Real plane sextic curves with smooth real part

Plan

1. Algebraic curves in $\mathbb{RP}^2$.
2. Singular real sextic curves.
3. Real sextics without real singular points.

1. Algebraic curves in $\mathbb{R}P^2$

Questions related to Hilbert's 16th problem.

$F \in \mathbb{R}[X_0, X_1, X_2]$ homogeneous polynomial of degree d

$\mathbb{R}F \subset \mathbb{R}P^2$ zero locus of F

$$\begin{array}{ccc} \mathbb{R}F & \subset & \mathbb{R}P^2 \\ \cap & & \cap \\ \mathbb{C}F & \subset & \mathbb{C}P^2 \end{array}$$

<u>Non-singular curves</u> : F does not have critical points in $\mathbb{C}^3 \setminus \{0\}$

Several classifications : • up to homeomorphism, $\mathbb{R}F$

• up to isotopy, $(\mathbb{R}P^2, \mathbb{R}F)$

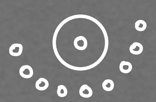
• up to rigid isotopy.

<u>Harnack inequality</u> (1876)

$$\# \text{ c.c. of } \mathbb{R}F \leq \frac{(d-1)(d-2)}{2} + 1$$

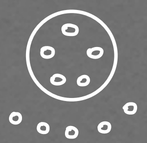
<u>Maximal curves</u>. $\# \text{ c.c. of } \mathbb{R}F = \dfrac{(d-1)(d-2)}{2} + 1.$

<u>Isotopy classification of maximal non-singular sextics</u>



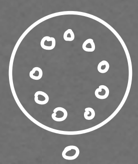
Harnack

$p = 10, n = 1$



Gudkov

$p = 6, n = 5$



Hilbert

$p = 2, n = 9$

Gudkov - Rokhlin congruence

For maximal non-singular curves of degree $d = 2k$ in $\mathbb{RP}^2$:

$$p - n \equiv k^2 \mod 8$$

p # even ovals

n # odd ovals

Halves $\mathbb{RP}^2_+ = \{ (x_0 : x_1 : x_2) \in \mathbb{RP}^2 \mid F(x_0, x_1, x_2) \geqslant 0 \}$

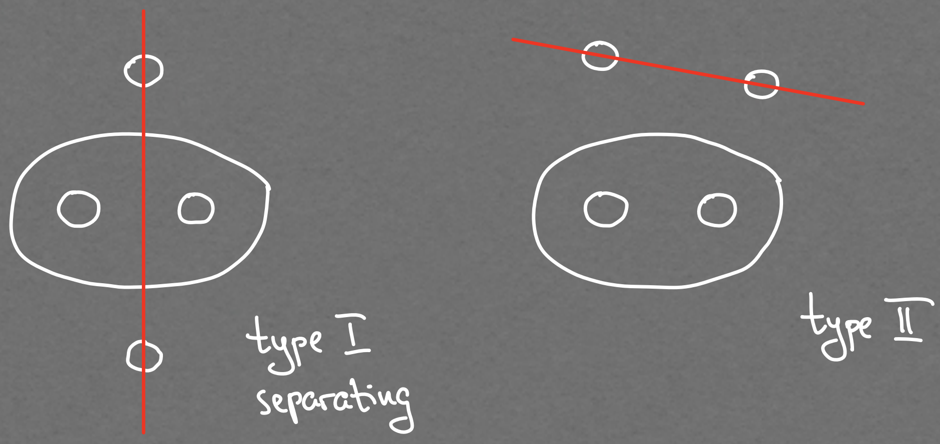
$\mathbb{RP}^2_- = \{ (x_0 : x_1 : x_2) \in \mathbb{RP}^2 \mid F(x_0, x_1, x_2) \leqslant 0 \}$

$p - n = \chi(\mathbb{RP}^2_+)$, $\quad n - p + 1 = \chi(\mathbb{RP}^2_-)$.

Rigid isotopy classification

Discriminant $\mathcal{D} \subset \mathbb{R}C_d \simeq \mathbb{RP}^N$, where $N = \frac{d(d+3)}{2}$

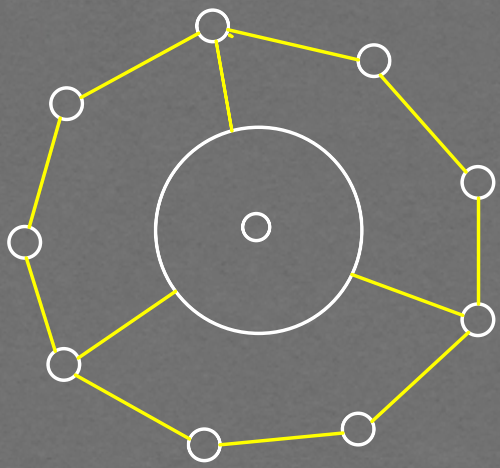
Connected components of $\mathbb{R}C_d \setminus \mathcal{D}$

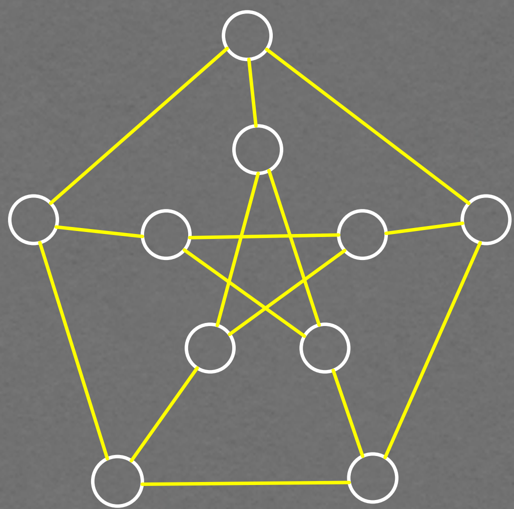


More generally, one can speak about equivariant equisingular deformation classification.

For sextic curves:

- non-singular (Nikulin)
- with one non-degenerate double point (I)
- with empty real part (joint work with A. Degtyarev)
- with smooth real part (joint work with A. Degtyarev)





2. Singular real sextic curves

Consider a simple (that is, with A-D-E singularities) sextic curve C given by a polynomial F.
The minimal resolution $X = X_C$ of the singularities of the double covering $X' \to \mathbb{C}P^2$ branched along $\mathbb{C}F$ is a K3-surface.

The lattice $L_X = H_2(X)$ is isomorphic to $2E_8 \oplus 3U$.
The classes of exceptional divisors span a root lattice $S \subset L_X$. It can be identified with the (abstract) set of singularities of C.

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Denote by $h = h_C$ the **polarization**, i.e., the class of the pullback of a line. One has $h^2 = 2$. Put $S_h = S \oplus \langle h \rangle \subset L_X$, and let $\tilde{S}_h$ be the primitive hull of S_h.

In the real case, two real structures σ^+ and σ^- on X.

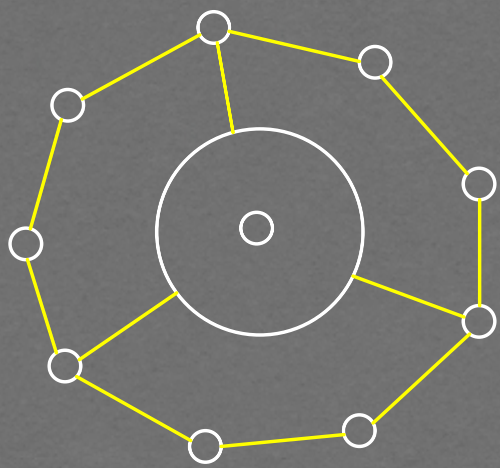
For a real sextic C without real singular points, one can define its **real homological type**:
the isomorphism class of the quadruple

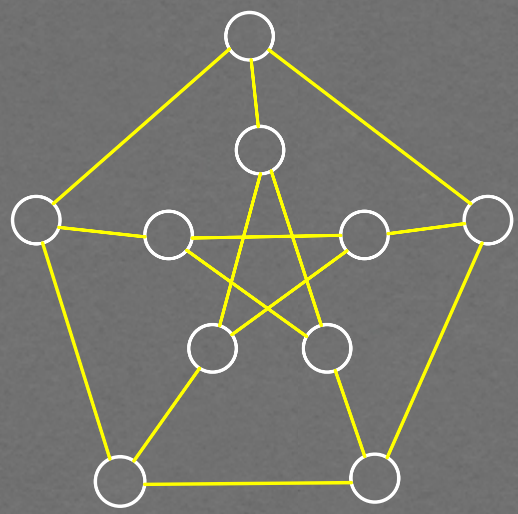
$$(L_X \supset \tilde{S}_h \ni h, \sigma_*^+).$$

Theorem (Degtyarev - I)

The real homological type of a simple sextic without real singular points determines its equivariant equisingular deformation class.

Remark This is not true in the presence of real singular points.





The proof uses the global Torelli theorem and the surjectivity of the period map for K3-surfaces, as well as results of Saint-Donat.

Additional statements

- Any empty oval of a real sextic with smooth real part can be contracted.

- Extremal congruences

  Locally irreducible singularities: A_{2k}, E_6, E_8

Theorem (Degtyarev - I)

For any maximal curve C of even degree in $\mathbb{R}P^2$ such that the singularities of C are non-real and of types A_{2k}, E_6, E_8, one has

$$2\chi(\mathbb{R}P^2_-) = \sigma(X) + 8\,\mathfrak{J}(S) \bmod 16$$

où X is the minimal resolution of singularities of the double covering $X' \xrightarrow{2:1} \mathbb{C}P^2$ branched along the complex point set of C, and $\mathfrak{J}(S)$ is the additive Jacobi symbol associated with the collection $2S$ of singularities of C:

$$\mathfrak{J}(S) = \begin{cases} 0 \bmod 2, & \text{if } \det S = \pm 1 \bmod 8, \\ 1 \bmod 2, & \text{if } \det S = \pm 3 \bmod 8. \end{cases}$$

One has

$$\mathfrak{J}(A_2) = \mathfrak{J}(A_4) = \mathfrak{J}(E_6) = 1 \bmod 2$$
$$\mathfrak{J}(A_6) = \mathfrak{J}(A_8) = \mathfrak{J}(E_8) = 0 \bmod 2$$