Nonnegative Polynomials versus Sums of Squares

Bernd Sturmfels
UC Berkeley

Based on a joint paper with G. Blekherman, J. Hauenstein, J.C. Ottem and K. Ranestad and on earlier work of G. Blekherman
Two Warm-up Questions

We know from *high school* that a quadric $ax^2 + bx + c$ is non-negative if and only if $a \geq 0$, $c \geq 0$ and $4ac - b^2 \geq 0$.

Let’s study this convex cone for polynomials of degree four:

$$C = \{(a, b, c, d, e) \in \mathbb{R}^5 \mid \forall x \in \mathbb{R} : ax^4 + bx^3 + cx^2 + dx + e \geq 0 \}$$

1. The Zariski closure of its boundary $\partial C$ is a hypersurface. Find the degree and defining polynomial of this hypersurface.

2. Determine an inequality representation of the dual cone $C^\vee$. 
Question 1: Algebraic boundary of nonnegative quartics

The boundary $\partial C$ consists of all nonnegative polynomial

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

such that $\exists \alpha \in \mathbb{R} : f(\alpha) = 0.$

The root $\alpha$ is necessarily a double root: $f(\alpha) = f'(\alpha) = 0.$
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Hence the discriminant of the quartic $f(x)$ vanishes:

$$256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4$$
$$+ 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e$$
$$- 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2$$

The Zariski closure of $\partial C$ is the irreducible hypersurface of degree 6 defined by this polynomial. We regard it as a hypersurface in $\mathbb{P}^4$. 
The dual cone $C^\vee$ is the cone spanned by the rational normal curve

$$C^\vee = \mathbb{R}_{\geq 0}\{(1, x, x^2, x^3, x^4) : x \in \mathbb{R}\} \subset \mathbb{R}^5.$$
Question 2: Duality in convex geometry

The dual cone $C^\vee$ is the cone spanned by the rational normal curve

$$C^\vee = \mathbb{R}_{\geq 0}\{ (1, x, x^2, x^3, x^4) : x \in \mathbb{R} \} \subset \mathbb{R}^5.$$ 

Its natural inequality representation is as the cone of $3 \times 3$-Hankel matrices that are positive semidefinite:

$$C^\vee = \{ (u_0, u_1, u_2, u_3, u_4) \in \mathbb{R}^5 : \begin{bmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix} \succeq 0 \}$$

In pure math, Hankel matrices are known as *catalecticants*. In applied math, Hankel matrices are known as *moment matrices*.

Now, let the lecture begin....
A Tale of Two Cones

Fix the real vector space of homogeneous polynomials in \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) of degree \( 2d \). In this space, consider the convex cone \( P_{n,2d} \) of all non-negative polynomials and the subcone \( \Sigma_{n,2d} \) of all polynomials that are sums of squares.

Are they equal?
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Are they equal?
Yes, if \( d = 1 \) (quadrics) or \( n = 2 \) (binary forms).

Example \((n = d = 2)\): The binary quartic

\[
f = 2x_1^4 - 6x_1^3x_2 + 9x_1^2x_2^2 - 6x_1x_2^3 + 2x_2^4
\]

is non-negative. To prove this, we write \( sdp \)

\[
f = (x_1^2 - x_2^2)^2 + (x_1^2 - 3x_1x_2 + x_2^2)^2.
\]
Yes, if \( d = 2 \) and \( n = 3 \):

Every non-negative ternary quartic can be written as a sum of three squares of ternary quadrics.

This involves some beautiful 19\(^{th}\) century geometry...

... as explained in [D.Plamann, B.St and C.Vinzant: Quartic curves and their bitangents, Journal of Symbolic Computation 46 (2011) 712-733.]
Theorem (Hilbert, 1888): The containment of convex cones

\[ \Sigma_{n,2d} \subset P_{n,2d} \]

is strict if and only if \((n \geq 3 \text{ and } d \geq 3) \text{ or } (n \geq 4 \text{ and } d \geq 2)\).

What does this mean for the algebraic boundaries of these cones?

The algebraic boundary of \( P_{n,d} \) is an irreducible hypersurface of degree \( n(2d - 1)^{n-1} \), namely the discriminant. This discriminant is one irreducible component also in the algebraic boundary of \( \Sigma_{n,d} \).

Today we examine the two borderline cases:

- Sextic curves in the plane \((n = 3, d = 3)\)
- Quartic surfaces in 3-space \((n = 4, d = 2)\)

The ambient spaces are \( \mathbb{P}^{27} \) and \( \mathbb{P}^{34} \) respectively.
The Lax-Lax Quartic

**Exercise:** The polynomial

\[
(a - b)(a - c)(a - d)(a - e) \\
+ (b - a)(b - c)(b - d)(b - e) \\
+ (c - a)(c - b)(c - d)(c - e) \\
+ (d - a)(d - b)(d - c)(d - e) \\
+ (e - a)(e - b)(e - c)(e - d)
\]

is non-negative but it is not a sum of squares.

[Anneli Lax and Peter Lax: On sums of squares, Linear Algebra and its Applications 20 (1978) 71–75]

This represents a point in \( P_{4,4} \setminus \Sigma_{4,4} \).

What’s the matter with this quartic surface in \( \mathbb{P}^3 \)?

**Pop Quiz:** *Tell us all you know about K3 surfaces.*
Theorem

The algebraic boundary of $\Sigma_{3,6}$ has a unique non-discriminant component. It has degree 83200 and is the Zariski closure of the sextics that are sums of three squares of cubics.

The algebraic boundary of $\Sigma_{4,4}$ has a unique non-discriminant component. It has degree 38475 and is the Zariski closure of the quartics that are sums of four squares of quadrics.

Both hypersurfaces define Noether-Lefschetz divisors in moduli spaces of K3 surfaces.

Q: What’s the point of numbers like 83200 and 38475?
Boundaries of SOS Cones

Theorem

The algebraic boundary of $\Sigma_{3,6}$ has a unique non-discriminant component. It has degree $83200$ and is the Zariski closure of the sextics that are sums of three squares of cubics.

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Q: What’s the point of numbers like 83200 and 38475 ?
A: Think about the historical importance of the number 3264.

Extreme Non-Negative Polynomials

A Gromov-Witten Number:

Theorem

The Zariski closure of the set of extreme rays of $\mathbb{P}_{3,6} \setminus \Sigma_{3,6}$ is the Severi variety of rational sextic curves in the projective plane $\mathbb{P}^2$. This Severi variety has dimension 17 and degree 26312976 in the $\mathbb{P}^{27}$ of all sextic curves.
Extreme Non-Negative Polynomials

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An Unknown Number:

Theorem

The Zariski closure of the set of extreme rays of $P_{4,4} \setminus \Sigma_{4,4}$ is the variety of quartic symmetroids in $\mathbb{P}^3$, that is, surfaces whose defining polynomial is the determinant of a symmetric 4 × 4-matrix of linear forms.

This variety has dimension 24 in the $\mathbb{P}^{34}$ of all quartic surfaces.
A Dual Characterization

... of the non-discriminant component in $\partial \Sigma_{3,6}$:

$$\begin{bmatrix}
a_{006} & a_{015} & a_{024} & a_{033} & a_{105} & a_{114} & a_{123} & a_{204} & a_{213} & a_{303} \\
a_{015} & a_{024} & a_{033} & a_{042} & a_{114} & a_{123} & a_{132} & a_{213} & a_{222} & a_{312} \\
a_{024} & a_{033} & a_{042} & a_{051} & a_{123} & a_{132} & a_{141} & a_{222} & a_{231} & a_{321} \\
a_{033} & a_{042} & a_{051} & a_{060} & a_{132} & a_{141} & a_{150} & a_{231} & a_{240} & a_{330} \\
a_{105} & a_{114} & a_{123} & a_{132} & a_{204} & a_{213} & a_{222} & a_{303} & a_{312} & a_{402} \\
a_{114} & a_{123} & a_{132} & a_{141} & a_{213} & a_{222} & a_{231} & a_{312} & a_{321} & a_{411} \\
a_{123} & a_{132} & a_{141} & a_{150} & a_{222} & a_{231} & a_{240} & a_{321} & a_{330} & a_{420} \\
a_{204} & a_{213} & a_{222} & a_{231} & a_{303} & a_{312} & a_{321} & a_{402} & a_{411} & a_{501} \\
a_{213} & a_{222} & a_{231} & a_{240} & a_{312} & a_{321} & a_{330} & a_{411} & a_{420} & a_{510} \\
a_{303} & a_{312} & a_{321} & a_{330} & a_{402} & a_{411} & a_{420} & a_{501} & a_{510} & a_{600}
\end{bmatrix}$$

Theorem

The above Hankel matrices of rank $\leq 7$ constitute a rational projective variety of dimension 21 and degree 2640. Its dual is a hypersurface, the Zariski closure of sums of three squares of cubics.
Proof

Consider the Grassmannian $\text{Gr}(3,10)$ of 3-dim’l subspaces $F$ in the 10-dimensional space $\mathbb{R}[x_1, x_2, x_3]_3$ of ternary cubics.

This Grassmannian is rational and its dimension equals 21.

The global residue in $\mathbb{P}^2$ specifies a rational map $F \mapsto \text{Res}\langle F \rangle$ from $\text{Gr}(3,10)$ into $\mathbb{P}((\mathbb{R}[x_1, x_2, x_3]_6)^*) \simeq \mathbb{P}^{27}$. Its base locus is the resultant of three ternary cubics, so $\text{Res}\langle F \rangle$ is well-defined whenever the ideal $\langle F \rangle$ is a complete intersection in $\mathbb{R}[x_1, x_2, x_3]$.

The value $\text{Res}\langle F \rangle(P)$ of this linear form on a ternary sextic $P$ is the image of $P$ modulo the ideal $\langle F \rangle$. It can be computed via Gröbner basis normal form. Our map $F \mapsto \ell$ is birational because it has an explicit inverse: $F = \text{kernel}(H_\ell)$. The inverse simply maps the rank 7 Hankel matrix representing $\ell$ to its kernel.

The degree is from [Harris-Tu 1994] using Cohen-Macaulayness.
A Dual Characterization

... of the non-discriminant component in $\partial \Sigma_{4,4}$:

\[
\begin{bmatrix}
 a_{0004} & a_{0013} & a_{0022} & a_{0103} & a_{0112} & a_{0202} & a_{1003} & a_{1012} & a_{1102} & a_{2002} \\
 a_{0013} & a_{0022} & a_{0031} & a_{0112} & a_{0121} & a_{0211} & a_{1012} & a_{1021} & a_{1111} & a_{2011} \\
 a_{0022} & a_{0031} & a_{0040} & a_{0121} & a_{0130} & a_{0220} & a_{1021} & a_{1030} & a_{1120} & a_{2020} \\
 a_{0103} & a_{0112} & a_{0121} & a_{0202} & a_{0211} & a_{0301} & a_{1102} & a_{1111} & a_{1201} & a_{2101} \\
 a_{0112} & a_{0121} & a_{0130} & a_{0211} & a_{0220} & a_{0310} & a_{1111} & a_{1120} & a_{1210} & a_{2110} \\
 a_{0202} & a_{0211} & a_{0220} & a_{0301} & a_{0310} & a_{0400} & a_{1201} & a_{1210} & a_{1300} & a_{2200} \\
 a_{1003} & a_{1012} & a_{1021} & a_{1102} & a_{1111} & a_{1201} & a_{2002} & a_{2011} & a_{2101} & a_{3001} \\
 a_{1012} & a_{1021} & a_{1030} & a_{1111} & a_{1120} & a_{1210} & a_{2011} & a_{2020} & a_{2110} & a_{3010} \\
 a_{1102} & a_{1111} & a_{1120} & a_{1201} & a_{1210} & a_{1300} & a_{2101} & a_{2110} & a_{2200} & a_{3100} \\
 a_{2002} & a_{2011} & a_{2020} & a_{2101} & a_{2110} & a_{2200} & a_{3001} & a_{3010} & a_{3100} & a_{4000}
\end{bmatrix}
\]

Theorem

The above Hankel matrices of rank $\leq 6$ constitute a rational projective variety of dimension 24 and degree 28314. Its dual is a hypersurface, the Zariski closure of sums of 4 squares of quadrics.
Q: Can we trust Maulik and Pandharipande?

A: It never hurts to double-check. We independently verified the asserted degrees using Bertini.

Bertini is an amazing piece of numerical software for algebraic geometry (and its many applications), due to Daniel Bates, Jonathan Hauenstein, Andrew Sommese and Charles Wampler. Try it tonight!

We computed the degrees of the irreducible variety of interest by intersecting with a generic linear space of complementary dimension, thus obtaining finitely many points over $\mathbb{C}$. Bertini finds numerical approximations of these points.
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Solution to the Pop Quiz

K3 stands for *Kummer, Kähler and Kodaira.*

*Definition:* K3 surfaces are complete smooth surfaces that have trivial canonical bundle and are not abelian surfaces.

Two standard models of algebraic K3 surfaces are

- smooth quartic surfaces in $\mathbb{P}^3$,
- double covers of $\mathbb{P}^2$ branched along a sextic curve.
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- smooth quartic surfaces in $\mathbb{P}^3$,
- double covers of $\mathbb{P}^2$ branched along a sextic curve.

**Noether-Lefschetz Theorem:**
General K3 surfaces in these families have Picard group $\mathbb{Z}$.

Max Noether (1882): Every irreducible curve on a general quartic surface $S$ is the intersection of $S \subset \mathbb{P}^3$ with another surface in $\mathbb{P}^3$.

Noether-Lefschetz divisors correspond to exceptional K3 surfaces:
- $S$ contains quartic elliptic curves, say $f = \det \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$,
- $S$ is the general surface of degree $(3,2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. 
Summary

Recent advances in *convex optimization* have led to a strong interest in understanding Hilbert’s inclusion

\[ \Sigma_{n,2d} \subset P_{n,2d}. \]

The varieties we wish to understand are:

- the Zariski closure of the extreme rays in \( P_{n,2d} \setminus \Sigma_{n,2d} \),
- the algebraic boundary of \( \partial \Sigma_{n,2d} \setminus \partial P_{n,2d} \),
- the projective duals to these varieties.

*This talk:* The smallest cases \( \Sigma_{3,6} \) and \( \Sigma_{4,4} \). We discovered

- a Severi variety and a variety of symmetroids,
- the two Noether-Lefschetz divisors on the previous slide,
- varieties defined by rank constraints on Hankel matrices.