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SEMICONTINUITY OF BETTI NUMBERS & SINGULAR SETS

Seminar in Real and Complex Geometry
Tel Aviv 2-12-2021

"SINGULARITIES"

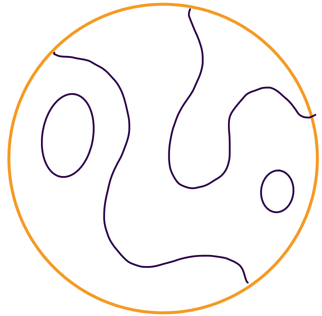
$$\psi : M \to \mathbb{R}^k$$

$$x \in M : \begin{cases} e\left(x, \psi(x), \psi'(x), \dots, \psi^{(r)}(x)\right) = 0 \\ \vdots \\ e_c\left(x, \psi(x), \psi'(x), \dots, \psi^{(r)}(x)\right) = 0 \end{cases}$$

$Z = \{ \text{solutions} \}$ = "singularity"

EXAMPLES

1. "ZEROES" $Z(\psi) := \psi^{-1}(0)$

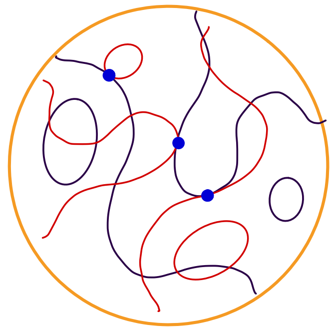


- If ψ homogeneous polynomial

$$\begin{tikzcd} Z = \{\psi = 0\} \arrow[d, "2:1"] & \subset & M = S^m \arrow[d, "2:1"] \\ Z_{\mathbb{P}}(\psi^1) \cap \dots \cap Z_{\mathbb{P}}(\psi^k) & \subset & \mathbb{RP}^m \end{tikzcd}$$

2. $Z = \{p : Z(\psi_1) \,\&\, Z(\psi_2) \text{ tangent at } p\}$

$$\psi = (\psi_1, \psi_2)$$

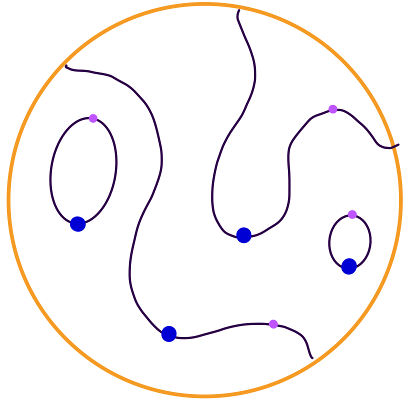


3. "Critical points":

$$Z = \mathrm{Crit}(\psi) = \{p : d_p\psi \text{ is not surj.}\}$$

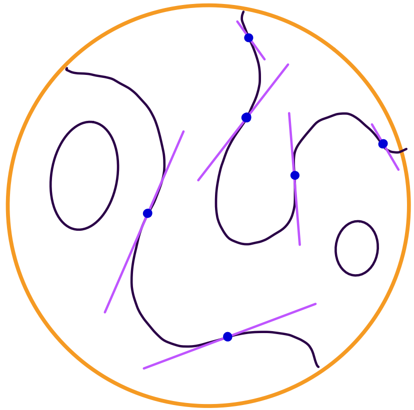
4. $Z_0 = \{\psi = 0\}$ $\quad g : M \longrightarrow \mathbb{R}$

$$Z = \left\{ x \in \mathrm{Crit}\left(g|_{Z_0}\right) : \begin{array}{l} \\ \text{MORSE INDEX} = \lambda \end{array} \right\}$$



RMK: $\underline{b_\lambda(Z_0) \le \# Z}$

5. $$Z = \left\{ x \in \{\psi = 0\} : \begin{array}{l} \text{The} \\ \text{II fundamental} \\ \text{form of } \{\psi=0\} \\ \text{is degenerate} \end{array} \right\}$$



• Type-W SINGULARITIES

$$D := \mathbb{D}^m = \{x \in \mathbb{R}^m : |x| \leq 1\}$$

- Jet space: $J^r(D, \mathbb{R}^k) := D \times \mathbb{R}[\underline{x}]^k_{\deg \leq r}$

$$\Psi$$

- $f \in C^r(D, \mathbb{R}^k) \rightsquigarrow j^r_x f = \left(x, \begin{array}{c}\text{TAYLOR POLYN. OF} \\ \text{ORDER } r \text{ AT } x\end{array}\right)$

$$j^r f : D \xrightarrow{C^0} J^r(D, \mathbb{R}^k) \supset W$$

- DEF: $Z_W(f) \doteq (j^r f)^{-1}(W) \subset D$

"type-W singularity of f"

- $$\varphi : M \longrightarrow \mathbb{R}^k$$

$$x \in M : \begin{cases} e\Big(\overbrace{x,\ \varphi(x),\ \varphi'(x),\ \dots,\ \varphi^{(r)}(x)}^{j^r_x \varphi}\Big) = 0 \quad (\text{or } \leq) \\ \vdots \\ e_c\Big(x,\ \varphi(x),\ \varphi'(x),\ \dots,\ \varphi^{(r)}(x)\Big) = 0 \quad (\text{or } \leq) \end{cases}$$

$$Z_W(\varphi) = \Big\{ x : j^r_x \varphi \in \overset{W}{\overset{\parallel}{\{e=0\}}} \Big\}$$

- $$j^r \varphi : M \, (= D^m) \longrightarrow J^r(M, \mathbb{R}^k) \supseteq W$$

- Usually, $W \subset J^r(\mathbb{D}^m, \mathbb{R}^n) \cong \mathbb{R}^N$ is
  - algebraic or (polynomials eqs.)

    $\Downarrow$
  - or semialgebraic (f. $\cup, \cap$ pol. eq. and ineq.)

    $\Downarrow$
  - or Whitney stratified ($\cup$ of smooth mfds glued properly)

DEF: $Z_W(f)$ is non-dg iff $j^r f \pitchfork W$ ($\forall$ stratum)

$\Longrightarrow$ $Z_W(f)$ is Whitney stratified $\subset \mathbb{D}^m$

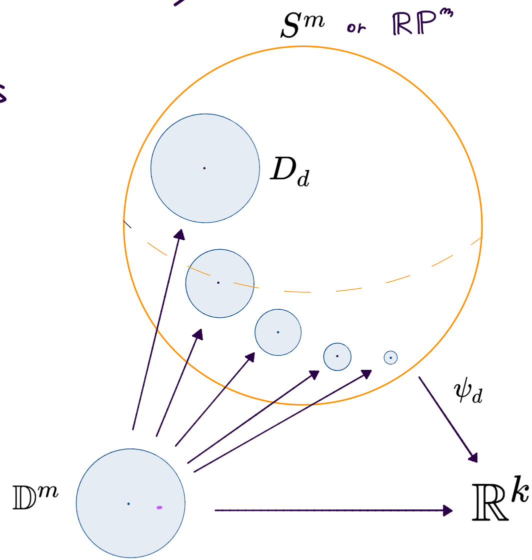
e.g. ($p \in \mathrm{Crit}(f)$ is non-dg iff $\det d_p^2 f \neq 0$)

2 Random Remarks :

1. $\psi_d : S^m \to \mathbb{R}^k$ Kostlan random pol.

$\left(\text{Probability } \mathbb{P} \text{ on } \mathbb{R}[x_0, \dots, x_{m+1}]^k_{(d)} \right)$

$D_d :=$ Any sequence of affine disks with radius $\frac{1}{\sqrt{d}}$

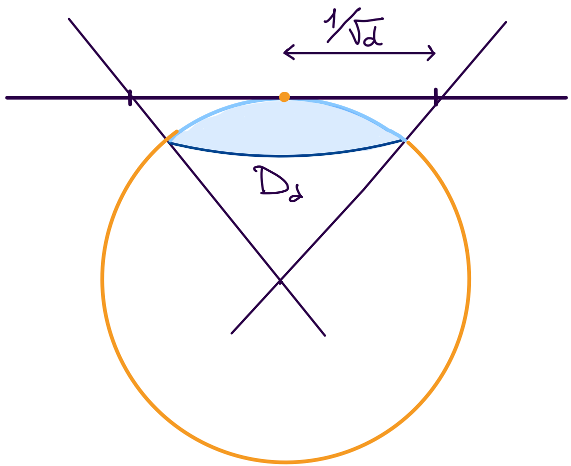


FACT:

$$\boxed{\begin{array}{l} \psi_d|_{D_d} \xrightarrow[d \to +\infty]{\text{law-}\mathcal{C}^\infty} f|_{\mathbb{D}} \\ \text{s.t. } \operatorname{supp}[f] = \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k) \end{array}}$$

- $\underset{\text{open}}{U} \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k) \implies \mathbb{P}\left\{ \psi_d|_{D_d} \in U \right\} \geq c \quad \forall d$ big

Meaning: Let $j_d : \mathbb{D}^m \hookrightarrow S^m$ be an embedding onto an affine disk $D_d = j_d(\mathbb{D}^m) \subset S^m$ ("spherical" would also work)



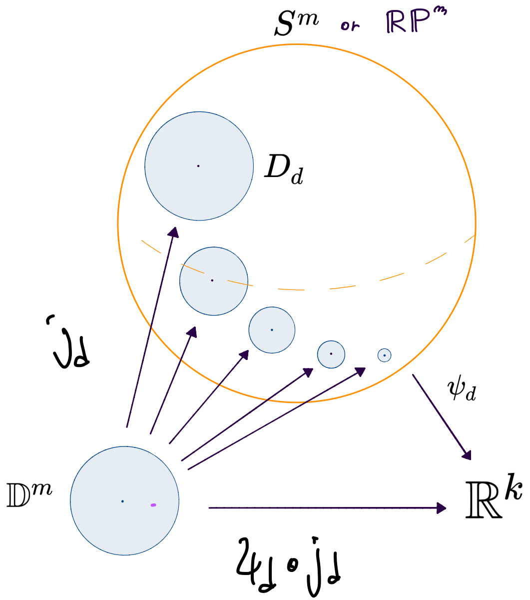
$$j_d(u) = R_d \circ j_1\left(\frac{u}{\sqrt{d}}\right)$$

where $R_d \in SO(m+1)$ is arbitrary

$$j_1(u) := \frac{(u,1)}{\sqrt{1+|u|^2}} \qquad \forall\, u \in \mathbb{D}^m$$

- Consider the random smooth maps:

$$\psi_d \circ j_d : \mathbb{D}^m \longrightarrow \mathbb{R}^k$$



The Kostlan probability $\mathbb{P}_d$ on $\mathbb{R}[x_0, \dots, x_m]^k_{(d)}$ induces, via the function $\psi_d \longmapsto \psi_d \circ j_d$, a probability μ_d on $\mathcal{E}^\infty(\mathbb{D}^m, \mathbb{R}^k)$.

Here, $\mathcal{E}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ has the Borel σ-algebra of Whitney topology: the one defined by the collection of seminorms $\{\|\cdot\|_{\mathcal{E}^k}\}_{k \in \mathbb{N}}$

- Usually, one writes μ_d as:

$$\mu_d(U) = \mathbb{P}\{ \psi_d \circ j_d \in U \}$$

$\forall\ U \subset \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ Borel set.

- **FACT**: $\exists$ probability μ on $\mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$

s.t. 1. $\mu_d \Rightarrow \mu$

2. $\operatorname{supp}(\mu) = \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$

1. This is the NARROW or WEAK convergence of probability measures. More precisely it's the weak-* convergence of

$$\left(F \mapsto \int F \, d\mu_d \right) \in \mathcal{C}_b\left(\mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k) \right)^*$$

Equivalently it can be defined as follows:

" $\forall \, U \subset \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ open :

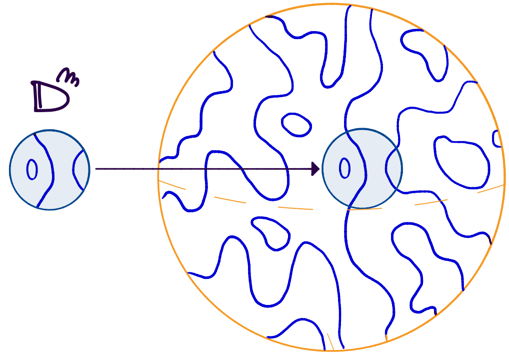
$$\liminf_{d \to +\infty} \mu_d(U) \geq \mu(U)$$

2. $\iff$ if $U \neq \emptyset$ & open, then $\mu(U) > 0$.

• Moreover,

$$b\big(Z_W(\psi_d)\cap D_d\big) \longrightarrow b\big(Z_W(f)\big)$$

both in <u>law</u> and in <u>$\mathbb{E}$</u>.



2. (DIATTA – LERARIO, 2018)

With $\mathbb{P}\sim 1$, $\quad Z(\psi_d) \simeq_{\text{isotopy}} Z(p_{d'})$

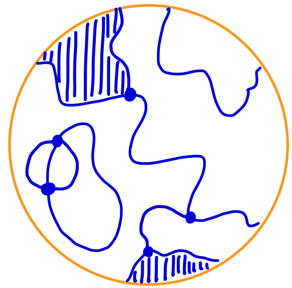
where $p_{d'}$ is a pol. of deg $d' = O\big(\sqrt{d\log d}\,\big)$

(ANCONA, 2020) Similar result in a more general setting.

SETTING:

$$\mathbb{D}^m \xrightarrow{f} \mathbb{R}^k$$

$Z_W(f)$



- (SMOOTH MAP) $f \in C^\infty(\mathbb{D}^m, \mathbb{R}^k)$
- (SINGULARITY TYPE) $W \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ Whitney stratified
- (SINGULARITY) $Z_W(f) = (j^r f)^{-1}(W)$

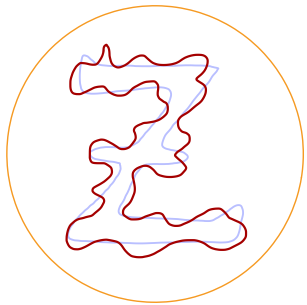
s.t. $j^r f \pitchfork W$ ($\forall$ stratum, $\Rightarrow j^r f|_{\partial D} \pitchfork W$)

PB: Topology of $Z_W(f)$: Betti nbs $b(Z_W(f))$

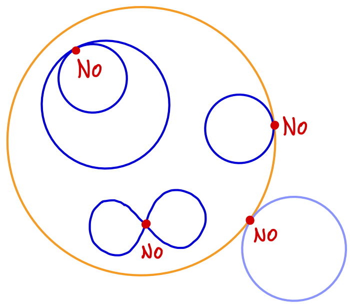
- The technique is: Approximate f with a pol. p_d $f \quad f_t \quad p_d$

(i) Thom isotopy Lemma

for $j^r f_t : D \longrightarrow J^r(D, \mathbb{R}^k)$:



$$\boxed{\text{If } j^r f_t \pitchfork W, \ \forall\, t \in [0,1] \text{ then } \left(\mathbb{D}, j^r f_t^{-1}(W)\right) \simeq \text{constant isotopy type}}$$



(ii) True if $\| j^r p_d - j^r f \|_{C^1} < \varepsilon(f)$.

- Make both quantitative and ...

THEOREM 1 (LERARIO - S., 2020) Let $f \in C^{r+2}(\mathbb{D}^m, \mathbb{R}^k)$ (with $j^r f \pitchfork W$) $W \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ closed and semialgebraic, then

$$\exists\, p_d \text{ s.t.}\ \cdot \ (\mathbb{D}^m, Z_W(p_d)) \underset{\text{isotopy}}{\cong} (\mathbb{D}^m, Z_W(f))$$

$$\cdot\ d \leq c(m,k,r) \max\left\{ \boxed{\frac{\|f\|_{C^{r+2}}}{\operatorname{dist}_{C^{r+1}}(f, \Delta_W)}},\ r+1 \right\}$$

where $\Delta_W = \{ g \in C^{r+1} : j^r g \not\pitchfork W \}$

COROLLARY: Let $f \in C^{r+2}(\mathbb{D}^m, \mathbb{R}^k)$ s.t. $J^r f \pitchfork W$.
Let $W \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ closed semialgebraic, then

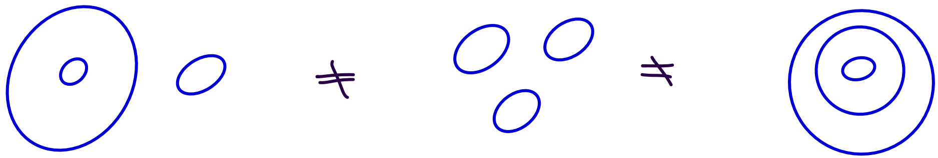
$$b(Z_W(f)) \leq C_W \left(\max\left\{ \frac{\|f\|_{C^{r+2}}}{\mathrm{dist}_{C^{r+1}}(f, \Delta_W)}, \; r+1 \right\} \right)^m$$

Proof: Thom-Milnor bound $\Rightarrow$ $Z_W(p_d) \leq C_W d^m$ $\square$

- Q: Is this good?

Isotopy type $(\mathbb{D}^m, Z_W(f))$:

Same Betti n.
$b_i(Z_W(f))$



• HYPERSURFACES

$D = \mathbb{D}^m$

$f \in \mathcal{C}^\infty(D, \mathbb{R}) \qquad Z(f) = f^{-1}(0)$

- $W_0 := D \times \{0\} \subset J^0(D,\mathbb{R}) = D \times \mathbb{R}$

- $\Delta_0 = D \times \{0\} \times \{0\} \subset J^1(D,\mathbb{R}) = D \times \mathbb{R} \times \mathbb{R}^m$

Note: $g \in \Delta_{W_0}$ iff $\exists\, x \in D$ s.t. $j^{r+1}_x g \in \Delta_0$

- $$\mathrm{dist}_{C^1}(f, \Delta_{W_0}) = \inf_{g \in \Delta_{W_0}} \sup_{x \in D} \| j^1_x g - j^1 f \|$$

FACT: $$\dots = \inf_{x \in D} \sqrt{|f(x)|^2 + |d_x f|^2}$$

- "Boundary assumption": $Z(f) \subset \operatorname{int} D$

  ($\Rightarrow f|_{\partial D} \pitchfork W$)

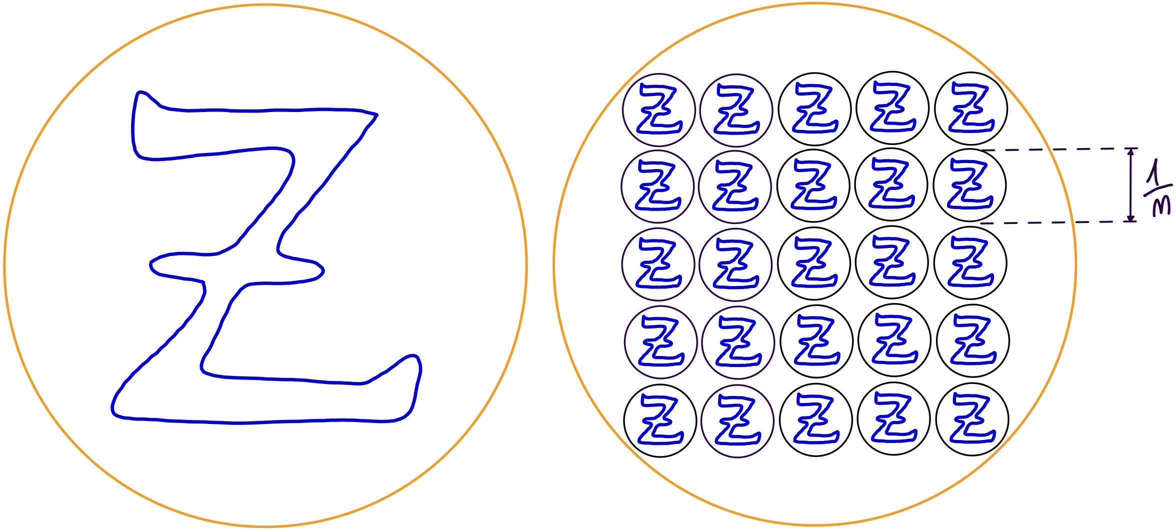
- $\delta(f) := \min\left\{ \operatorname{dist}_{C^1}(f, \Delta_0),\ \max_{\partial D} |f| \right\}$

- $\delta(f) > 0$ iff 0 is regular value

  & $Z(f) \subset \operatorname{int}(D)$

COR: $f \in C^2$, $\delta(f) > 0$, then

$$\boxed{b(Z(f)) \leq c(m) \left(\frac{\|f\|_{C^2}}{\delta(f)} \right)^m}$$



$Z(f)$ $\qquad$ $Z(f_n)$

$$b(Z(f_n)) \sim n^n \sim \left(\frac{\|f_n\|_{C^1}}{\delta(f_n)} \right)^n \quad \text{as } n \to +\infty$$

- Locally $\quad f_n(x) = f(n(x-c_i))$

- $\|f_n\|_\infty = \|f\|, \quad \|f_n'\|_\infty = n\|f'\|_\infty, \quad \|f_n''\|_\infty = n^2\|f''\|_\infty$

- $\delta(f_n) = \inf_x \left(|f(nx)|^2 + n^2|f'(nx)|^2\right)^{\frac{1}{2}} \xrightarrow[n\to+\infty]{} \min_{\mathrm{Crit}(f)} |f|$

$$\left(\frac{\|f_n\|_{C^1}}{\delta(f_n)}\right)^m \sim n^m \sim b\left(Z_W(f_n)\right) \underset{\text{COR}}{\ll} \left(\frac{\|f_n\|_{C^2}}{\delta(f_n)}\right)^m \sim n^{2m}$$

- THM2 (LERARIO - S., 2020)

$$f \in C^1(\mathbb{D}^m, \mathbb{R}), \quad \delta(f) > 0$$

$$b(Z(f)) \leq c(m) \left(\frac{\|f\|_{C^1}}{\delta(f)} \right)^m$$

- COR: Let $Z \subset \mathrm{int}(D)$ $\mathcal{C}^1$ hypersurface s.t.

$$\rho := \text{REACH} = \sup \left\{ \varepsilon > 0 : \exists!\ \pi : B_\varepsilon(Z) \longrightarrow Z \right\} > 0$$

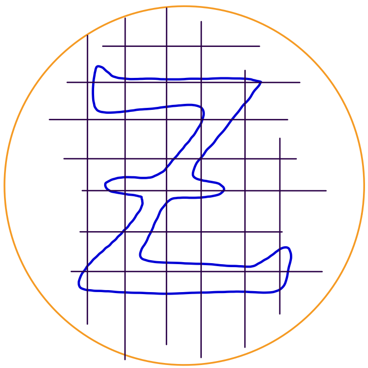
$$b(Z) \leq c \left(1 + \frac{1}{\rho} + \cancel{\frac{S_m}{\rho^2}} \right)^m$$

- Similar results:

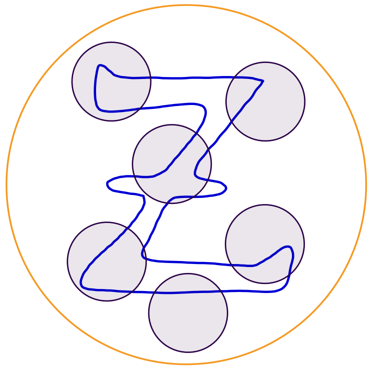
(Yomdin '83) $\Rightarrow$ $$b(Z(f)) \le B\left(\frac{\|f\|_{C^1}}{\delta(f)}\right)$$

(Niyogi, Smale & Weinberger 2006) $\Rightarrow$ $$b(Z) \le c\,\frac{1}{\rho^m}$$

METHOD:



or



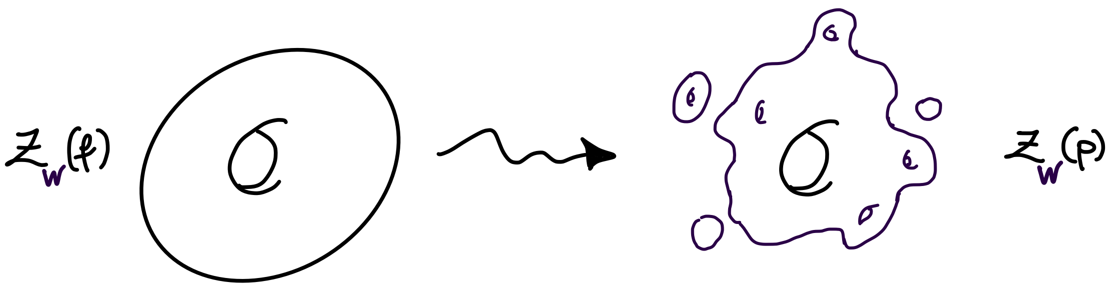
- What about $Z_W(f) = j^r f^{-1}(W)$?

FIRST METHOD : Thom Isotopy lemma

$$j^r p \sim j^r f \text{ in } C^1 \implies Z_W(f) \simeq Z_W(p)$$ too much

NEW IDEA : Allow changes in the topology, but C^0-small

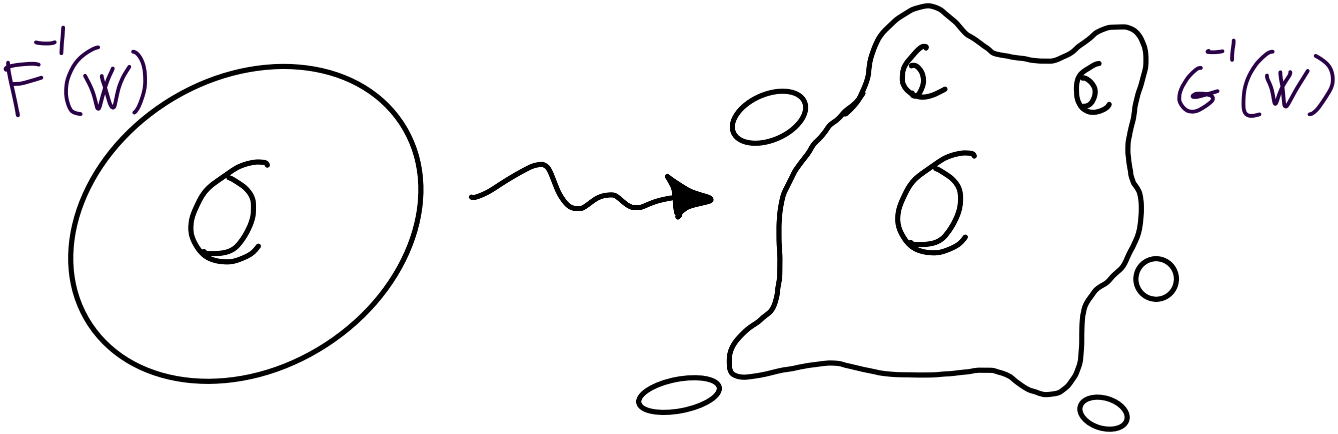
$$j^r p \sim j^r f \text{ in } C^0 \overset{\text{THM}}{\implies} b(Z(f)) \leq b(Z(p))$$



THM Semicontinuity of Betti numbers (S. 2019)

- $F, G : M \longrightarrow N \supset W$ closed C^1 submfd
- $F, G \pitchfork W$
- $\mathrm{dist}_{C^0}(F, G) < \varepsilon(F)$

$\Longrightarrow b_i(F^{-1}(W)) \leq b_i(G^{-1}(W))$



(M, N are smooth mfds; M is compact; Betti numbers b_i are with $\mathbb{Z}$-coeff.)

THM (LERARIO - S. 2020) $\quad D = D^m$

$W \subset \mathbb{R}^N$ semialgebraic, smooth & compact

$F \in \mathcal{C}^1(D, \mathbb{R}^N)$ s.t. $F \pitchfork W$ & $F^{-1}(W) \subset \operatorname{int}(D)$

$$b\left(F^{-1}(W)\right) \le c_W \max\left\{ \frac{\|F\|_{C^1}}{\hat{\delta}_W(F)}, r \right\}^m$$

- $\hat{\delta}_W(F) \sim \min\left\{ \operatorname{dist}_{C^1}(F, \Delta_W), \operatorname{reach}(W) \right\}$

- With $F = j^r f$ we obtain

$$\mathcal{b}\left(Z_W(f)\right) \le C_W \max\left\{ \frac{\|f\|_{C^{r+1}}}{\hat{\delta}_W(f)}, r \right\}^{m}$$

Q: Is this stupid?

$$\exists\, f \text{ s.t. } \mathcal{b}\left({j^r f}^{-1}(W)\right) \geqslant \varepsilon \ ?$$

$$\left(\mathcal{b}_0 \ge 1 \text{ easy}\right) \qquad \mathcal{b}_0 \geqslant 2 \ ?$$

$$\mathcal{b}_i \geqslant 1 \ ?$$

- Certainly $\exists\ F : D \longrightarrow \underset{\substack{\cup \\ W}}{J^r(D,\mathbb{R}^k)} \cong \mathbb{R}^N$

  s.t. $b_i\left(F^{-1}(W)\right) \geq m$

  but $F = j^r f$ has to be "holonomic":

$$j^r_x f = \left(x, f(x), f'(x), f''(x), \ldots\right)$$

THM (LERARIO–S, 2019)

Given $m \in \mathbb{N}$ $\exists f$ s.t. $j^r f \pitchfork W$ and

$$b_i\left(Z_W(f)\right) \geq m \quad \forall i \leq \dim Z$$

sketch of the proof: $(\bullet)$ Take $F : \mathbb{D}^m \to J^r(\mathbb{D}^m, \mathbb{R}^k)$

s.t. $b_i(F^{-1}(W)) \geq m$

$(\bullet\bullet)$ <u>HOLONOMIC APPROXIMATION</u> (Eliashberg - Mishachev - Gromov)

$\exists f \in \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ s.t. $j^r f \underset{\mathcal{C}^0\text{- close}}{\sim} F \circ h$

& $h \in \mathrm{Aut}(\mathbb{D}^m)$

(Here I am cheating: it is more complicated)

$(\because)$ Semicontinuity thm: $b_i(j^r f^{-1}(W)) \geq b_i(F^{-1}(W)) \geq m$ $\square$

- Open problem:

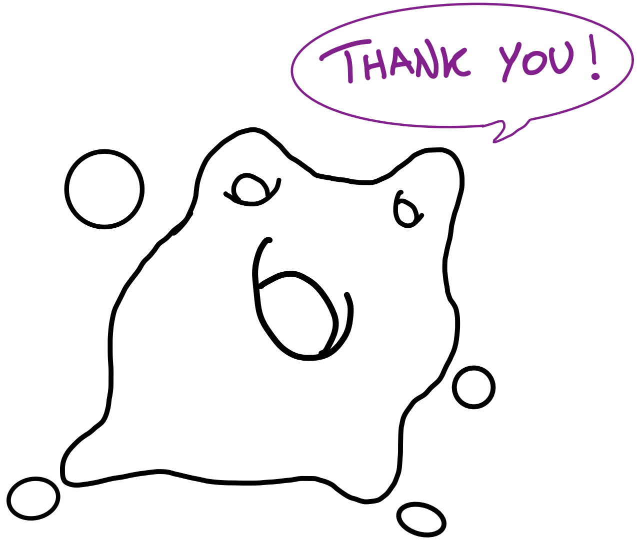
Often $W \subset J^n(D, \mathbb{R}^k)$ is stratified

$$\left(\text{e.g. } Z_W(f) = \{ p \in D : \ \operatorname{rank}(d_p f) \leq c \} \right)$$

THM SEMICONTINUITY OF BETTI NUMBERS

$$\left. \begin{array}{l} f : M \longrightarrow N \supset W \quad \begin{array}{l}\text{closed} \\ \text{smooth}\end{array} \\ f, g \pitchfork W \\ \operatorname{dist}_{C^0}(g, f) < \varepsilon(f) \end{array} \right\} \Longrightarrow b_i(f^{-1}(W)) \leq b_i(g^{-1}(W))$$

Is it true if $W \subset N$ is stratified?


THANK YOU!