Refined invariants in tropical, complex, and real enumerative geometries

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Background

Real and Complex enumeration of curves

$$N_{d,g} := \left|\left\{ C \in \mathbb{P}^2_{\mathbb{C}} \mid C \text{ is a genus } g \text{ curve of degree } d \right\}\right|$$

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Welschinger invariant:

$$W_{d,s} := \sum_{C \subset \mathbb{P}^2_{\mathbb{R}}} w(C)$$

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Tropical geometry

Let K be a field complete with respect to a non-archimedean valuation $val : K^\times \to \mathbb{R}$.

For a plane algebraic curve $C \subset \mathbb{P}^2_K$ the tropicalization is

$$trop(C) := \overline{\{(-val(x), -val(y)) \mid (x, y) \in C \cap (K^\times)^2\}}.$$

Motivation and past results

Mikhalkin's Correspondence Theorem

Theorem [Mikhalkin '05].

Let $d \geq 1$, $g \geq 0$, $n = 3d + g - 1$, and let $p_1, \ldots, p_n \in \mathbb{P}^2_K$ be generic points. Then tropicalization induces a finite-to-one map

$$\{\, C \mid C \text{ is a genus } g \text{ curve of degree } d \text{ passing through } p_1, \ldots, p_n \,\}$$

$$\downarrow$$

$$\left\{ \Gamma \;\middle|\; \begin{array}{l} \Gamma \text{ is a tropical curve of genus } g \text{ and degree } d \text{ passing} \\ \text{through } trop(p_1), \ldots, trop(p_n) \end{array} \right\}.$$

The (signed) count of preimages for a given tropical curve Γ can be computed by taking products of certain weights associated to the vertices of Γ.

Real Counts with Non-Totally Real Point Conditions

Theorem [Shustin '06].

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Theorem [Gathmann, Markwig, Schroeter '13].
The above computation is equal to the count (with certain multiplicities $\mu_{\mathbb{R}}(\Gamma)$ for each curve Γ) of degree d rational tropical curves through $3d-1-s$ points, with the first s points on the vertices.

The multiplicities $\mu_{\mathbb{R}}(\Gamma)$ are product of terms corresponding to vertices of Γ.

Definition [Gottsche, Schroeter '19, following Block and Gottsche '16].

Let $s \in \mathbb{N} \cup \{0\}$ and let $p_1, \ldots, p_{3d-1-s} \in \mathbb{P}^2_K$ be generic points. For a rational tropical curve Γ, of degree d, that passes through $p_1, \ldots, p_{3d-1-s}$ and with $p_1, \ldots, p_s$ on its vertices,

$$RB_q(\Gamma) := \prod_{V \in \Gamma^0 \cap \boldsymbol{p}} [\mu(V)]_q^+ \cdot \prod_{V \in \Gamma^0 \setminus \boldsymbol{p}} [\mu(V)]_q^-$$

where

$$[a]_q^{\pm} := \frac{q^{a/2} \pm q^{-a/2}}{q^{1/2} \pm q^{-1/2}} \in \mathbb{Z}\left(q^{1/2}\right).$$

Refined Invariants

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$[\mu(V_1)]_q^+ [\mu(V_2)]_q^-$

Theorem [Gottsche, Schroeter '16].

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$$RB_q(d, 0, s, \boldsymbol{p}) := \sum_{\Gamma} RB_q(\Gamma)$$

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2. *For $q=1$ we get*

$$RB_1(d,0,s) = \left\langle \tau_0(2)^s \tau_1(2)^{3d-1-s} \right\rangle_{0,d}$$

is the descendant GW invariant [Markwig, Rau '09]. This is also equal to the number of degree d curves in the plane passing through $3d-1-s$ points, with the first s points having prescribed tangency direction [Graber, Kock, Pandharipande '02].

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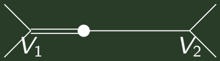
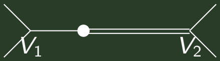
3. $$RB_{-1}(d, 0, s) = W_{d,3d-1-s}.$$

Definition of refined elliptic broccoli invariants

To get an invariance in genus 1 we need to consider *collinear cycles*





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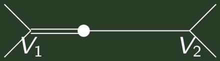

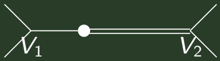

and to assign the set of curves with the same image in $\mathbb{R}^2$ the weight [Schroeter, Shustin '16]

$$\Psi_q^{(2)}(w, \mu(V_1), \mu(V_2)) \cdot \prod_{\substack{V \in \Gamma^0 \setminus \boldsymbol{p} \\ V \notin \{V_1, V_2\}}} [\mu(V)]_q^- \cdot \prod_{V \in \Gamma^0 \cap \boldsymbol{p}} [\mu(V)]_q^+ .$$

Our Results

Simple curves

Definition.

A tropical curve Γ is called *simple* if all its vertices of valency higher than 3 are contained in one of the following configurations:

The *fragments* of a simple curve are

Definition of the Refined Count

Definition.

The refined weight of a simple curve Γ is defined as

$$\mathrm{RB}_q(\Gamma) = \prod_{V \in \Gamma^0 \cap \boldsymbol{p}} [\mu(V)]_q^+ \cdot \prod_{\substack{V \in \Gamma^0 \setminus \boldsymbol{p} \\ \mathrm{Val}(V)=3 \\ \mu(V) \neq 0}} [\mu(V)]_q^- \times$$

$$\times \prod_{(E_1,E_2) \in \mathcal{C}_{\mathrm{central}}(\Gamma)} \varphi_q^{(0)}(\mathrm{wt}(E_1), \mathrm{wt}(E_2)) \times$$

$$\times \prod_{(V,E_1,E_2) \in \mathcal{C}_{\text{non-central}}(\Gamma)} \varphi_q^{(1)}(\mathrm{wt}(E_1), \mathrm{wt}(E_2), \mu(V))$$

Theorem [Shustin, S. '24].

Let $g \in \mathbb{N} \cup \{0\}$, $d \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ and $p_1, \ldots, p_{3d+g-1-s} \in \mathbb{P}^2_K$ be generic points. Suppose that one of the following assumption holds:

1. *Either $g \leq 1$,*
2. *or $s \leq 1$,*
3. *or the points $\boldsymbol{p}$ are in Mikhalkin position.*

Then all the degree d tropical curves passing through $\boldsymbol{p}$ with $p_1, \ldots, p_{3d+g-1-s}$ on the vertices are simple and the sum

$$RB_q(d, g, s) := \sum_{\Gamma} RB_q(\Gamma)$$

is independent of the choice of points.

Relation to other definitions of refined count

- Our count recovers the refined count of Schroeter and Shustin for $g = 1$.

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- Our count recovers the refined count of Schroeter and Shustin for $g = 1$.
- Our count also agree with the one defined in terms of floor diagrams by Mevel. This provides a computation tool for our invariant, as well as alternative proof of the invariance for points in Mikhalkin position.

Theorem [Shustin, S. '24].

Suppose that either $g = 0$, or the points $\boldsymbol{p}$ are in Mikhalkin position. Then

$$RB_1(d, g, s)$$

is the number of degree d curves in the plane passing through $3d + g - 1 - s$ points, with the first s points having a prescribed tangent direction.

Value at $q = 1$ - Correspondence

The tropicalization induces a finite to one map

$$\left\{ C \,\middle|\, \begin{array}{l} C \text{ is a genus } g \text{ curve of degree } d \text{ passing through} \\ x_1, \ldots, x_{3d+g-1-s} \text{ and having prescribed tangent direc-} \\ \text{tions at } x_1, \ldots, x_s \end{array} \right\}$$

$$\downarrow$$

$$\left\{ \Gamma \,\middle|\, \begin{array}{l} \Gamma \text{ is a tropical curve of genus } g \text{ and degree } d \text{ passing} \\ \text{through } p_1, \ldots, p_{3d+g-1-s} \text{ s.t. } p_1, \ldots, p_s \text{ are on ver-} \\ \text{tices of } \Gamma \end{array} \right\},$$

and the number of preimages for a given tropical curve Γ is the value of its refined weight at $q = 1$.

Value at $q = -1$

Problem: There exists tropical point conditions (even in Mikhalkin position) for which the signed number of real curves passing through the given point configuration is different.

Solution: Put the points with prescribed tangent directions on the boundary divisor.

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Solution: Put the points with prescribed tangent directions on the boundary divisor.
The tropicalization will have fixed unbounded ends of higher weight.

Refined weight for points on the boundary divisor

Definition.

The refined weight of a tropical curve Γ with points on the boundary divisor is defined as

$$\mathrm{RB}_q(\Gamma) = \prod_{V \in \Gamma^0} [\mu(V)]_q^- \prod_{E \in \Gamma^1_\infty} \frac{1}{[\mathrm{wt}(E)]_q^-}.$$

Refined weight for boundary conditions

Theorem [Shustin, S. '26+].

Let $RB_q^{\partial}(d,g,s) := \sum_{\Gamma} RB_q(\Gamma)$*, summing over tropical curves of genus* g*, degree* d *passing through* $3d+g-1-s$ *points, with the first* s *on the boundary divisor and weight* 2 *edges.*

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- ***(Invariance)*** $RB^{\partial}_q(d,g,s)$ *is invariant under point configuration.*
- ***(***$q=1$***)*** *Counts degree* d *curves in* $\mathbb{P}^2_K$ *through* $3d+g-1-s$ *points in* $(\mathbb{C}^\times)^2$*, tangent to the boundary divisor at* s *prescribed points.*

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Let $RB^{\partial}_q(d,g,s) := \sum_{\Gamma} RB_q(\Gamma)$*, summing over tropical curves of genus* g*, degree* d *passing through* $3d+g-1-s$ *points, with the first* s *on the boundary divisor and weight 2 edges.*

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($q=1$**)** *Counts degree* d *curves in* $\mathbb{P}^2_K$ *through* $3d+g-1-s$ *points in* $(\mathbb{C}^{\times})^2$*, tangent to the boundary divisor at* s *prescribed points.*

($q=-1$**)** *Counts degree* d *real curves in* $\mathbb{P}^2_K$ *through* $3d+g-1-s$ *points in* $(\mathbb{R}^{\times})^2$*, meeting the boundary at* s *pairs of complex conjugate points, plus additional unspecified boundary points.*

Future work

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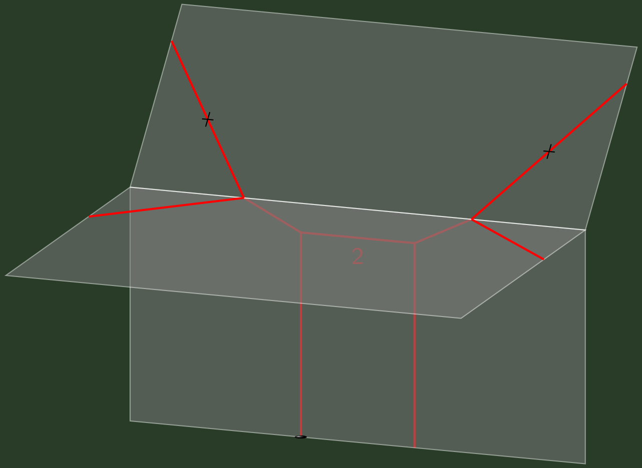
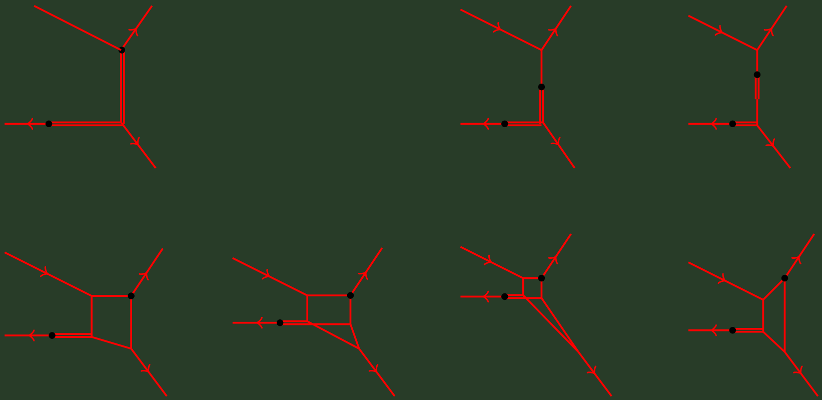
- We are looking at the interpertation of $RB_q^{\partial}(d, g, s)$ in terms of Mikhalkin's quantum indices.
- It is interesting to investigate an interpertation of both $RB_q(d, g, s)$ and $RB_q^{\partial}(d, g, s)$ as a certain generating series for log-GW invariants with lambda class insertions, as in the works of Bousseau and of Kennedy-Hunt, Shafi, and Urundolil Kumaran.

Questions?

Modification of centrally embedded collinear cycle

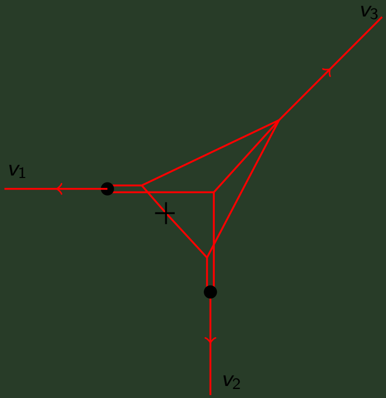


Challenges in higher genus - locallity of the weight



Challenges in higher genus - non simple curves

In higher genus we can have a non simple curve passing through a generic point configuration, for example:



$$\varphi_q^{(0)}(w_1, w_2) = \frac{[w_1]_q^- [w_2]_q^-}{[w_1 + w_2]_q^-},$$

$$\varphi_q^{(1)}(w_1, w_2, \nu) = [w_1 \nu]_q^- [w_2 \nu]_q^- - \frac{[w_1]_q^- [w_2]_q^-}{[w_1 + w_2]_q^-} [(w_1 + w_2)\nu]_q^-$$

$$\Psi_z^{(2)}(m,\nu_1,\nu_2)=\frac{1}{\left(z-z^{-1}\right)^3\left(z+z^{-1}\right)}\times$$
$$\times\left[\frac{2\left(z^{\nu_2 m}-z^{-\nu_2 m}\right)\left(z^{\nu_1 m-1}-z^{1-\nu_1 m}\right)}{z-z^{-1}}-\right.$$
$$-\frac{2m\left(z^{\nu_2 m}-z^{-\nu_2 m}\right)\left(z^{\nu_1 m-m}-z^{m-\nu_1 m}\right)}{z^m-z^{-m}}+$$
$$+(m-1)\left(z^{\nu_1 m}-z^{-\nu_1 m}\right)\left(z^{\nu_2 m}+z^{-\nu_2 m}\right)-$$
$$-\frac{2\left(z^{\nu_2 m}-z^{-\nu_2 m}\right)\left(z^{\nu_1 m-\nu_1}-z^{\nu_1-\nu_1 m}\right)}{z^{\nu_1}-z^{-\nu_1}}-$$
$$\left.-\frac{2\left(z^{\nu_1 m}-z^{-\nu_1 m}\right)\left(z^{\nu_2 m-\nu_2}-z^{\nu_2-\nu_2 m}\right)}{z^{\nu_2}-z^{-\nu_2}}\right].$$

where $\mu(V_1)=m\nu_1$ and $\mu(V_2)=m\nu_2$.