Counting Rational Curves over Any Field

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1 Witt Invariants

K field, char $\neq 2$, V K-vs, $\dim V < \infty$, q quadratic form non-deg

DEF $$\tilde{W}(K) = \{q : V \to K\} / \text{isom}$$

semiring
$$q \oplus q' : V \oplus V' \to K$$
$$q \otimes q' : V \otimes V' \to K$$

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Witt ring

$$W(K) = \tilde{W}(K) / \underbrace{\langle h \rangle}_{\mathbb{Z} \cdot h}$$

EX

(a) $\langle a_1, \ldots, a_n \rangle = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

(b) $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \langle 1, -1 \rangle \cong \langle a, -a \rangle$ hyperbolic form
$a \in K^*$

$q \otimes h = \mathrm{rk}(q) \cdot h$

(c) $\hat{W}(\mathbb{C}) \overset{\mathrm{rk}}{\cong} \mathbb{N}$ $\qquad W(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$

(d) $W(\mathbb{R}) \underset{\mathrm{sgn}}{\cong} \mathbb{Z}$ $\qquad W(\mathbb{F}_2) \cong \mathbb{Z}/2\mathbb{Z}$

(e) $W(\mathbb{Q}) \cong W(\mathbb{R}) \oplus \bigoplus_p W(\mathbb{F}_p)$ Hasse-Minkowski

DEF $\mathcal{E}t_n(K) = \left\{ \begin{array}{l} A \text{ étale } K\text{-alg} \\ \text{of } \dim_K A = n \end{array} \right\} / \text{isom}$

$A \simeq L_1 \times L_2 \times \dots \times L_e$, $K \hookrightarrow L_i$ finite separable ext.

$n = \sum [L_i : K]$

EX $\underline{a} \in A$: $tr_A(a) = \text{trace}\left(A \xrightarrow{\cdot a} A\right) \in K$

$$Tr_A : A \longrightarrow K$$
$$a \longmapsto tr_A(a^2)$$

A étale $\iff$ Tr_A non-deg.

$k \longrightarrow L$ extension

$W(k) \longrightarrow W(L)$, $\mathcal{E}t_n(k) \longrightarrow \mathcal{E}t_n(L)$

$q \longmapsto q \otimes L$ $\quad A \longmapsto A \otimes L$

$W, \mathcal{E}t_n$: $\mathrm{Fields}_k^{\neq 2} \longrightarrow \mathrm{Sets}$

DEF An étale Witt invariant α of rank n (over k [base field]) is a natural transformation from $\mathcal{E}t_n$ to W.

DEF étale Witt invariant $\mathrm{Inv}_k(n)$, $\mathrm{Inv}^{>p}(n)$

For all $\left(\begin{smallmatrix} k \to \\ \text{or char} > p \end{smallmatrix}\right) K \to L$:

$$\begin{tikzcd} \alpha_K : \mathrm{Et}_n(K) \arrow[r] \arrow[d, "\otimes L"'] \arrow[dr, phantom, "\circlearrowleft"] & W(K) \arrow[d, "\otimes L"] \\ \alpha_L : \mathrm{Et}_n(L) \arrow[r] & W(L) \end{tikzcd}$$

EX $\mathrm{Tr} \in \mathrm{Inv}(n)$:

$$A \longmapsto \mathrm{Tr}_A$$
$$A \otimes L \longmapsto \mathrm{Tr}_A \otimes L$$

TMA (Serre, ...) $\qquad m = \lfloor \frac{n}{2} \rfloor$

$$\mathrm{Inv}_k(n) \underset{\substack{\uparrow \\ W(k)\text{-module}}}{\cong} W(k)^{m+1}$$

$\lambda_0, \ldots, \lambda_m$ basis

$\lambda_i(A) = \Lambda^i \mathrm{Tr}_A$

PROP (BRW) Exists basis $\beta_0, \ldots, \beta_m \in \mathrm{Inv}_k(n)$

s. th. $\beta_i(A_1 \times \ldots \times A_m (\times K)) = e_i(\mathrm{Tr}A_1, \ldots, \mathrm{Tr}A_m)$ $\quad$ i-th symm polyn

$A_1, \ldots, A_m \in \mathrm{Et}_2(K)$

$\beta_0 = \langle 1 \rangle \quad \beta_1 = \mathrm{Tr}$

$$\alpha = b_0\beta_0 + \dots + b_m\beta_m$$

$b_i \in W(k)$ beta-coefficients

DEF α *integral* $:\Leftrightarrow$ $b_i \in \mathbb{Z}$

$$\mathbb{Z} \to W(k), \quad n \mapsto n\langle 1\rangle$$

PROP (BRW) $k = \mathbb{Q}$

If α is integral, the *multireal values*

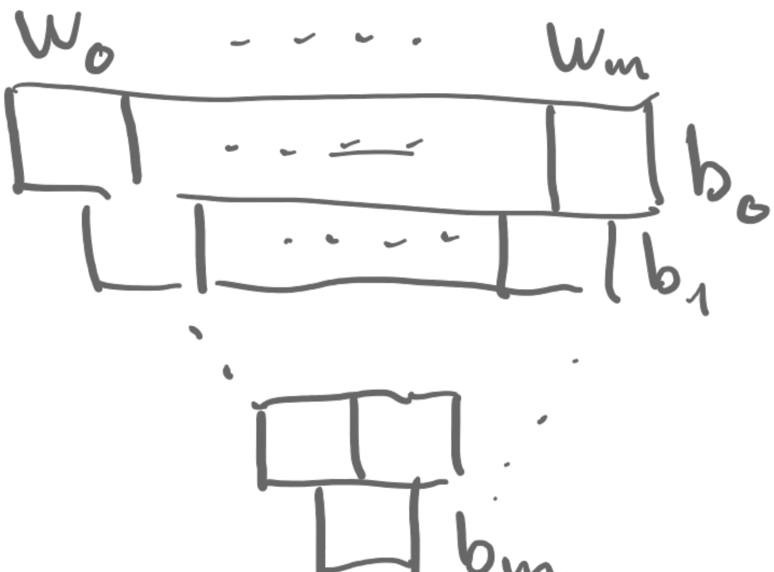
$0 \le s \le m \quad w_s = \alpha_{\mathbb{R}}(\mathbb{C}^s \times \mathbb{R}^{n-2s}) \in \mathbb{Z}$

determine α.

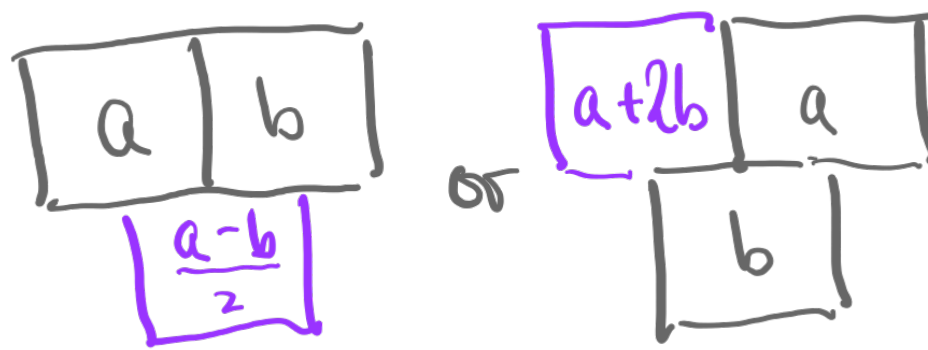
Proof

$$\mathrm{Tr}_{\mathbb{R}}(\mathbb{C}) = h = 0 \qquad \mathrm{Tr}_{\mathbb{R}}(\mathbb{R}^2) = \langle 1,1\rangle = 2$$

$$\beta_i^{\mathbb{R}}(\mathbb{C}^s \times \mathbb{R}^{n-2s}) = e_i(\underbrace{0,\dots,0}_{s},\underbrace{2,\dots,2}_{m-s}) = \binom{m-s}{i} 2^i$$



with recursion



TMA (BRW) $\alpha \in \mathrm{Inv}^{>p}(n)$, $q > p$ prime $\alpha_q \in \mathrm{Inv}_{\mathbb{F}_q}(n)$

If for all R complete DVR with $\mathcal{K} = R/\mathfrak{m}$, $K = \mathrm{Quot}(R)$ of char $> p$,

$$\begin{tikzcd} \mathrm{Et}_n(\mathcal{K}) \arrow[r, "\alpha_{\mathcal{K}}"] \arrow[d, "\text{smoothing}"] \arrow[dr, phantom, "\circlearrowleft"] & W(\mathcal{K}) \arrow[d] \\ \mathrm{Et}_n(K) \arrow[r, "\alpha_K"] & W(K) \end{tikzcd}$$

unramified

then $\alpha_\infty \in \mathrm{Inv}_{\mathbb{Q}}(n)$ determines α and all $b_i \in \mathbb{Z}[\langle a \rangle : a \leq p] \in W(\mathbb{Q})$

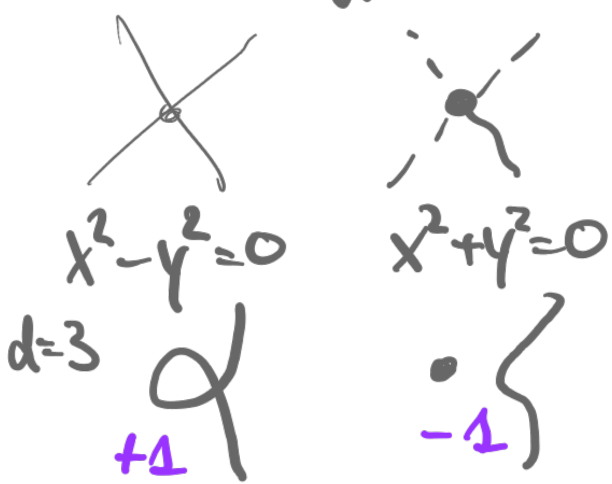
Ex $p = 3$.

$$\mathbb{Z}\{\langle 1\rangle, \langle 2\rangle, \langle 3\rangle\} = \mathbb{Z}\langle 1\rangle + \mathbb{Z}\langle 2\rangle + \mathbb{Z}\langle 3\rangle + \mathbb{Z}\langle 6\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$

2 WELSCHINGER INVARIANTS

$d > 0$, $n = 3d-1$, $0 \leq s \leq m = \lfloor \frac{n}{2} \rfloor$

$W_d^{\mathbb{P}^2}(s) =$ # rat'l curves $C \subset \mathbb{P}^2_{\mathbb{R}}$ of degree d passing through $\mathcal{P} = \{p_1, \dots, p_n\} \subset \mathbb{P}^2_{\mathbb{C}}$ with $\overline{\mathcal{P}} = \mathcal{P}$ and $\mathcal{P} \cap \mathbb{P}^2_{\mathbb{R}} = n-2s$



counted with Welschinger sign $s(C) = (-1)^{\#\times}$

EX $d=3$, $n=8$

s	0	1	2	3	4
$W_3(s)$	8	6	4	2	0

$d=4$, $n=11$

s	0	1	2	3	4	5
$W_4(s)$	240	144	80	40	16	0

THM (W) X real surface, $D \in \mathrm{Pic}^{\mathbb{R}}(X)$

$$W_D^X(s) - W_D^X(s+1) = 2 \cdot W_{D-2E}^{Bl_p X}(s)$$

EX $d = 4$

$\mathbb{P}^2$ $\quad 240 \quad 144 \quad 80 \quad 40 \quad 16 \quad 0$

$Bl_1\mathbb{P}^2$ $\quad 48 \quad 32 \quad 20 \quad 12 \quad 8$

$\vdots$ $\quad 8 \quad 6 \quad 4 \quad 2$

$1 \quad 1 \quad 1$

$0 \quad 0$

$Bl_5\mathbb{P}^2$ $\quad 0$

$$\omega_4^{\mathbb{P}^2} = 8\beta_1 + 2\beta_2 + \beta_3$$

DEF Welschinger-Witt invariant

$$W_d^{\mathbb{P}^2} = \sum_{i=0}^{m} W_{d-2E}^{\mathrm{Bl}_i\mathbb{P}^2}(m-i)\,\beta_i \in \mathrm{GW}(k)$$

$$E = E_1 + \dots + E_i$$

Unique integral Witt invariant with multireal values

$$w_s = W_d^{\mathbb{P}^2}(s)$$

3 QUADRATIC INVARIANTS

(Kass – Levine – Solomon – Wickelgren)

$d>0$, $n=3d-1$, $A \in \mathcal{E}t_n(K)$

$$Q^{\mathbb{P}^2}_{d,K}(A) = \sum_{[C]} \mu(C) \in W(K)$$

$[C]$ = Galois orbits of rat'l curve of deg. d in $\mathbb{P}^2_{\bar{K}}$ s.th. $P \subset C$.

$P \underset{K\text{-sch.}}{\cong} \mathrm{Spec}(A) \subset \mathbb{P}^2_{\bar{K}}$ $\quad \mu(C) \in W(K)$

THM (KLSW) $Q^{\mathbb{P}^2}_{d,K}$ is independent of P.

C curve ..., $[p]$ Galois orbit of nodes in $C(\bar{K})$

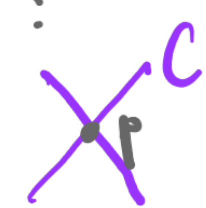
$K \underset{tr}{\subseteq} K_c \underset{nm}{\subseteq} K_p$ fields of def of C and p.

(1) $\mu(C) = \left[K_c \to K : a \mapsto tr_{K_c/K}\left(\nu(C)\, a^2\right) \right] \in W(K)$

(2) $\nu(C) = \prod_{[p]} nm_{K_p/K_c}(\delta(p)) \quad \in K_c^*$

(3) $\delta(p) \in K_p^*$ such that tangent cone at p:

$$x^2 - \delta(p)\, y^2 = 0$$



EX $K = \mathbb{R}$

$\mathbb{R} \subset \mathbb{C} \subset \mathbb{C}$

(1) $K_C = \mathbb{C} : \delta(p) = 1 \ \forall p \implies \nu(C) = 1$

$\implies \mu(C) = \mathrm{Tr}_{\mathbb{R}}(\mathbb{C}) = h = 0$

(2) $K_C = \mathbb{R}$ $\quad \mathbb{R} \subset \mathbb{R} \subset \mathbb{C}$

(a) $K_p = \mathbb{C} : \quad \delta(p) = 1 \implies nm(\delta(p)) = 1$

(b) $K_p = \mathbb{R} : \quad \delta(p) = +1$ or -1 $\quad \mathbb{R} \subset \mathbb{R} \subset \mathbb{R}$

$x^2 - y^2 = 0$   $x^2 + y^2 = 0$

$$\implies \mu(C) = \langle (-1)^{\#} \rangle = \langle s(C) \rangle \in W(\mathbb{R}) \cong \mathbb{Z}$$

COR (KLSW) $K=\mathbb{R}$ $\qquad A = \mathbb{C}^s \times \mathbb{R}^{n-2s}$

$$\mathcal{P} \cong \operatorname{Spec} A \implies \mathcal{P} \cap \mathbb{P}^2_{\mathbb{R}} = n-2s$$

$$\implies Q^{\mathbb{P}^2}_{d,\mathbb{R}}(A) = W^{\mathbb{P}^2}_d(s)$$

COR (BRW) $Q^{\mathbb{P}^2}_d$ is Witt invariant (with multireal values $W^{\mathbb{P}^2}_d(s)$)

"**Proof**" $K \subset L$, $[c] \otimes L = [c_1] \cup \dots \cup [c_\ell]$

$$\implies \mu(c) \otimes L = \mu(c_1) + \dots + \mu(c_\ell)$$

TMA (BRW) $\quad Q_d^{\mathbb{P}^2}$ is unramified (when defined, $p>3$)

$\rightsquigarrow \quad b_i \in \mathbb{Z}[\langle 2\rangle, \langle 3\rangle] = \mathbb{Z} \oplus \underline{\mathbb{Z}/2\mathbb{Z}} \oplus \underline{\mathbb{Z}/4\mathbb{Z}}$

TMA (BRW) $\quad Q_d^{\mathbb{P}^2}$ is integral

$$\Longrightarrow \quad Q_d^{\mathbb{P}^2} = \omega_d^{\mathbb{P}^2} = \sum_{i=0}^{m} W_{d-2E}^{\mathrm{Bl}_i \mathbb{P}^2}(m-i)\beta_i$$

$\rightsquigarrow \quad Q_{d,K}^{\mathbb{P}^2}(A)$ for K, A is determined by $W_d^{\mathbb{P}^2}(s)$

4 OTHER SURFACES

EX $X = \text{Fermat cubic} \subset \mathbb{P}^3$, $D = H = -K_X$

$$Q^X_D(A) = \langle 1, -3 \rangle + \mathrm{Tr}_A \in \mathrm{Inv}_{\geq 3}(2)$$
$$= \mathrm{Tr}_B + \mathrm{Tr}_A$$

BUT $X = Bl_R \mathbb{P}^2$, $R = \operatorname{Spec} B \subset \mathbb{P}^2$

s.th. $\mathrm{Tr}_B = \langle 1, -3 \rangle \ (+2h)$

$$B = \overset{\overset{L}{\shortparallel}}{\mathbb{Q}(\sqrt{-3})}^3 \rightsquigarrow \mathrm{Tr}_B = 3 \cdot \mathrm{Tr}_L = 3 \cdot \langle 2, -6 \rangle = \langle 1, -3 \rangle$$

MULTIVARIATE WITT-INVARIANTS

Simplest case:

Fix $m, d, e \geq 0$ s.th. $n = 3d - em - 1 \geq 0$

Given $A \in \mathcal{E}t_n(k)$, $B \in \mathcal{E}t_m(k)$, choose:

(1) $R \cong \operatorname{Spec} B \subset \mathbb{P}^2 \rightsquigarrow X = Bl_R \mathbb{P}^2$

(2) $P = \operatorname{Spec} A \subset X$

DEF $$\mathbb{Q}^m_{d,e}(A, B) = \mathbb{Q}^X_{dH - eE}(A)$$

(when defined)

Conj

(a) $Q^m_{d,e}$ only depends on A, B

(b) $Q^m_{d,e}$ is integral Witt invariant

$$Q^m_{d,e} = \sum_{i,j} b_{ij} \beta_{ij}, \qquad \beta_{ij}(A,B) = \beta_i(A) \cdot \beta_j(B)$$

(c) Multireal values are

$$w_{s,t} = W^{X_{m,t}}_{dH\text{-}eF}(s)$$

$X_{m,t} := Bl_R \mathbb{P}^2_{\mathbb{R}}$

with $R = \operatorname{Spec} \mathbb{C}^t \times \mathbb{R}^{m-2t}$

THM (Real Abramovich-Bertram)

$$W_D^{X_{m,t+1}}(s) - W_D^{X_{m,t}}(s) = 2 \cdot \sum_{l \geq 1} (-1)^l \, W_{D+lE_1-lE_2}^{X_{m,t}}(s)$$

COR $w_{d,e}^m \in \mathbb{Z}[\mathrm{Mor}(n,m)]$ con $w_{s,t} = \ldots$ exists.

TMA (BRW) Conjecture holds for $m \leq 3$ (and $\mathbb{P}^1 \times \mathbb{P}^1$)

Proof JMPR

Jaramillo-Markwig-Pauli-Röhrle

$$Q(A) = \sum_{\mathcal{D}} \mu(\mathcal{D})$$

floor diagrams

BRW $\mu(\mathcal{D})$ are integral comb of traces

EX $X = \mathbb{P}^1 \times \mathbb{P}^1 = \mathrm{Contr}_{L(R)} \mathrm{Bl}_R \mathbb{P}^2$

$m = 2$, $D = 3(L_1 + L_2) \approx 6H - 3E$

$t=1$: hyperboloid

1086 606 318 158 78 46

? 80 40 16

? 20 12

? 4

? 1

0

255 159 95 55 31 15

? 12 8

? 2

? 1

? 0

0

b_{i1}

$$\sum_\ell (-1)^\ell W^{\text{hyp.}}_{(3+\ell,8-\ell)}$$

$t=0$: ellipsoid

576 288 128 48 16 16

? 40 16 0

? 12 8

? 2

? 1

0

b_{i0}

$$\Rightarrow Q_{3,3}^{\mathbb{P}^1\times\mathbb{P}^1} = 16\beta_{00} + 8\beta_{20} + 2\beta_{30} + \beta_{40} + 15\beta_{01} + 8\beta_{11} + 2\beta_{21} + \beta_{31}$$

THANK YOU!