Tropical floor diagrams

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Outline

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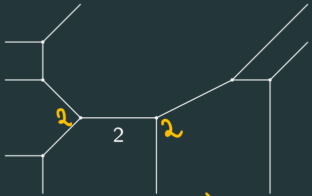
Floor diagrams

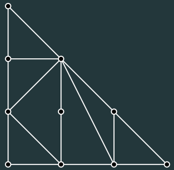
Real and refined invariants

Lattice path algorithm

Tropical curves - a reminder

A plane tropical curve is piecewise linear map with rational slopes from a finite metric graph to $\mathbb{R}^2$ satisfying the balancing condition. The degree is the multiset of directions of unbounded ends. We have *dual subdivision* of the Newton polygon. The genus is the first Betti number. The multiplicity $\mu(\Gamma)$ is the product $\prod_{V\in\Gamma^0}\mu(V)$ where $\mu(V)$ is the lattice area of the dual cell.





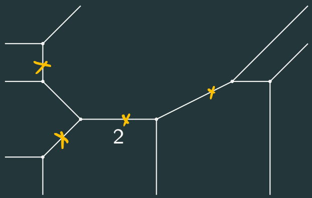
$\deg = 3\times\binom{-1}{0}, 3\times\binom{0}{-1}, 3\times\binom{1}{1} = \deg d$

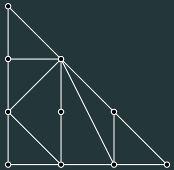
$\mu(\Gamma) = 4$

Tropical curves - a reminder

A marking of a tropical curve is a choice of finite sequence of points on it.

A combinatorial type of a tropical curve is the data of the graph, slopes of edges, and for each marking on which edge it resides.





Remark.

For simplicity we will focus on degrees $\Delta_d := \left\{ d \times \begin{pmatrix} -1 \\ 0 \end{pmatrix}, d \times \begin{pmatrix} 0 \\ -1 \end{pmatrix}, d \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ *which correspond to degree d curves in $\mathbb{P}^2$. All the results can be generalized to v-tranverse degrees, which include all main examples, such as $\mathbb{P}^1 \times \mathbb{P}^1$, Hirzebruch surfaces, and del-Pezzo surfaces.*

Correspondence theorem

Theorem [Mikhalkin].

Let $\boldsymbol{p}$ *be a sequence of* $3d+g-1$ *points in* $\mathbb{R}^2$ *in general (tropical) position and let* Γ *be a tropical curve of genus* g *and degree* Δ_d *passing through* $\boldsymbol{p}$*. For a sequence* $\boldsymbol{w}$ *of* $3d+g-1$ *points in* $(\mathbb{K}^\times)^2$ *that tropicalize to* $\boldsymbol{p}$*, there exist* $\mu(\Gamma)$ *algebraic curves of genus* g *and degree* d *in* $\mathbb{P}^2$ *that pass through* $\boldsymbol{w}$ *and tropicalize to* Γ*.*

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Corollary: The Gromov-Witten invariants of toric surfaces can be computed tropically.

Problem: Although the computation is finite, it is very hard to perform and analyze.

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Solution: Put the points in suitable position.

Lattice path

Definition.

A sequence of points $\langle p_i = (x_i, y_i) \in \mathbb{R}^2 \rangle$ is called horizontally stretched if for all $i < j$ we have $x_i < x_j$, $y_i < y_j$ and $x_j - x_i > C \cdot (y_j - y_i)$ where C is large enough constant (depending only on Δ).

Definition.

If p_i is contained in the interior of an edge of Γ, denote by E_i the corresponding edge of the dual subdivision. Denote by $P(\Gamma, \boldsymbol{p}) := \bigcup E_i$ the lattice path corresponding to Γ.

Proposition.

If $\boldsymbol{p}$ is horizontally stretched, $P(\Gamma, \boldsymbol{p})$ is homeomorphic to a segment with endpoints $(0,0)$ and $(d,0)$.

Examples

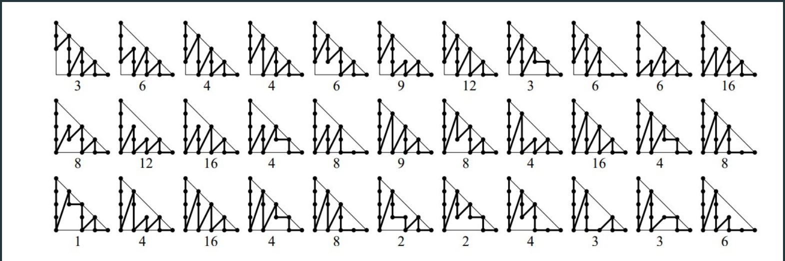
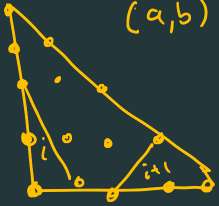


Figure 1: All lattice paths and their multiplicities for degree 4 and genus 1 (credit: Mikhalkin).

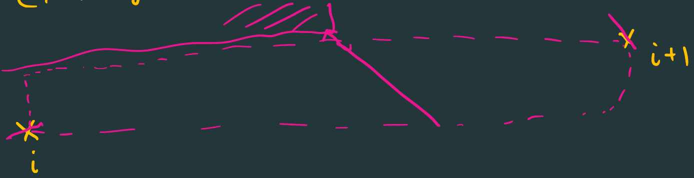
Sketch of the proof



$(a,b) < (c,d) \iff c > a$ or $c = a,\ d > b$

the endpoints of E_i are smaller than the endpoints of E_{i+1}

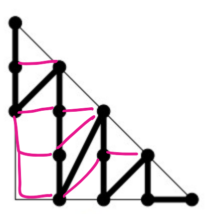
The cc of $\Gamma \setminus P$ are trees with exactly one unbounded end.



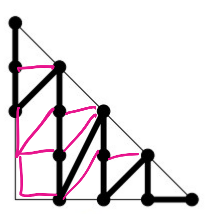
Lattice path algorithm

- We can enumerate all the lattice paths that can occur.
- For each lattice path we can enumerate the subdivisions containing this lattice path, by using the fact such a subdivision has to contain only triangles and parallelograms.
- In the final step we will need to filter reducible curves or curves of the wrong degree.

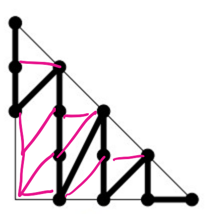
Example



(a)



(b)



(c)

Floor diagrams

Floor graph

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Let Γ be a planar tropical curve. A horizontal edge of Γ is called an elevator, a connected component of $\Gamma \setminus \bigcup\{\text{elevators}\}$ is called a floor.

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Definition.

Let Γ be a planar tropical curve. A horizontal edge of Γ is called an elevator, a connected component of $\Gamma \setminus \bigcup\{\text{elevators}\}$ is called a floor. The floor graph of Γ is a weighted directed graph with:

Vertices: Floors of Γ.

Edges: Bounded elevators of Γ oriented from left to right and with the same weight.

Definition.

A connected directed weight graph $\mathcal{D} = (V, E, w)$ is called a floor diagram if it is acyclic and for every vertex $v \in V$ it satisfies the divergence condition:

$$\operatorname{div}(v) \stackrel{\text{def}}{=} \sum_{\circ \xrightarrow{e} v} w(e) - \sum_{v \xrightarrow{e} \circ} w(e) \leq 1.$$

The degree of $\mathcal{D}$ is the cardinality of V, the genus of $\mathcal{D}$ is its first Betti number $g(\mathcal{D}) = 1 - \chi(\mathcal{D}) = 1 + |E| - |V|$.

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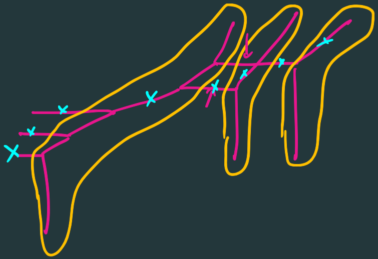
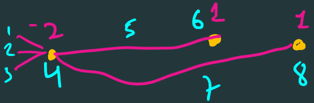
A tropical curve is floor decomposed if its floor graph is a floor diagram.

Proposition.

If $\boldsymbol{p}$ is horizontally stretched, every tropical curve (of degree d and genus g) passing through $\boldsymbol{p}$ is floor decomposed.

Example





Definition.

Let $\mathcal{D} = (V, E, w)$ be a floor diagram. The extended floor diagram $\tilde{\mathcal{D}} = (\widetilde{V}, \widetilde{E}, w)$ is obtained from $\mathcal{D}$ by adding $1 - \text{div}(v)$ vertices at infinity for each $v \in V$ and connecting them by an edge of weight 1 to v. A marking on $\mathcal{D}$ is a linear order on $V \cup \widetilde{E}$ s.t. every edge is greater than its source and smaller than its target.

Marked floor diagrams

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Correspondence

Proposition.

If $\boldsymbol{p}$ are horizontally stretched, there exist a multiplicity preserving bijection between tropical curves through $\boldsymbol{p}$ and marked floor diagrams (both of degree d and genus g).

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Corollary.

The Gromov-Witten invariant $N_{d,g}$ is equal to

$$N_{d,g} = \sum_{\mathcal{D}} \mu(\mathcal{D})\nu(\mathcal{D})$$

where the sum is over all floor diagrams and $\nu(\mathcal{D})$ denotes the number of markings of $\mathcal{D}$.

Example - smooth curves

Smooth curves have maximal genus among curves of degree d, namely

$$g_{\text{max}}(d) = \frac{(d-1)(d-2)}{2}.$$

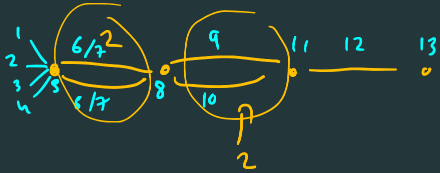
There is only one marked floor diagram of degree d: and genus g_{max}

Example - uninodal curves

There are 2 types of marked floor diagrams of genus $g_{\max} - 1$:

1. Floor diagrams with edge of weight 2. There are $1 + 2 + \cdots + (d-2) = \frac{(d-1)(d-2)}{2}$ of those, each with multiplicity 4.

$d = 4$



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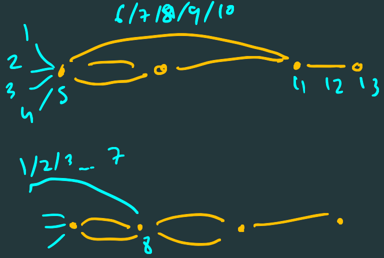

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Summing all the contributions we get the well known formula for the degree of the discriminant:

$$N_{d,g_{\max}-1} = 3(d-1)^2.$$

Node polynomials

Using the same ideas it is possible to prove:

Theorem [Fomin, Mikhalkin, '09].

For a fixed $\delta \in \mathbb{Z}_{\geq 0}$ there exist a polynomial $N_\delta(d) \in \mathbb{Q}[d]$ of degree 2δ s.t. $N_{d,g_{\max}-\delta} = N_\delta(d)$ for all d large enough.

Real and refined invariants

Welschinger invariants

- It is possible to compute the Welschinger weight of a floor decomposed tropical curve from its marked floor diagram.
- For totally real configuration of points we will get $\mu^{\mathbb{R}}(\mathcal{D}) = \begin{cases} 0 & \mu(\mathcal{D}) \in 2\mathbb{Z} \\ 1 & \mu(\mathcal{D}) \in 1 + 2\mathbb{Z} \end{cases}$ and in particular does not depend on the marking. We get

$$W_d = \sum_{\mu(\mathcal{D}) \equiv 1 \mod 2} \nu(\mathcal{D}).$$

- For configuration of points containing conjugate pairs, the real multiplicity depends on the marking.

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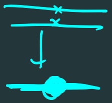

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Theorem [Arroyo, Brugalle, de Medrano, '08].

There exist a recursive formula that computes Welschinger invariants for any number of conjugate pair of points.

Block-Göttsche invariants

- All this discussion transfers as is to the computation of Block-Göttsche invariants (i.e. refined invariants without points of high multiplicity).
- The refined weight of a floor diagram is

$$\mu_y(\mathcal{D}) := \prod_{e\in E} \left([w(e)]_y\right)^2 .$$

Theorem [Block, Göttsche, '16].

There exist a polynomial $N_\delta(d;y) \in \mathbb{Q}[y^{\pm 1}][d]$ of degree 2δ in d s.t.

$$BG_y(d, g_{\max} - \delta) = N_\delta(d;y)$$

for all large enough values of d.

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