Broccoli curves of genus 0 and maybe 1

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Tel Aviv, November 1, 2012

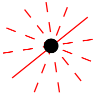
Enumerative geometry

Aim is to count geometric objects like curves, surfaces, etc. such that

- they satisfy certain conditions like passing through given points, tangency conditions, have degree d and genus g, etc.
- their count gives a finite number N(conditions).

Example

- Severi degree $N(d,\delta)$: # of complex plane curves C of degree d with δ nodes through $3d-1+(\binom{d-1}{2}-\delta)$ generic points.
- For $d \geq \delta+2$ ($\Rightarrow C$ irred.) the Severi degree $N(d,\delta)$ is called Gromov-Witten number.
- Lines ($d=1$) through 2 points in $\mathbb{P}^2$:

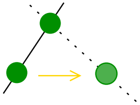


$P_1 = P_2$

$\rightsquigarrow$ infinite number of lines!

Main feature:

The number $N(d,\delta)$ is invariant, i.e. does not depend on the position of the points as long as they are in general position.



Theorem (Mikhalkin's Correspondence Theorem)

$$N(d,\delta) = N_{trop}(d,\delta).$$

That is: the number of complex plane nodal resp. tropical plane nodal curves of degree d with δ nodes through a fixed set ω of $3d-1+(\binom{d-1}{2}-\delta)$ points in general position is equal.

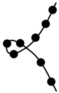
Idea:

- Degeneration: complex curves $\to$ tropical curves
- Define multiplicity $\mathrm{mult}_{\mathbb{C}}(C)$ of a tropical curve C combinatorially $\rightsquigarrow$

$$N_{trop}(d,\delta) = \sum_{C \text{ through } \omega} \mathrm{mult}_{\mathbb{C}}(C) = \sum_{C \text{ through } \omega} \prod_{v \text{ vertex in } C} \mathrm{mult}_{\mathbb{C}}(v).$$

Example

$d=3,\ \delta=1:$



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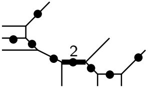


$\mathrm{mult}_{\mathbb{C}}(C) = 4$

$N(3,1) =$ 12 $= N_{trop}(3,1)$

Real plane curves

Definition

A real plane curve consists

- of the defining polynomial $f \in \mathbb{R}[x, y, z]$
- and the zero set in $\mathbb{P}^2_{\mathbb{C}}$.

Consequences:

- If $z \in \mathbb{P}^2_{\mathbb{C}}$ is in $V(f) \Rightarrow \bar{z} \in V(f)$.
- We can choose real or pairs of complex conjugate points as fixing conditions in a real enumerative problem.

Problem:
Counting nodal real plane curves of degree d with δ nodes through r real points and s pairs of complex conjugate points s.t. $r + 2s = 3d - 1 + \binom{d-1}{2} - \delta$ does not lead to an invariant number!

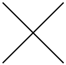
Example (A. I. Degtyarev, V. M. Kharlamov, 2000)

The number of rational cubics through 8 real points is $8, 10$ or 12 depending on the position of the points.

Reason: Two different types of real nodes:

$$x^2 + y^2 = 0$$

isolated node



$$x^2 - y^2 = 0$$

Idea (J. Y. Welschinger): Count real rational curves with weights ± 1!

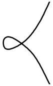
Definition (Welschinger numbers)

C real plane rational curve of degree d through a set ω of r real and s pairs of complex conjugate points in general position.

$$m(C) := \#\text{ isolated nodes in } C,$$

$$W(d,r,s) := \sum_{C \text{ through } \omega} (-1)^{m(C)}.$$

Example



$m(C) = 0$

$m(C) = 1$

Example (Comparison between complex and real invariants)

d	1	2	3	4
$N(d, \binom{d-1}{2})$	1	1	12	620
$W(d, 3d-1, 0)$	1	1	8	240

Theorem (J. Y. Welschinger, 2005)

The numbers $W(d,r,s)$ are invariant.

But: No method of computing these numbers.

If we just consider real curves passing through real points ($s=0$):

Tropical Welschinger numbers:
Same curves as for $N_{trop}(d,\delta)$ but new multiplicity $\text{mult}_{\mathbb{R}}(C)$.
For a fixed point configuration ω we then have:

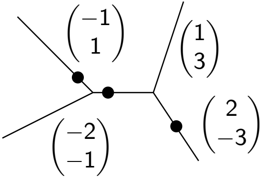
$$W_{trop}(d,3d-1,0) := \sum_{C \text{ through } \omega} \text{mult}_{\mathbb{R}}(C).$$

The multiplicity $\text{mult}_{\mathbb{R}}(C)$

Definition

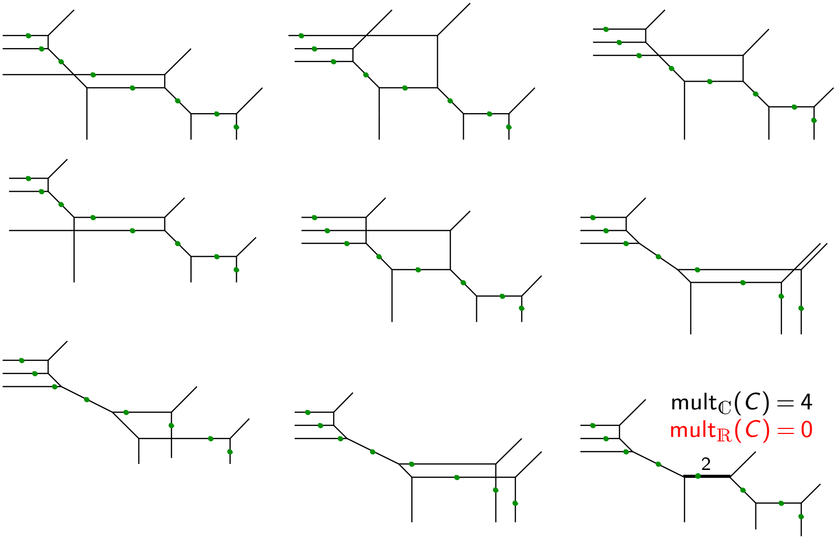
$$\text{mult}_{\mathbb{R}}(C) := \begin{cases} 0, & \text{if } \text{mult}_{\mathbb{C}}(C) \equiv 0 \mod 4, \\ 1, & \text{if } \text{mult}_{\mathbb{C}}(C) \equiv 1 \mod 4, \\ 0, & \text{if } \text{mult}_{\mathbb{C}}(C) \equiv 2 \mod 4, \\ -1, & \text{if } \text{mult}_{\mathbb{C}}(C) \equiv 3 \mod 4. \end{cases}$$

Example



$\text{mult}_{\mathbb{C}}(C) = 27$
$\text{mult}_{\mathbb{R}}(C) = -1$

The (bigger) example of cubics...



$$W_{trop}(3,8,0) = 8$$

Theorem (Correspondence Thm., G. Mikhalkin, 2005)

$$W(d, 3d-1, 0) = W_{trop}(d, 3d-1, 0).$$

On the complex side we have:

Theorem (A. Gathmann, H. Markwig, 2007)

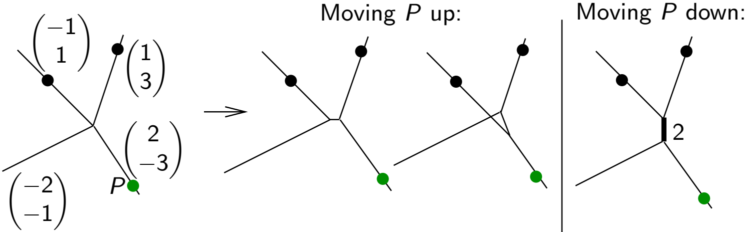
Tropical proof of the invariance of $N_{trop}(d, \binom{d-1}{2})$.

$\rightsquigarrow$ Construction of tropical moduli spaces.

Theorem (I. Itenberg, V. Kharlamov, E. Shustin, 2009)

- *Tropical proof of the invariance of the numbers* $W_{trop}(d, 3d-1, 0)$
- *Caporaso-Harris type formula for* $W_{trop}(d, 3d-1, 0)$
- *Construction of new tropical invariants*

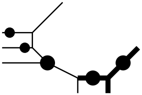
Idea behind the local invariance proof



$\text{mult}_{\mathbb{C}}(C):$	$3\cdot 9=27$	$5\cdot 1=5$	$4\cdot 8=32$
$\text{mult}_{\mathbb{R}}(C):$	-1	1	0

Complex point conditions

Allow now the curves to pass through complex points (i.e. not necessary $s = 0$).
$\rightsquigarrow$ can define $W(d,r,s)$.
Tropical side: Need new tropical curves to define $W_{trop}(d,r,s)$.



Theorem (J. Y. Welschinger, 2005)

The numbers $W(d,r,s)$ are invariant.

Theorem (Correspondence Thm., E. Shustin, 2006)

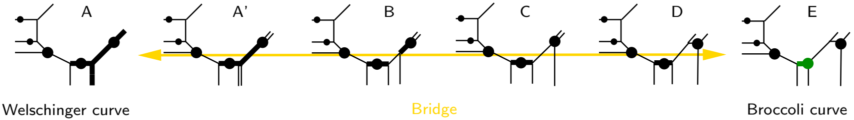
$$W(d,r,s) = W_{trop}(d,r,s).$$

But: There is no tropical proof of the invariance of $W_{trop}(d,r,s)$ known yet!

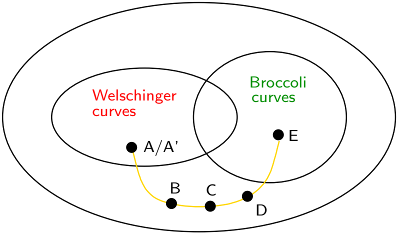
One reason: the local invariance proof fails.

The origin of broccoli curves

Idea: Modify tropical Welschinger numbers s.t. they equal under certain conditions Welschinger numbers and s.t. their invariance can be proved locally!
$\rightsquigarrow$ Define *broccoli curves* and count them instead!



Corresponding picture in the moduli space $\mathscr{M}^{\text{lab}}_{0,2+3,\text{trop}}(\mathbb{R}^2, \Delta_3)$:

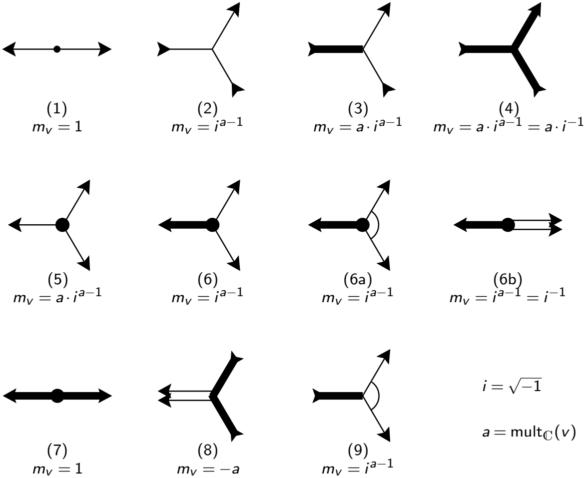


Curve which is broccoli and Welschinger:

Definition (Pseudo definitions)

- *Welschinger curve:* Fat markings lie on vertices or on edge of even weight; the connected component of even edges meets the connected component of odd edges in one vertex,
- *broccoli curve:* fat markings should lie on vertex s. t. at least one even edge is adjacent to this vertex; if one connected component of even edges meets a component of odd edges in k vertices, then $k-1$ of them should have a fat marking.

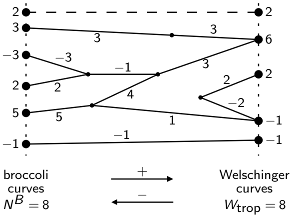
The true definition is based on vertex types:



Results

Theorem (A. Gathmann, H. Markwig, F. S., 2011)

The broccoli numbers $N^B(d, r, s)$ are invariant.
Certain broccoli invariants equal tropical Welschinger invariants $W_{trop}(d, r, s)$.



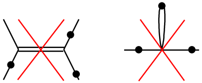
Main features of broccoli invariants:

- Classical counterpart not known yet.
- We can define relative broccoli invariants $\rightsquigarrow$ recursive formulas, e.g. Caporaso-Harris.
- Generalization to higher genus possible.

Broccoli curves of genus 1 (work in progress)

Construction:

- Start with broccoli vertex types and construct curves of genus 1, i.e. with a loop.
- Eliminate in the corresponding moduli space cells where the expected dimension $(|\Delta|-1)+1$ does not coincide with the actual dimension of the cell.



is strongly related to the question of

Invariance of the corresponding numbers:

- Are there new vertex types appearing while moving markings around?
- Noninvariance would make broccoli curves of genus 1 useless.

A very partial result:

Theorem (Local picture)

Let $\Delta \subset \mathbb{R}^2$ be a lattice triangle, whose lattice points are either corners of Δ or in the interior $int(\Delta)$. Assume there are n lattice points in $int(\Delta)$.

Consider a broccoli curve dual to Δ consisting of one 3-valent vertex and passing through one fat marking P and one thin marking P'.

Then it holds for the broccoli multiplicities $mult_B(C_l)/mult_B(C_r)$ of broccoli curves C_l/C_r of genus 1 appearing when moving P' to the left/right hand side and passing through P and P':

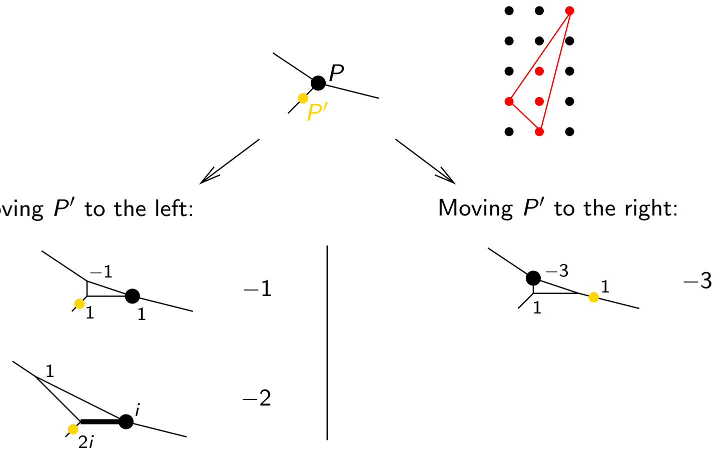
$$\sum_{C_l} mult_B(C_l) = \sum_{C_r} mult_B(C_r) \qquad \text{and} \qquad \left|\sum_{C_l} mult_B(C_l)\right| = \frac{n(n+1)}{2}.$$

Remarks:

- No new vertex types should be introduced for this theorem.
- An analogous statement holds when P is moved instead.

- Codimenson-1 cells in the moduli space correspond to curves with a 4-valent vertex or with a contracted loop.
- n equals the number of resolutions of the contracted loop.
- Curves appearing as resolutions may count 0.

Example ($n = 2$)



תודה על תשומת הלב שלך