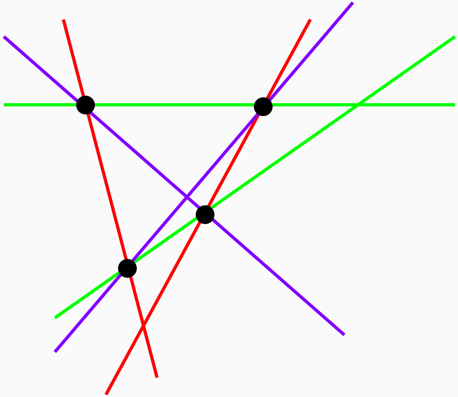

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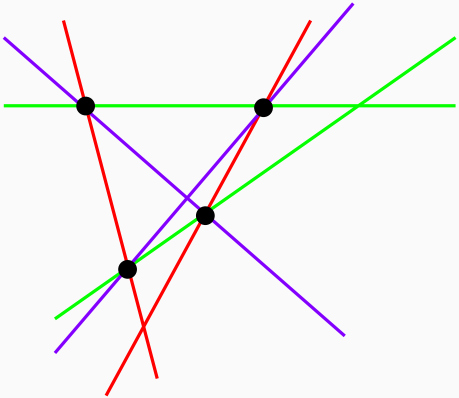

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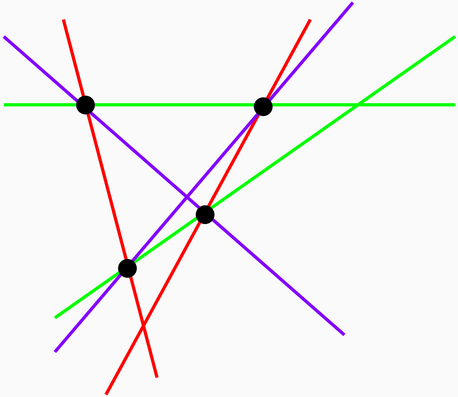

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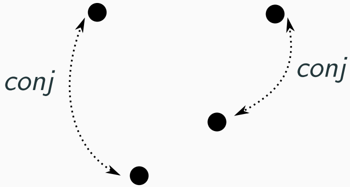

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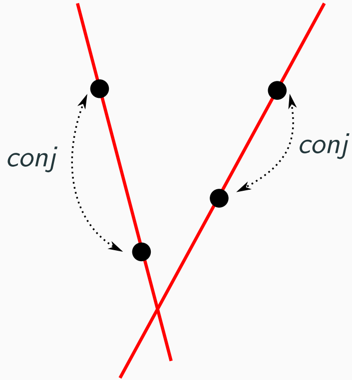

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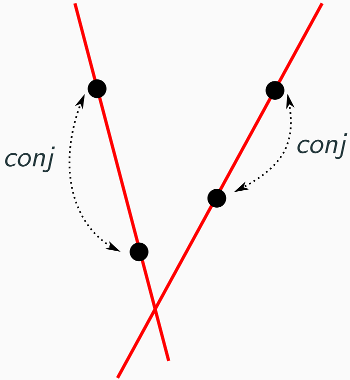

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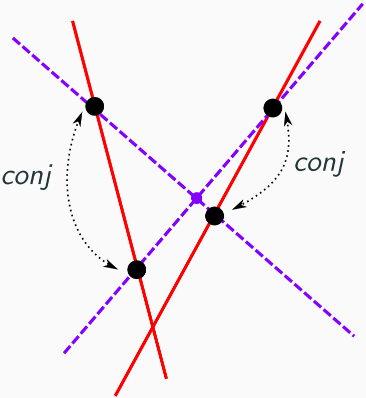

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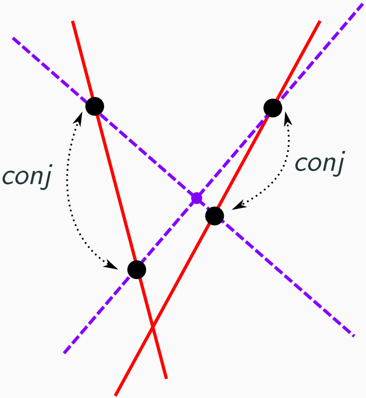

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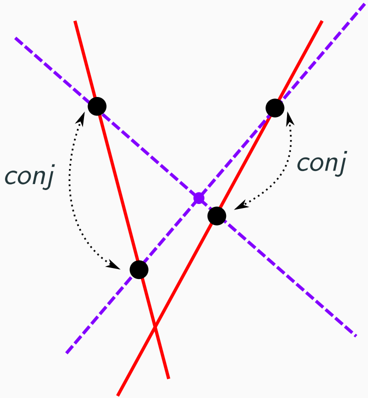

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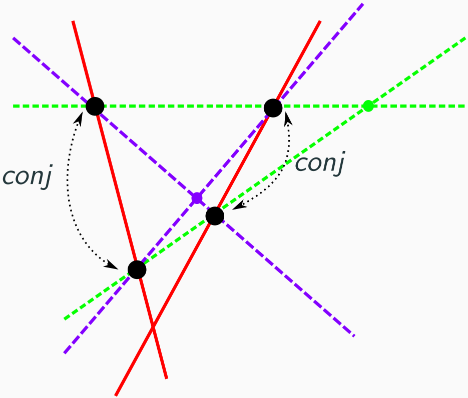

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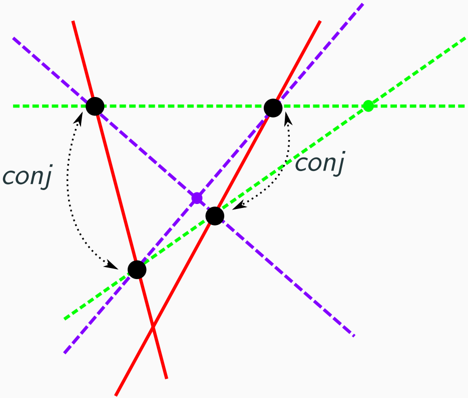

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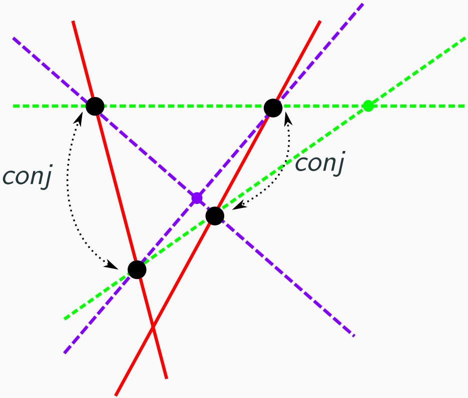

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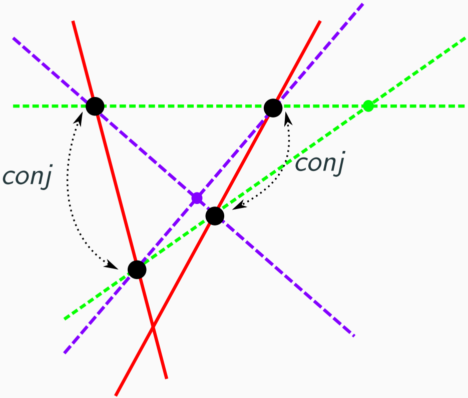

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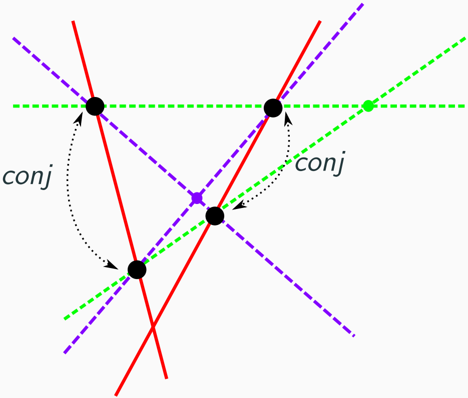

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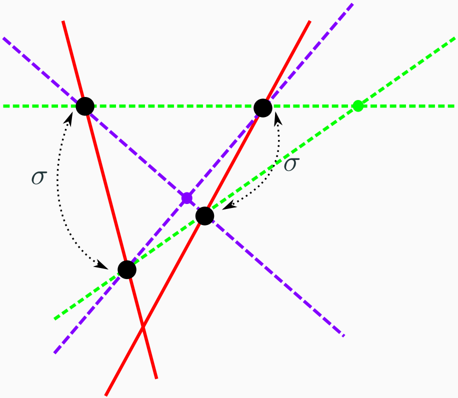

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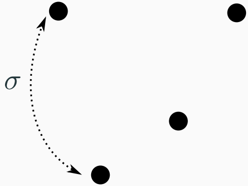

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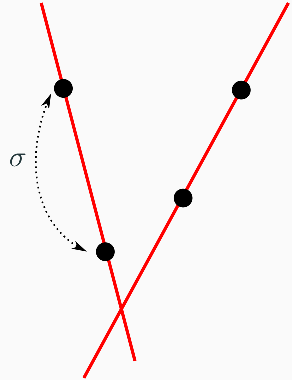

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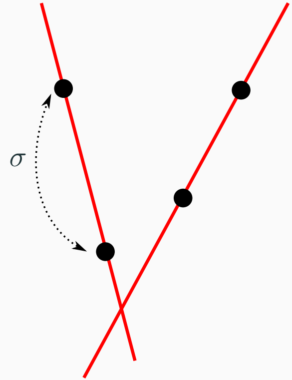

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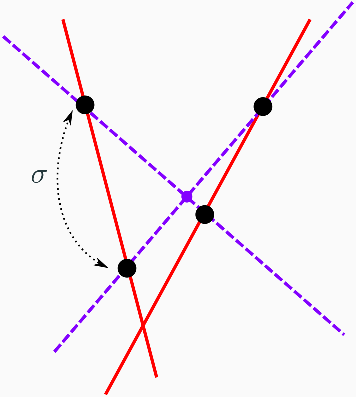

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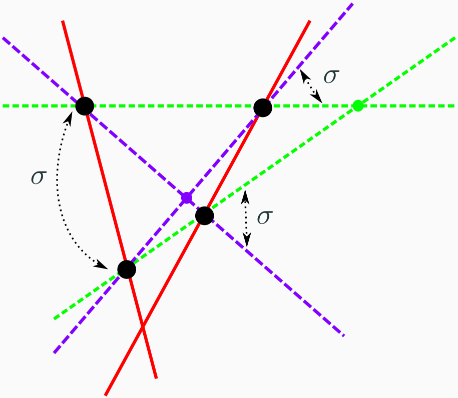

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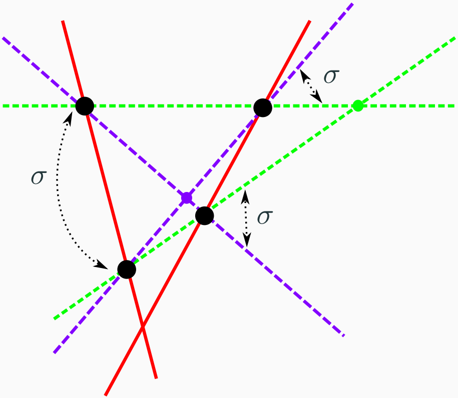

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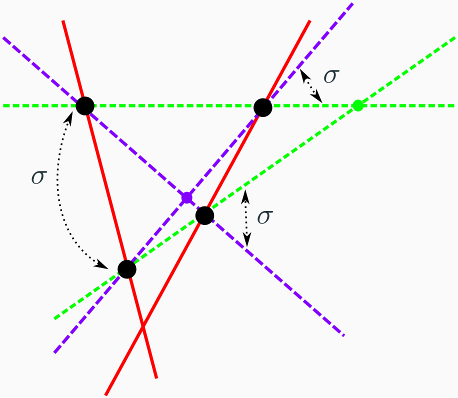

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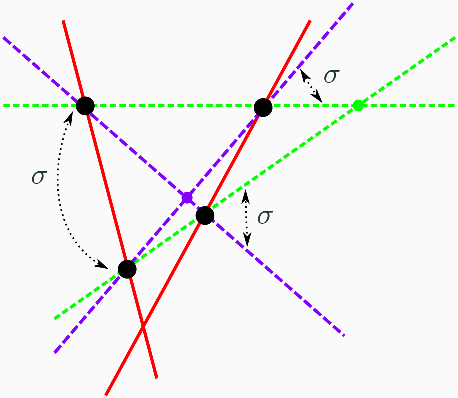

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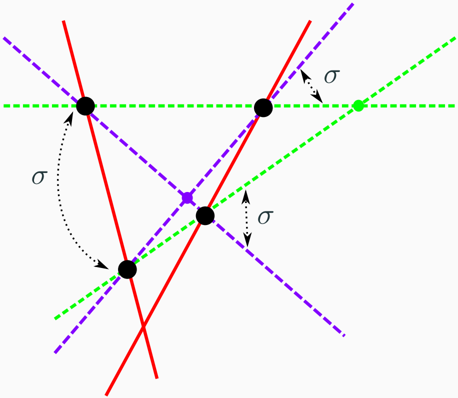

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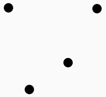
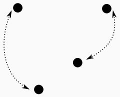


$$3 < 1 >$$



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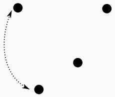

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Grothendieck-Witt ring of a field

$\{$non-degenerate quadratic forms on $k\}/$equivalence is a commutative monoid for the direct sum.

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$GW(k)$ is a ring for the product generated by

$$< a >< b >=< ab > .$$

Trace form

Given $[L : k] = m$, there exists an additive map

$$Tr_{L/k} : \quad GW(L) \longrightarrow GW(k)$$

defined by the composition

$$L \xrightarrow{\quad q \quad} L \xrightarrow{\quad tr_{L/k} \quad} k$$

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Note : $rk(Tr_{L/k}(q)) = m \times rk(q)$

Quadratic multiplicity of a rational curve

- C is a rational nodal curve in $\mathbb{P}^2(k)$ defined by a polynomial $f \in k[x, y]$:

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$car(K) = 0$

Let $d \geq 1$, and $\sigma = \{L_1, \ldots, L_n\}$ be k-extensions such that

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$$N_{\mathbb{P}^2,\sigma}(d) = \sum_C \mu_k(C)$$

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Replace $\mathbb{P}^2$ with a k-rational del Pezzo surface S. Replace d by $D \in Pic_k(S)$, and let $\sigma = \{L_1, \ldots, L_n\}$ be k-extensions such that

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Remark
When $k = \mathbb{R}$, this is a reformulation of a weak version of Welschinger invariants.

Invariants of quadrics ($\mathbb{C}$–Abramovich-Bertram, $\mathbb{R}$–B-Puignau)

$$Q_\delta : \qquad x^2 - y^2 + z^2 - \delta w^2 = 0 \qquad \delta \in k^*/(k^*)^2$$

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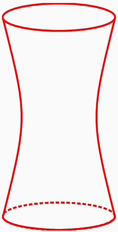

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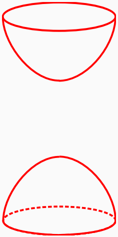

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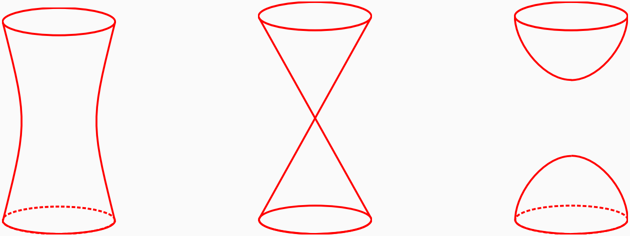

$x^2 - y^2 + z^2 - w^2$ | $x^2 - y^2 + z^2$ | $x^2 - y^2 + z^2 + w^2$

Q_1 | Q_0 | Q_{-1}

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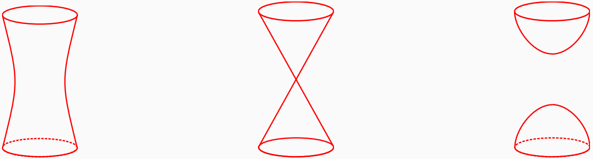

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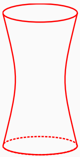

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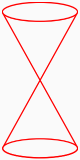

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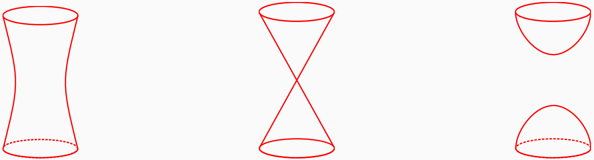

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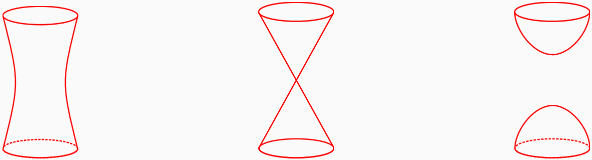

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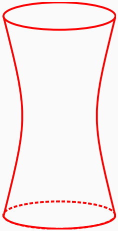
Spectacular fact : a suitable combination of these expressions makes all non-invariant terms disappear !

From Q_0 to Q_δ

Consider $x^2 - y^2 + z^2 - \delta w^2 = 0$ as a nodal quadric $\mathcal{Q}$ in $\mathbb{A}^4$.



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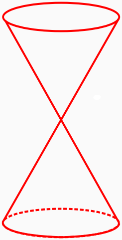

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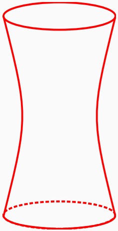

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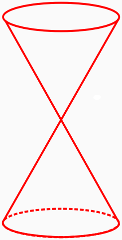

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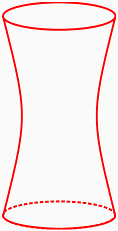

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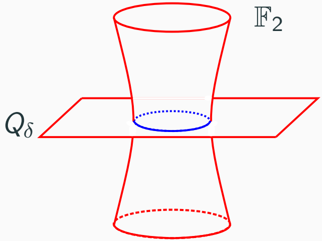

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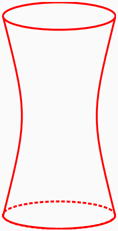

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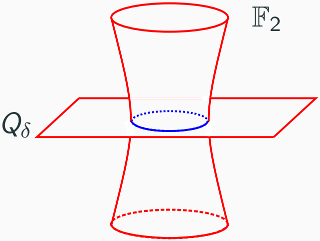

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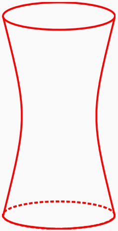

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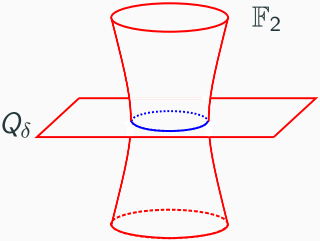

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   - ✓ $k = \mathbb{R}$ (B)
   - ✓ S toric and $\sigma = \{k, \ldots, k\}$ (Jaramillo Puentes - Pauli)
   - ✓ Q_δ and $\sigma = \{k, \ldots, k\}$ (B - Wickelgren)

Comments – speculations

1. An enlightning geometric proof ?

$$N_{Q_\delta,\sigma}(a,a) - N_{Q_\gamma,\sigma}(a,a) = (\chi(Q_\delta/k) - \chi(Q_\gamma/k)) \sum_{j\geq 1} (-1)^j N_{Q_1,\sigma}(a+j, a-j)$$

2. The field k doesn't play much role in known values of $N_{S,\sigma}$.

   Vague conjecture : There exist universal expressions for $N_{S,\sigma}$ in terms of the action of $Gal(\overline{k} : k)$ on $Pic(S \otimes \overline{k})$ and $\{k \subset L \subset \overline{k}\}$.

   - ✓ $k = \mathbb{R}$ (B)
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   Exemple $N_{\mathbb{P}^2,\{k^5,\mathfrak{S}_6\}}(4) = 218 < 1 > +190 < -1 > +17\,\mathrm{Tr}(L^5)$
   (B-Rau-Wickelgren) $+ 4\,\mathrm{Tr}(L^{4,2}) + \mathrm{Tr}(L^4) + \mathrm{Tr}(L^{3,3})$