Broccoli curves of genus 0 and maybe 1

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Enumerative geometry

Aim is to count geometric objects like curves, surfaces, etc. such that

- they satisfy certain conditions like passing through given points, tangency conditions, have degree \( d \) and genus \( g \), etc.
- their count gives a finite number \( N(\text{conditions}) \).

Example

- Severi degree \( N(d, \delta) \): \# of complex plane curves \( C \) of degree \( d \) with \( \delta \) nodes through \( 3d - 1 + \left( \binom{d-1}{2} - \delta \right) \) generic points.
- For \( d \geq \delta + 2 \) (\( \Rightarrow C \) irred.) the Severi degree \( N(d, \delta) \) is called Gromov-Witten number.
- Lines \( (d = 1) \) through 2 points in \( \mathbb{P}^2 \):

\[ P_1 = P_2 \]

\( \sim \) infinite number of lines!
Main feature:
The number $N(d, \delta)$ is invariant, i.e. does not depend on the position of the points as long as they are in general position.
Theorem (Mikhalkin’s Correspondence Theorem)

\[ N(d, \delta) = N_{trop}(d, \delta). \]

That is: the number of complex plane nodal resp. tropical plane nodal curves of degree \(d\) with \(\delta\) nodes through a fixed set \(\omega\) of \(3d - 1 + \left(\left(\frac{d - 1}{2}\right) - \delta\right)\) points in general position is equal.

Idea:

- Degeneration: complex curves \(\rightarrow\) tropical curves
- Define multiplicity \(\text{mult}_C(C)\) of a tropical curve \(C\) combinatorially \(\rightsquigarrow\)

\[
N_{trop}(d, \delta) = \sum_{C \text{ through } \omega} \text{mult}_C(C) = \sum_{C \text{ through } \omega} \prod_{v \text{ vertex in } C} \text{mult}_C(v). 
\]

Example

\(d = 3, \ \delta = 1:\)

\[
N(3, 1) = 12 = N_{trop}(3, 1).
\]
Real plane curves

Definition

A real plane curve consists

- of the defining polynomial $f \in \mathbb{R}[x,y,z]$
- and the zero set in $\mathbb{P}^2_{\mathbb{C}}$.

Consequences:

- If $z \in \mathbb{P}^2_{\mathbb{C}}$ is in $V(f)$ $\Rightarrow \bar{z} \in V(f)$.
- We can choose real or pairs of complex conjugate points as fixing conditions in a real enumerative problem.

Problem:

Counting nodal real plane curves of degree $d$ with $\delta$ nodes through $r$ real points and $s$ pairs of complex conjugate points s.t. $r + 2s = 3d - 1 + \binom{d-1}{2} - \delta$ does not lead to an invariant number!
Example (A. I. Degtyarev, V. M. Kharlamov, 2000)

The number of rational cubics through 8 real points is 8, 10 or 12 depending on the position of the points.

Reason: Two different types of real nodes:

\[ x^2 + y^2 = 0 \]

isolated node

\[ x^2 - y^2 = 0 \]

Idea (J. Y. Welschinger): Count real rational curves with weights ±1!
**Definition (Welschinger numbers)**

A real plane rational curve of degree $d$ through a set $\omega$ of $r$ real and $s$ pairs of complex conjugate points in general position.

\[
m(C) := \# \text{ isolated nodes in } C,
\]

\[
W(d, r, s) := \sum_{C \text{ through } \omega} (-1)^{m(C)}.
\]

**Example**

\[
\begin{align*}
\bigcap & \quad m(C) = 0 \\
\cup & \quad m(C) = 1
\end{align*}
\]

**Example (Comparison between complex and real invariants)**

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(d, \binom{d-1}{2})$</td>
<td>1</td>
<td>1</td>
<td>12</td>
<td>620</td>
</tr>
<tr>
<td>$W(d, 3d - 1, 0)$</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>240</td>
</tr>
</tbody>
</table>
Theorem (J. Y. Welschinger, 2005)

The numbers $W(d, r, s)$ are invariant.

But: No method of computing these numbers.

If we just consider real curves passing through real points ($s = 0$):

Tropical Welschinger numbers:
Same curves as for $N_{trop}(d, \delta)$ but new multiplicity $\text{mult}_R(C)$.
For a fixed point configuration $\omega$ we then have:

$$W_{trop}(d, 3d - 1, 0) := \sum_{C \text{ through } \omega} \text{mult}_R(C).$$
The multiplicity \( \text{mult}_R(C) \)

**Definition**

\[
\text{mult}_R(C) := \begin{cases} 
0, & \text{if } \text{mult}_C(C) \equiv 0 \mod 4, \\
1, & \text{if } \text{mult}_C(C) \equiv 1 \mod 4, \\
0, & \text{if } \text{mult}_C(C) \equiv 2 \mod 4, \\
-1, & \text{if } \text{mult}_C(C) \equiv 3 \mod 4. 
\end{cases}
\]

**Example**

\[
\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
\begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -3 \end{pmatrix}
\]

\( \text{mult}_C(C) = 27 \)
\( \text{mult}_R(C) = -1 \)
The (bigger) example of cubics...

\[ W_{\text{trop}}(3,8,0) = 8 \]

\[ \text{mult}_\mathbb{C}(C) = 4 \]
\[ \text{mult}_\mathbb{R}(C) = 0 \]
Theorem (Correspondence Thm., G. Mikhalkin, 2005)

\[ W(d, 3d - 1, 0) = W_{trop}(d, 3d - 1, 0). \]

On the complex side we have:

Theorem (A. Gathmann, H. Markwig, 2007)

*Tropical proof of the invariance of \( N_{trop}(d, \binom{d-1}{2}) \).*

\[ \supseteq \text{Construction of tropical moduli spaces.} \]

Theorem (I. Itenberg, V. Kharlamov, E. Shustin, 2009)

- *Tropical proof of the invariance of the numbers \( W_{trop}(d, 3d - 1, 0) \)*
- *Caporaso-Harris type formula for \( W_{trop}(d, 3d - 1, 0) \)*
- *Construction of new tropical invariants*
Idea behind the local invariance proof

\[
\begin{align*}
\begin{pmatrix}
-1 \\
1
\end{pmatrix} & \quad \begin{pmatrix}
1 \\
3
\end{pmatrix} & \quad \begin{pmatrix}
2 \\
-3
\end{pmatrix} \\
\begin{pmatrix}
-2 \\
-1
\end{pmatrix}
\end{align*}
\]

\[
\text{mult}_C(C) : \\
\text{mult}_R(C) :
\]

Moving \( P \) up:

\[
\begin{align*}
3 \cdot 9 &= 27 \\
5 \cdot 1 &= 5
\end{align*}
\]

Moving \( P \) down:

\[
\begin{align*}
4 \cdot 8 &= 32 \\
-1 &
\end{align*}
\]

1

0
Complex point conditions

Allow now the curves to pass through complex points (i.e. not necessary $s = 0$).
\[ \rightsquigarrow \text{can define } W(d, r, s). \]

**Tropical side:** Need new tropical curves to define $W_{trop}(d, r, s)$.

\[ \begin{align*}
\text{Theorem (J. Y. Welschinger, 2005)} \\
\text{The numbers } W(d, r, s) \text{ are invariant.}
\end{align*} \]

\[ \begin{align*}
\text{Theorem (Correspondence Thm., E. Shustin, 2006)} \\
W(d, r, s) = W_{trop}(d, r, s).
\end{align*} \]

But: There is no tropical proof of the invariance of $W_{trop}(d, r, s)$ known yet!

One reason: the local invariance proof fails.
The origin of broccoli curves

Idea: Modify tropical Welschinger numbers s.t. they equal under certain conditions Welschinger numbers and s.t. their invariance can be proved locally!

\[ \rightsquigarrow \] Define \textit{broccoli curves} and count them instead!

Corresponding picture in the moduli space \( \mathcal{M}^{\text{lab}}_{0,2+3,\text{trop}}(\mathbb{R}^2, \Delta_3) \):
Curve which is broccoli and Welschinger:

\[ \sim \]

**Definition (Pseudo definitions)**

- **Welschinger curve**: Fat markings lie on vertices or on edge of even weight; the connected component of even edges meets the connected component of odd edges in one vertex,
- **broccoli curve**: fat markings should lie on vertex s. t. at least one even edge is adjacent to this vertex; if one connected component of even edges meets a component of odd edges in \( k \) vertices, then \( k - 1 \) of them should have a fat marking.
The true definition is based on vertex types:

(1) \( m_v = 1 \)

(2) \( m_v = i^{a-1} \)

(3) \( m_v = a \cdot i^{a-1} \)

(4) \( m_v = a \cdot i^{a-1} = a \cdot i^{-1} \)

(5) \( m_v = a \cdot i^{a-1} \)

(6) \( m_v = i^{a-1} \)

(6a) \( m_v = i^{a-1} \)

(6b) \( m_v = i^{a-1} = i^{-1} \)

(7) \( m_v = 1 \)

(8) \( m_v = -a \)

(9) \( m_v = i^{a-1} \)

\( i = \sqrt{-1} \)

\( a = \text{mult}_\mathbb{C}(v) \)
Results

Theorem (A. Gathmann, H. Markwig, F. S., 2011)

The broccoli numbers $N^B(d, r, s)$ are invariant.
Certain broccoli invariants equal tropical Welschinger invariants $W_{\text{trop}}(d, r, s)$.

Main features of broccoli invariants:

- Classical counterpart not known yet.
- We can define relative broccoli invariants $\rightsquigarrow$ recursive formulas, e.g. Caporaso-Harris.
- Generalization to higher genus possible.
Broccoli curves of genus 1 (work in progress)

Construction:

- Start with broccoli vertex types and construct curves of genus 1, i.e. with a loop.
- Eliminate in the corresponding moduli space cells where the expected dimension \(|\Delta| - 1\) + 1 does not coincide with the actual dimension of the cell.

\[
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{diagram1.png}} \\
\text{\includegraphics[width=0.2\textwidth]{diagram2.png}}
\end{array}
\]

is strongly related to the question of

Invariance of the corresponding numbers:

- Are there new vertex types appearing while moving markings around?
- Noninvariance would make broccoli curves of genus 1 useless.
A very partial result:

**Theorem (Local picture)**

Let $\Delta \subset \mathbb{R}^2$ be a lattice triangle, whose lattice points are either corners of $\Delta$ or in the interior $\text{int}(\Delta)$. Assume there are $n$ lattice points in $\text{int}(\Delta)$.

Consider a broccoli curve dual to $\Delta$ consisting of one 3-valent vertex and passing through one fat marking $P$ and one thin marking $P'$.

Then it holds for the broccoli multiplicities $\text{mult}_B(C_l)/\text{mult}_B(C_r)$ of broccoli curves $C_l/C_r$ of genus 1 appearing when moving $P'$ to the left/right hand side and passing through $P$ and $P'$:

$$\sum_{C_l} \text{mult}_B(C_l) = \sum_{C_r} \text{mult}_B(C_r) \quad \text{and} \quad |\sum_{C_l} \text{mult}_B(C_l)| = \frac{n(n+1)}{2}.$$

**Remarks:**

- No new vertex types should be introduced for this theorem.
- An analogous statement holds when $P$ is moved instead.
• Codimension-1 cells in the moduli space correspond to curves with a 4-valent vertex or with a contracted loop.
• $n$ equals the number of resolutions of the contracted loop.
• Curves appearing as resolutions may count 0.

Example ($n = 2$)
Thank you for your help.

תודה על תשובת הלב שלך.