Tropical refined curve counting & mirror symmetry

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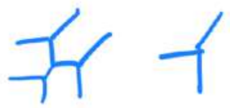
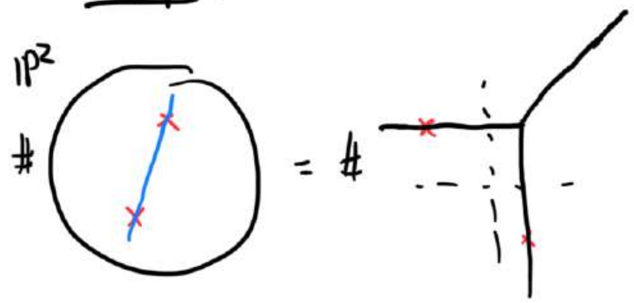
Mikhalkin 03'



$N_{0,d}$ = # rat'l degree d curves in $\mathbb{P}^2$ through $3d-1$ points $=$ weighted # rat'l tropical curves in $\mathbb{R}^2$ of "deg d" through $3d-1$ points $= N^{trop}_{0,d}$

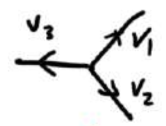
Example: $(d=1)$

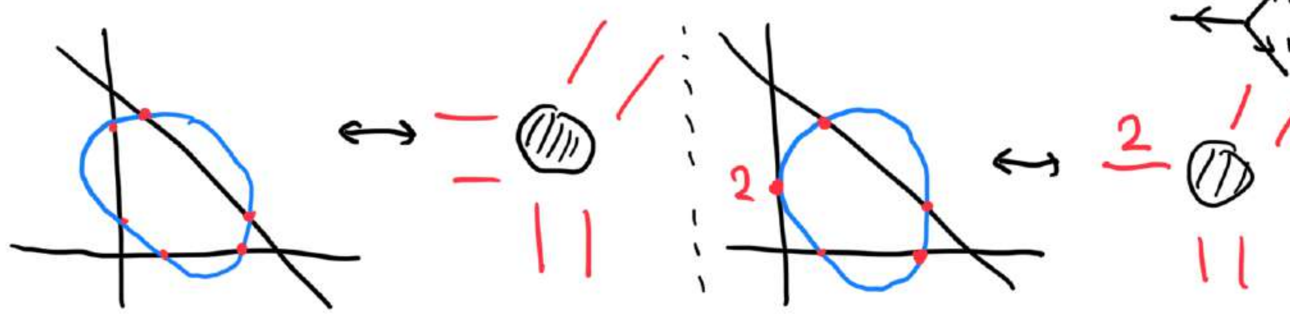


Remark: This is not really a theorem about curves in $\mathbb{P}^2$ but really about curves in $\mathbb{P}^2$ w/ tangencies to the coordinate lines.

- "piecewise linear graphs in $\mathbb{R}^2$
- rat'l means (parametrized by a) graph of genus zero.
- "deg d" means d unbdd edges in the directions $(-1,0), (0,-1)$ & $(1,1)$

- balanced: for each vertex v: sum of out going vectors is zero
- weight: each tropical curve h counted w/ multiplicity $m_h = \prod_{v:\text{vertices}} m_v$, $m_v = \left| \begin{matrix} v_1 & v_2 \end{matrix} \right|$





Here are two directions in which to generalise this theorem:

algebro-geometric ⟷ tropical

① more general constraints ⟷ higher valency tropical curves

② higher genus curves ⟷ counting tropical curves w/ multiplicity $m_h \in \mathbb{Q}[q^{\pm 1/2}]$

We will see a generalisation which does both of these things...

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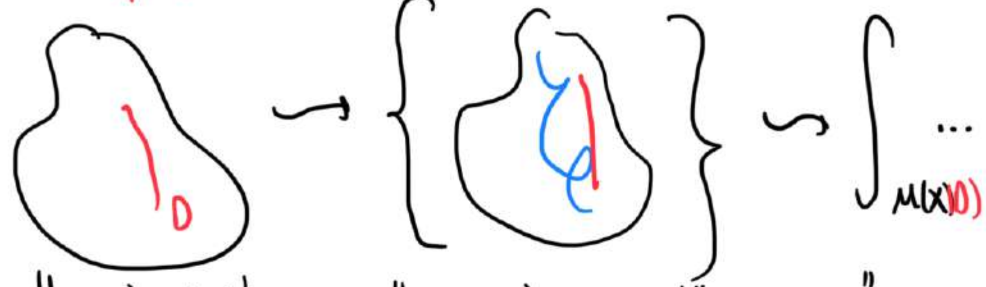
but the LHS will need to be replaced by (logarithmic) Gromov–Witten invariants

(log) Gromov–Witten theory

(X, D) $\qquad\qquad$ $\mathcal{M}(X|D)$ $\qquad\qquad$ log GW invariants



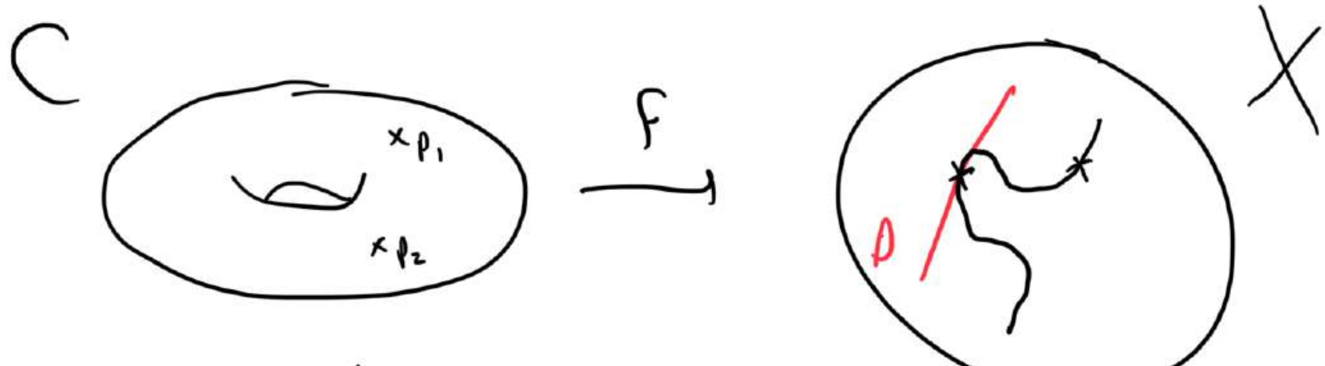
smooth proj. variety
$D \subseteq X$ divisor

"space of curves in X"
(of fixed degree & genus)
w/ fixed tangencies at D

"curve counts"

The $(\ldots)$ encode geometric constraints on the curve in X in the form of cohom. classes on $\mathcal{M}(X)$. For us

$$\mathcal{M}(X) = \mathcal{M}^{\log}_{g,n}(X|D,d) \simeq \left\{ \begin{array}{l} \bullet\ C \text{ is a curve of genus } g \\ \bullet\ p_1, \ldots, p_n \in C \text{ distinct} \\ \bullet\ f : C \to X \text{ parametrisation of degree } d \\ \bullet\ \text{w/ certain tangencies to } D \text{ fixed at some of} \\ \quad \text{the } p_i \end{array} \right\}$$



(the compactification) $\mathcal{M}^{\log}_{g,n}(X|D,d)$ is called the moduli space of stable logarithmic maps [ACGS].

Sometimes (log) GW invariants coincide w/ actual curve counts, for example

$$N_{0,d} = \int [3d-1 \text{ pt constraints}]$$

$$N_{0,d} \;\; \int_{[\mathcal{M}^{log}_{0,3d-1}(\mathbb{P}^2,d)]}$$

but often this is not the case! People still care...

1) They can help with enumerative counts
2) They show up in mirror symmetry
3) They can help to understand related moduli spaces e.g. $\overline{\mathcal{M}}_{g,n}$
4) Better properties (deformation invariant).

Recall we were trying to generalize $N_{0,d}$

- For each $i=1,\dots,n$ there is a line bundle on the moduli space $\mathcal{M}^{log}_{g,n}(X|D,d)$ where the fibre over $(C,p_1,\dots p_n) \xrightarrow{f} (X,D)$ is $T^*_{p_i}C$. Let ψ_i be its first chern class
- there is a rk g vector bundle over $\mathcal{M}^{log}_{g,n}(X|D,d)$ where the fibre over $(C,p_1,\dots,p_n) \xrightarrow{f} (X,D)$ is $\underbrace{H^0(C,\omega_C)}_{g \text{ dim}}$. Let λ_g be its top (g^{th}) chern class

fix $g \geq 0$ & $n \leq 3d-1$, & $k_1,\dots k_n$ integers with $\sum_{i=1}^n k_i + n = 3d-1$

$$\text{Let } N^{\underline{k}}_{g,d} := \int_{[\mathcal{M}^{log}_{g,n}(\mathbb{P}^2,d)]^{vir}} (-1)^g \lambda_g \, [\, \overbrace{n \text{ pt conditions}}^{ev_1^*pt \cup \dots \cup ev_n^* pt} \,] \cdot \psi_1^{k_1} \cdots \psi_n^{k_n}$$

when $g=0$ & $\underline{k}=\underline{0}$ $\Rightarrow$ $n=3d-1$

$$\Rightarrow N^{\underline{0}}_{0,d} = N_{0,d} = \int_{[\mathcal{M}^{log}_{0,3d-1}(\mathbb{P}^2,d)]} [3d-1 \text{ pt conditions}]$$

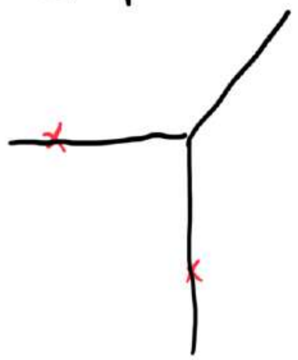
The generalisation of the Mikhalkin theorem will involve $N^{\underline{k}}_{g,d}$ $\forall g$.

Tropical side

We will still consider rational tropical curves (in $\mathbb{R}^2$) of "deg d", but instead through $n \leq 3d-1$ points where at each point we specify the valency.

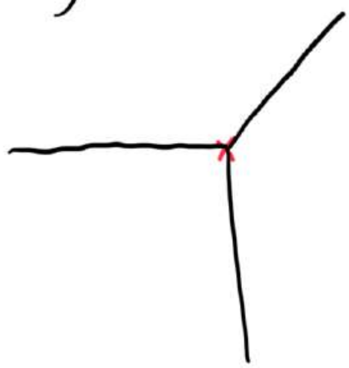
Example

$\exists!$ tropical line through 2 points



also..

$\exists!$ tropical line through a single point with valency 3 at that point



$T_{\underline{k},d}$ in general, fix $n \leq 3d-1$ & $k_1, \dots, k_n$ s.t. $\sum_{i=1}^{n} k_i + n = 3d-1$ we will consider rat'l "deg d" tropical curves through n points where the valency at the i^{th} point is $k_i + 2$.

In our example $d=1$, $n=1$, $k_1 = 1$ so that $1+1 = 3 \cdot 1 - 1$

Multiplicity:

Recall before we had for each vertex an integer m_v & the multiplicity was $\prod_v m_v$.

Now, fix $h \in T_{\underline{k},d}$. We will now assign a multiplicity $m_v(q) \in \mathbb{Q}[q^{\pm\frac{1}{2}}]$

Th (Blechman–Shustin): $\exists\, m_v(q)$ for each vertex v s.t.

Theorem ([illegible])

$$N^{trop}_{0,\underline{k},d}(q) := \sum_{h \in T_{\underline{k},d}} \prod_{V \in h} m_V(q)$$ is independent of the choice of points $(p_1, \dots, p_n)$.

Example:  multiplicity is $\frac{1}{2}\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)$.

What is the multiplicity?

- for a trivalent vertex (unpointed) V () $\rightsquigarrow (-i)\left(q^{\frac{m(V)}{2}} - q^{\frac{-m(V)}{2}}\right)$ $\left(m(V) = \left| \begin{matrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{matrix} \right|\right)$
- for an m-valent vertex (pointed) V () $\rightsquigarrow \dfrac{1}{2^{m-2}} \dfrac{1}{(m-1)!} \displaystyle\sum_{\sigma \in S_{m-1}} \prod_{j=1}^{m-2} \left[v_{\sigma(j+1)} \wedge \sum_{l=1}^{j} v_{\sigma(l)} \right]_+$

(where $[n]_+ = q^{\frac{n}{2}} + q^{-\frac{n}{2}}$)

Linking the algebraic & tropical

Theorem [KH-S-UK] fix $n \leq 3d-1$ & $k_1, \dots, k_n$ w/ $n + k_1 + \dots + k_n = 3d-1$

$$\boxed{\sum_{g \geq 0} N^{\underline{k}}_{g,d}\, u^{2g-2+3d-\sum_{i=1}^{n} k_i} = N^{trop}_{0,\underline{k},d}(q)}$$

$\hookrightarrow q = e^{iu}$

Pf involves

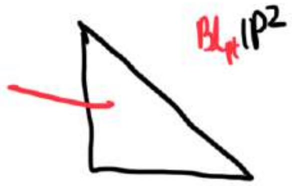
- degeneration
- intersection theory on $\overline{M}_{g,n}$
- integrable hierarchies

(More generally this works for any toric surface and any tangency conditions)

Rmk: when $\underline{k} = 0$ this recovers a theorem of [Bousseau] (which more generally applies for higher genus tropical curves)

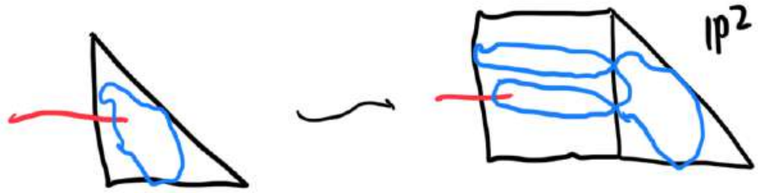
Application (quantum mirrors)

What if we have a non-toric surface?

e.g. 

is no longer a toric surface (so we can't apply the above thm)
it is an example of a log CY surface
i.e. smooth proj. surface w/ $\underbrace{\text{anticanonical divisor}}_{\text{strict transform of coordinate lines}}$

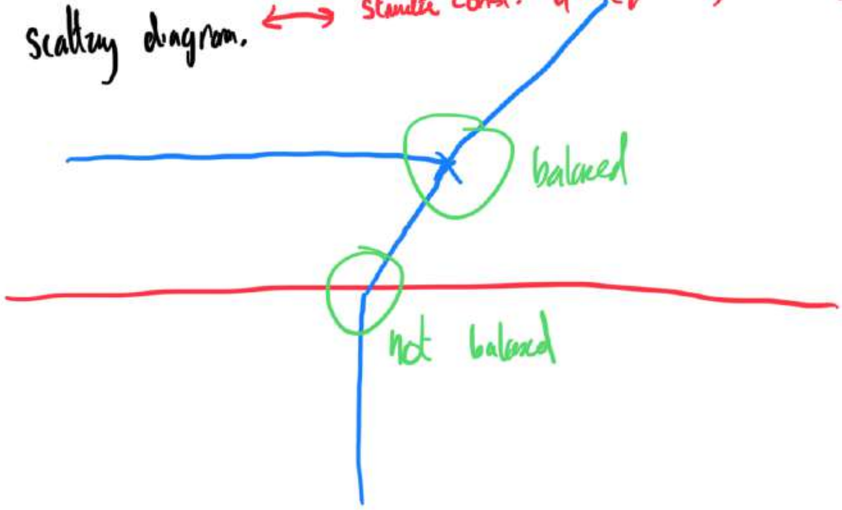
However there is a degeneration



Deformation invariance of log GW invariants says that
log GW invariants of  can be computed in terms of

- log GW invariants of  $\longleftrightarrow$ tropical correspondence thm
- log GW invariants of  $\longleftrightarrow$ simpler computation

$\rightsquigarrow$ output is that log GW invariants get encoded in a comb. gadget called (quantum) scattering diagram. $\longleftrightarrow$ structure const. of (quantum) mirror algebra of log CY surface



$\rightsquigarrow$ log GW invariants of log CY surface $\longleftrightarrow$ structure constants of quantum mirror algebra