Skeins, clusters and wavefunctions

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Talk based on joint projects with Mingyuan Hu, Linhui Shen, and Eric Zaslow.

The chromatic Lagrangian (`arXiv:2302.00159` w. L.Shen, E.Zaslow)

Skeins, clusters, and wavefunctions (`arXiv:2312.10186` w. M.Hu, E.Zaslow)

See also closely related work by Scharitzer-Shende `arXiv:2312.10625`.

Overview

Goal: compute open Gromov-Witten generating functions for a class of non-compact Lagrangians in $\mathbb{C}^3$.

Strategy: derive some *equations* satisfied by the generating function (using symplectic geometry), and then solve those equations (using low dimensional topology/quantum algebra).

The 'quantum algebra' we use to solve these equations seems interesting in its own right – should be part of a generalization of the theory of *cluster algebras.*

Open Gromov-Witten theory setup

Let (X, ω, J) be a 6-dimensional Calabi-Yau manifold (e.g $\mathbb{C}^3$, T^*S^3) and L a (Maslov-0) Lagrangian submanifold in X.

Let $\Sigma = \Sigma_{g,h}$ be a Riemann surface of genus g with h boundary components.

Open Gromov-Witten theory seeks to 'count' holomorphic maps

$$u : (\Sigma, \partial\Sigma) \longrightarrow (X, L).$$

Open Gromov-Witten theory

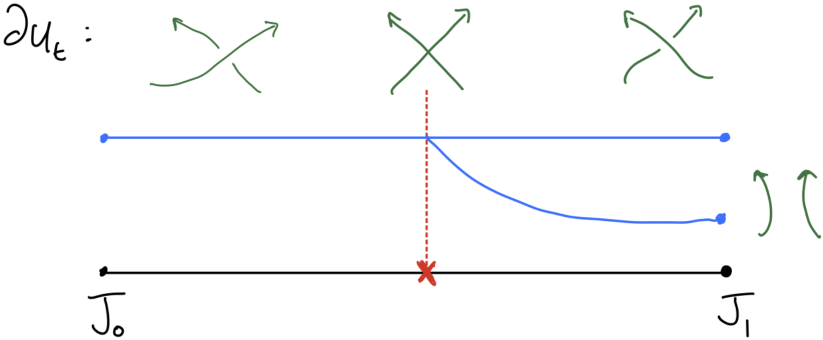
Let's fix the homology class $[\partial u] \in H_1(L; \mathbb{Z})$.

The expected dimension of the moduli space of such maps is zero, so we hope to define a numerical invariant counting such maps.

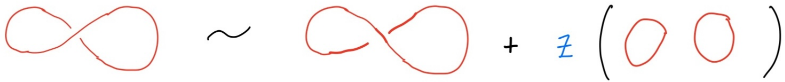
For a generic almost complex structure J, all maps u are rigid and embedded so we can indeed count them.

Dependence on generic almost complex structure

But these counts can jump in 1-parameter families as we pass through non-generic J:



Note that this jumping is not even homogeneous with respect to Euler characteristic of Σ!

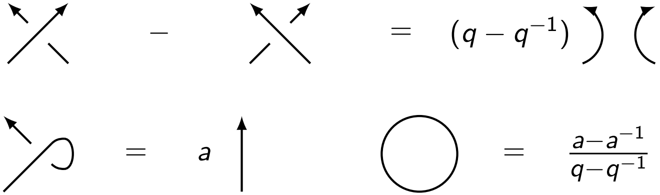


So maybe to get something well-defined we should sum over all topological types of domains $\Sigma_{g,h}$, weighting maps by $z^{\chi(\Sigma)}$.

Ekholm-Shende's skeins on branes

Ekholm and Shende observed that this jumping behavior is of a kind familiar from knot theory, where it appears as a *skein relation*.

Let L be an oriented 3-manifold. Its *HOMFLYPT skein module* is the $\mathbb{C}(a,q)$-module $Sk(L)$ spanned by isotopy classes of framed oriented links in L, modulo the relations



HOMFLYPT Skeins

Example 1: $Sk(\mathbb{R}^3) = \mathbb{C}(a,q) \cdot \langle \emptyset \rangle$. The element of $\mathbb{C}(a,q)$ corresponding to a link L is called its HOMFLYPT polynomial.

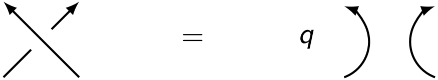
HOMFLYPT Skeins

If Σ is a surface, then $\mathrm{Sk}(\Sigma \times I)$ is an **algebra** by stacking cylinders $I \sqcup I \hookrightarrow I$.

And if L is a 3-manifold with $\partial L = \Sigma$, $\mathrm{Sk}(L)$ is a **module** over $\mathrm{Sk}(\Sigma \times I)$.

For each $N \in \mathbb{N}$, the skein module has a quotient corresponding to reduction to $U_q(\mathfrak{gl}_N)$ skeins.

For $N = 1$, this reduction is the so-called 'linking skein': extra relations $a = q$ and



allows to resolve all crossings, so class of link l only depends on image in $H_1(\Sigma, \mathbb{Z})$

E.g.

$$Sk_{N=1}(T^2) \simeq \mathbb{C}(q)\langle X^{\pm 1}, Y^{\pm 1}\rangle, \quad XY = q^2 YX,$$

where $X = P_{1,0}$, $Y = P_{0,1}$ is a **quantum torus algebra** (deformed group ring of lattice $H_1(T^2; \mathbb{Z}^2)$).

Linking skein

Same is true of linking skein for general surface Σ:

$$[l_1][l_2] = q^{2(l_1,l_2)}[l_2][l_1]$$

where $(\cdot,\cdot)$ is intersection form.

HOMFLYPT skein algebra of the cylinder

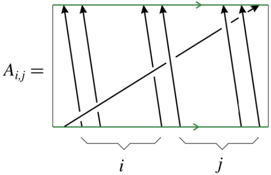
If Σ is a cylinder, the skein algebra $\mathrm{Sk}(\Sigma \times \mathbb{R})$ is commutative, and isomorphic to the tensor product $\Lambda_{a,q}(x) \otimes \Lambda_{a,q}(y)$ of two copies of the ring $\Lambda_{a,q}$ of symmetric functions in infinitely many variables.

Each copy $\Lambda_{a,q}$ is freely generated over $\mathbb{C}(a,q)$ by the power sum symmetric functions

$$P_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k \geq 1$$

The isomorphism identifies $P_1(x)$ (resp. $P_1(y)$) with the simple closed curve winding clockwise (resp. counterclockwise) around the cylinder.

More generally, if we introduce skeins



then P_k corresponds to their linear combination

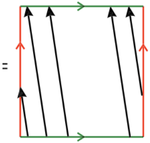
$$P_k = \frac{1}{(k)_q} \sum_{i=0}^{k-1} A_{i,k-i-1}$$

HOMFLYPT Skein algebra of torus

Example 2: (Morton-Samuelson '17) The skein algebra of the 2-torus $\mathrm{Sk}(T^2)$ is generated by elements $P_{\mathbf{x}}, \mathbf{x} \in \mathbb{Z}^2$ with relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (q^d - q^{-d}) P_{\mathbf{x}+\mathbf{y}}, \quad d = \det(\mathbf{x}|\mathbf{y})$$

For (m, n) coprime, $P_{(m,n)}$ is the skein corresponding to the (m, n)-curve $l_{m,n}$ on the torus.



In general, $P_{k(m,n)}$ is the element corresponding to the k-th power sum in the skein of the cylinder obtained by thickening $l_{m,n}$.

Ekholm-Shende partition function

Theorem: (Ekholm-Shende) The partition function

$$\Psi_L = \sum_u a^{(u,L)} z^{\chi(u)} [\partial u] \in \widehat{Sk}(L)$$

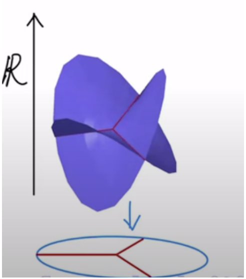
is independent of the choice of J.

Question: how to compute Ψ_L?

Goal for rest of talk: answer this for a class of cylindrical Lagrangians L in $X = \mathbb{C}^3$, which are asymptotic to $\Lambda \times \mathbb{R}$ where $\Lambda \subset S^5$ is a Legendrian surface in $S^5 = \partial_\infty \mathbb{C}^3$.

Cubic graph Lagrangians

The surfaces Λ_Γ we consider sit inside $J^1 S^2 \subset S^5$, with front projections Λ'_Γ sin $S^2 \times \mathbb{R}$ encoded by the combinatorial data of a trivalent graph Γ on S^2:



The map $\Lambda_\Gamma \to S^2$ is a 2:1 branched cover, with branch points at the vertices of Γ.

So the preimage of an edge e gives a loop ℓ_e on Λ_Γ.

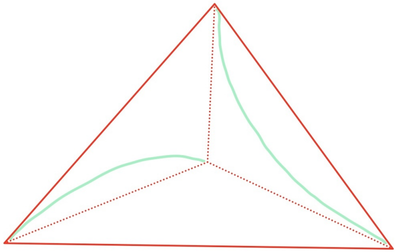
Cubic graph Lagrangians

The Lagrangians L we consider are double covers of B^3, branched over a tangle with endpoints = vertices of Γ, and are topologically genus g handlebodies (where # faces of $\Gamma = g + 3$).

Aganagic-Vafa brane

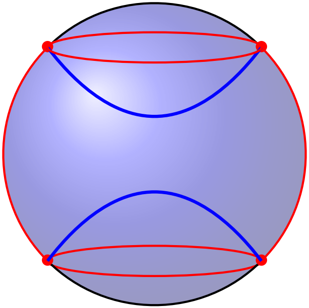
If Γ is the tetrahedron graph with tangle shown below, Λ_Γ is the Clifford torus in S^5. The Lagrangian L is non exact, and has topology of a solid torus, known as the Aganagic-Vafa toric brane:

$$L_{AV} = \{\sqrt{r^2+\epsilon^2}e^{2\pi i s}, re^{2\pi i t}, re^{-2\pi i(s+t)}\} \subset \mathbb{C}^3.$$



Necklace graph and Chekanov torus

As another example, consider the ‘necklace with two beads’ graph, with the following tangle:

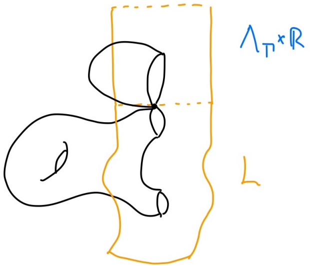


The corresponding L_Γ is the Chekanov Lagragian, which is exact.

Recursion for the partition function from SFT

Ekholm-Shende: suppose $L \approx \Lambda \times \mathbb{R}$ is cylindrical, and fix a Reeb chord for the Legendrian Λ. Then this data defines a 1-dimensional moduli space.

Its boundary corresponds to curves of the form



So we get a relation of the form

$$(\#\text{curves at infinity })\star(\#\text{curves in L }) = 0,$$

i.e.

$$A\cdot\Psi = 0,$$

where $A \in Sk(\Lambda\times\mathbb{R}), \quad \Psi \in Sk(L)$.

Recursion for the partition function from SFT stretching

In our case, there's a Reeb chord for each face f of Γ, and the argument above gives a relation

$$A_f \cdot \Psi_{L_\Gamma} = 0$$

The element $A_f \in \mathrm{Sk}(\Lambda_\Gamma \times \mathbb{R})$ can be computed combinatorially using Morse flow trees (Schartizer-Shende '23).

Recursion for the AV brane

For the AV brane, all relations are equivalent to

$$A_{AV} = P_{1,1} + P_{0,1} - [\bigcirc] \in Sk(T^2).$$

Linking skein recursion for the AV brane

After reducing to the linking skein, $Sk(T^2)$ becomes the quantum torus $XY = q^2 YX$, and $Sk(L)$ becomes the ring of Laurent polynomials in X, with the action

$$(Y \cdot \psi)(X) = \psi(q^{-2}X).$$

The reduced AV recursion is equivalent to

$$(Y + q^{-1}XY - 1) \cdot \psi = 0$$

i.e. the *q-difference equation*

$$\psi(q^{-2}X) = (1 + q^{-1}X)\psi(X).$$

Recursion for the AV brane

In the case of the necklace graph and its Chekanov filling, the recursion reads

$$(P_{0,1} - [\bigcirc]) \cdot \Psi_{Chek} = 0,$$

which in the linking skein reduces to the QDE

$$\psi(q^2 X) = \psi(X).$$

This equation has the unique solution

$$\Psi_{Chek} \equiv 1.$$

(Indeed, we could have seen this without using the recursion since L_{Chek} is **exact**!)

AV solution via mutation

We can solve the linking skein version of the AV recursion relation using the quantum dilogarithm

$$\phi(X) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} X^n}{n(q^n - q^{-n})}\right) = \prod_{m\geq 0} (1 + q^{2m+1} X)^{-1},$$

an element of the (completed) linking skein of the cylinder.

The key point is that the dilogarithnm *conjuagtes* A_{Check} to A_{AV}:

$$A_{AV} = \phi(X) \cdot A_{Chek} \cdot \phi(X)^{-1}.$$

So we can solve the QDE

$$A_{AV} \cdot \psi_{AV} = (q^{-1}XY + Y - 1) \cdot \psi_{AV} = 0$$

by conjugating it back to the *trivial* QDE

$$A_{Chek} \cdot \psi_{Chek} = (Y - 1) \cdot \psi_{Chek} = 0$$

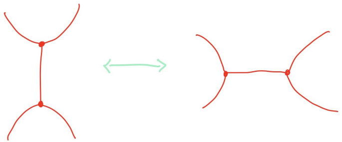
The solutions are related by acting with $\phi(X)$:

$$\begin{aligned} \psi_{AV} &= \phi(X) \cdot \psi_{Chek} \\ &= \phi(X) \cdot 1 \\ &= \phi(X). \end{aligned}$$

In fact, can use the same trick to solve the linking skein QDE for any cubic planar graph Lagrangian.

Solution via mutation

Theorem (SSZ '23). The system of linking skein QDE associated to any Γ has a unique solution ψ_Γ. If graphs Γ, Γ' differ by the performing the following local transformation at an edge e



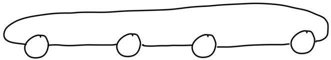
then conjugation by $\phi(X_e)$ sends $A_{f,\Gamma}$ to $A_{f',\Gamma'}$, and hence the solutions are related by

$$\psi_{\Gamma'} = \phi(X_e) \cdot \psi_\Gamma,$$

Solution via mutation

But any two trivalent planar graphs on S^2 with the same number of vertices can be connected by a sequence of such moves at edges $e_1, \ldots, e_m$.

Hence, we can transform any Γ back to the higher genus analog of the Chekanov Lagrangian

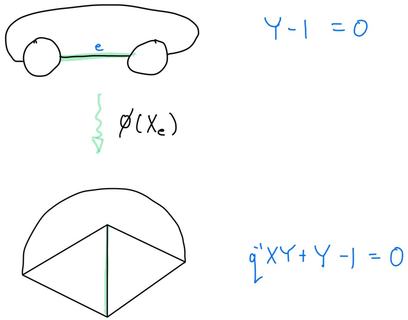


which is also exact and has $\Psi = 1$. So we get a formula for the partition function:

$$\psi_\Gamma = \phi(e_m) \cdots \phi(e_1) \cdot 1.$$

Solution by mutation

AV solution in this context:



$$\psi_{AV} = \phi(X) \cdot \psi_{Chek} = \phi(X)$$

Cluster interpretation

The map between completed quantum tori given by conjugation by $\Phi(X)$ is a familiar object in the theory of cluster algebras: it is exactly a *quantum cluster mutation.*

In the classical limit $q \to 1$ we have a holomorphic symplectic *cluster variety* $\mathcal{X}$ covered by algebraic torus charts, one for each Γ, and the mutations define the gluing maps between these charts.

Cluster interpretation

The classical limit of the recursion relations in each chart defining Ψ_Γ define an ideal sheaf on $\mathcal{X}$, cutting out a compact holomorphic Lagrangian $\mathcal{M} \subset \mathcal{X}$.

Hence we can think of the collection of partition functions $(\Psi_\Gamma)_\Gamma$ as defining a 'global section' of a 'coherent sheaf' on the quantum cluster variety.

HOMFLYPT version

Question: Can we solve the full HOMFLYPT skein-valued recursions (not just their $N = 1$ reductions) in a similar way, using a skein-theoretic analog of cluster mutation?

To do this, we need a skein analog of the quantum dilogarithm $\Phi(\ell)$, where ℓ is the simple closed loop on Λ_Γ corresponding to an edge e.

Skein mutation

HSZ '23: define the **skein dilogarithm** to be

$$\Phi(\ell) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} P_{n\ell}}{n(q^n - q^{-n})}\right) \in Sk(\Lambda_\Gamma),$$

where the $P_{n\ell}$ are the elements in the skein of the cylinder obtained by thickening ℓ corresponding to the power sum symmetric functions.

We use conjugation by $\Phi(\ell)$ to define the notion of **skein mutation** corresponding to flipping the edge ℓ of Γ.

Partition functions via skein mutation

Theorem (HSZ '23). The system of HOMFLYPT skein recursion relations associated to any Γ has a unique solution Ψ_Γ. If graphs Γ, Γ' differ by the a flip at e, skein mutation sends $A_{f,\Gamma}$ to $A_{f',\Gamma'}$, and the solutions to the two systems are related by

$$\Psi_{\Gamma'} = \Phi(\ell_e) \cdot \Psi_\Gamma.$$

Hence if $e_1, \dots, e_m$ is any sequence of edge flips taking Γ_{neck} to Γ, we have

$$\Psi_\Gamma = \Phi(e_m) \cdots \Phi(e_1) \cdot 1.$$

Skein Pentagon

Theorem (HSZ 23)

The pentagon identity for the skein dilogarithm holds in $Sk(T^2)$:

$$Q_{1,0}Q_{0,1} = Q_{0,1}Q_{1,1}Q_{1,0}$$

This result was later significantly improved by Hu:

Theorem (HSZ 23)

The pentagon identity for the skein dilogarithm holds in the skein of the punctured torus $Sk(T^2 - D^2)$:

In particular, using the surjection from $Sk(T^2 - D^2)$ to the elliptic Hall algebra, Hu gives a purely topological proof of the pentagon identities of Zenkevich and Garsia-Mellit (originally proved using Macdonald theory).