Nonnegative Polynomials
versus
Sums of Squares

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Based on a joint paper with
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and on earlier work of G. Blekherman

Two Warm-up Questions

We know from *high school* that a quadric $ax^2 + bx + c$ is non-negative if and only if $a \geq 0$, $c \geq 0$ and $4ac - b^2 \geq 0$.

Let's study this convex cone for polynomials of degree four:

$$C = \left\{ (a, b, c, d, e) \in \mathbb{R}^5 \mid \forall\, x \in \mathbb{R} : ax^4 + bx^3 + cx^2 + dx + e \geq 0 \right\}$$

1. The Zariski closure of its boundary ∂C is a hypersurface.
   Find the degree and defining polynomial of this hypersurface.

2. Determine an inequality representation of the dual cone $C^\vee$.

Question 1: Algebraic boundary of nonnegative quartics

The boundary ∂C consists of all nonnegative polynomial

$$f(x) \;=\; ax^4 + bx^3 + cx^2 + dx + e$$

such that $\exists \alpha \in \mathbb{R} \,:\, f(\alpha) = 0$.

The root α is necessarily a double root: $f(\alpha) = f'(\alpha) = 0$.

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Hence the *discriminant* of the quartic $f(x)$ vanishes:

$$\begin{gathered} 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 \\ +144ab^2ce^2 - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e \\ -4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2 \end{gathered}$$

The Zariski closure of ∂C is the irreducible hypersurface of degree **6** defined by this polynomial. We regard it as a hypersurface in $\mathbb{P}^4$.

Question 2: Duality in convex geometry

The dual cone $C^\vee$ is the cone spanned by the rational normal curve

$$C^\vee \;=\; \mathbb{R}_{\geq 0}\{\,(1, x, x^2, x^3, x^4)\;:\; x \in \mathbb{R}\} \quad \subset \quad \mathbb{R}^5.$$

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Its natural inequality representation is as the cone of
3×3-Hankel matrices that are positive semidefinite:

$$C^\vee \;=\; \left\{\,(u_0,u_1,u_2,u_3,u_4)\in\mathbb{R}^5\;:\;\begin{bmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix}\succeq 0\,\right\}$$

In pure math, Hankel matrices are known as *catalecticants*.
In applied math, Hankel matrices are known as *moment matrices*.

Now, let the lecture begin....

A Tale of Two Cones

Fix the real vector space of homogeneous polynomials in $\mathbb{R}[x_1, x_2, \ldots, x_n]$ of degree $2d$. In this space, consider the convex cone $P_{n,2d}$ of all non-negative polynomials and the subcone $\Sigma_{n,2d}$ of all polynomials that are sums of squares.

Are they equal ?

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Are they equal ?

Yes, if $d = 1$ *(quadrics) or* $n = 2$ *(binary forms).*

Example $(n = d = 2)$: The binary quartic

$$f \;=\; 2x_1^4 - 6x_1^3x_2 + 9x_1^2x_2^2 - 6x_1x_2^3 + 2x_2^4$$

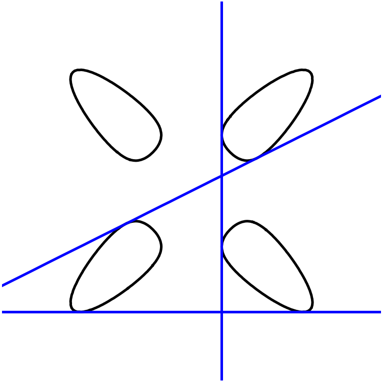
is non-negative. To prove this, we write[*sdp*]

$$f \;=\; (x_1^2 - x_2^2)^2 + (x_1^2 - 3x_1x_2 + x_2^2)^2.$$

Plane Quartics

Yes, if $d = 2$ and $n = 3$:
Every non-negative ternary quartic can be written^{sdp} *as a sum of three squares of ternary quadrics.*

This involves some beautiful 19^{th} century geometry...



... as explained in [D.Plaumann, B.St and C.Vinzant: Quartic curves and their bitangents, Journal of Symbolic Computation **46** (2011) 712-733.

124 Years Ago

Theorem (Hilbert, 1888): *The containment of convex cones*

$$\Sigma_{n,2d} \subset P_{n,2d}$$

is strict if and only if $(n \geq 3$ *and* $d \geq 3)$ *or* $(n \geq 4$ *and* $d \geq 2)$.

What does this mean for the *algebraic boundaries* of these cones?

The algebraic boundary of $P_{n,d}$ is an irreducible hypersurface of degree $n(2d-1)^{n-1}$, namely the discriminant. This discriminant is one irreducible component also in the algebraic boundary of $\Sigma_{n,d}$.

Today we examine the two borderline cases:

- Sextic curves in the plane $(n = 3, d = 3)$
- Quartic surfaces in 3-space $(n = 4, d = 2)$

The ambient spaces are $\mathbb{P}^{27}$ and $\mathbb{P}^{34}$ respectively.

The Lax-Lax Quartic

Exercise: The polynomial

$$\begin{array}{rc} & (a-b)(a-c)(a-d)(a-e) \\ + & (b-a)(b-c)(b-d)(b-e) \\ + & (c-a)(c-b)(c-d)(c-e) \\ + & (d-a)(d-b)(d-c)(d-e) \\ + & (e-a)(e-b)(e-c)(e-d) \end{array}$$

is non-negative but it is not a sum of squares.

[Anneli Lax and Peter Lax: On sums of squares,
Linear Algebra and its Applications **20** (1978) 71–75]

This represents a point in $P_{4,4} \backslash \Sigma_{4,4}$.
What's the matter with this quartic surface in $\mathbb{P}^3$?

Pop Quiz: *Tell us all you know about K3 surfaces.*

Boundaries of SOS Cones

Theorem

The algebraic boundary of $\Sigma_{3,6}$ *has a unique non-discriminant component. It has degree* 83200 *and is the Zariski closure of the sextics that are sums of three squares of cubics.*

The algebraic boundary of $\Sigma_{4,4}$ *has a unique non-discriminant component. It has degree* 38475 *and is the Zariski closure of the quartics that are sums of four squares of quadrics.*

Both hypersurfaces define Noether-Lefschetz divisors in moduli spaces of K3 surfaces.

Q: What's the point of numbers like 83200 and 38475 ?

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Q: What's the point of numbers like 83200 and 38475 ?
A: Think about the historical importance of the number 3264.

Our numbers are coefficients of certain modular forms in [D. Maulik and R. Pandharipande: Gromov-Witten Theory and Noether-Lefschetz Theory, `arXiv:0705.1653`]

Extreme Non-Negative Polynomials

A Gromov-Witten Number:

Theorem

The Zariski closure of the set of extreme rays of $P_{3,6} \setminus \Sigma_{3,6}$ *is the* **Severi variety** *of rational sextic curves in the projective plane* $\mathbb{P}^2$*.*

This Severi variety has dimension 17 *and degree* 26312976 *in the* $\mathbb{P}^{27}$ *of all sextic curves.*

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An Unknown Number:

Theorem

The Zariski closure of the set of extreme rays of $P_{4,4} \backslash \Sigma_{4,4}$ *is the variety of* **quartic symmetroids** *in* $\mathbb{P}^3$*, that is, surfaces whose defining polynomial is the determinant of a symmetric* 4×4*-matrix of linear forms.*

This variety has dimension 24 *in the* $\mathbb{P}^{34}$ *of all quartic surfaces.*

A Dual Characterization

... of the non-discriminant component in $\partial\Sigma_{3,6}$:

$$\begin{bmatrix}
a_{006} & a_{015} & a_{024} & a_{033} & a_{105} & a_{114} & a_{123} & a_{204} & a_{213} & a_{303} \\
a_{015} & a_{024} & a_{033} & a_{042} & a_{114} & a_{123} & a_{132} & a_{213} & a_{222} & a_{312} \\
a_{024} & a_{033} & a_{042} & a_{051} & a_{123} & a_{132} & a_{141} & a_{222} & a_{231} & a_{321} \\
a_{033} & a_{042} & a_{051} & a_{060} & a_{132} & a_{141} & a_{150} & a_{231} & a_{240} & a_{330} \\
a_{105} & a_{114} & a_{123} & a_{132} & a_{204} & a_{213} & a_{222} & a_{303} & a_{312} & a_{402} \\
a_{114} & a_{123} & a_{132} & a_{141} & a_{213} & a_{222} & a_{231} & a_{312} & a_{321} & a_{411} \\
a_{123} & a_{132} & a_{141} & a_{150} & a_{222} & a_{231} & a_{240} & a_{321} & a_{330} & a_{420} \\
a_{204} & a_{213} & a_{222} & a_{231} & a_{303} & a_{312} & a_{321} & a_{402} & a_{411} & a_{501} \\
a_{213} & a_{222} & a_{231} & a_{240} & a_{312} & a_{321} & a_{330} & a_{411} & a_{420} & a_{510} \\
a_{303} & a_{312} & a_{321} & a_{330} & a_{402} & a_{411} & a_{420} & a_{501} & a_{510} & a_{600}
\end{bmatrix}$$

Theorem

The above Hankel matrices of rank ≤ 7 *constitute a rational projective variety of dimension* 21 *and degree* 2640*. Its dual is a hypersurface, the Zariski closure of sums of three squares of cubics.*

Proof

Consider the Grassmannian $\mathrm{Gr}(3,10)$ of 3-dim'l subspaces F in the 10-dimensional space $\mathbb{R}[x_1,x_2,x_3]_3$ of ternary cubics.

This Grassmannian is rational and its dimension equals 21.

The *global residue* in $\mathbb{P}^2$ specifies a rational map $F \mapsto \mathrm{Res}_{\langle F \rangle}$ from $\mathrm{Gr}(3,10)$ into $\mathbb{P}((\mathbb{R}[x_1,x_2,x_3]_6)^*) \simeq \mathbb{P}^{27}$. Its base locus is the resultant of three ternary cubics, so $\mathrm{Res}_{\langle F \rangle}$ is well-defined whenever the ideal $\langle F \rangle$ is a complete intersection in $\mathbb{R}[x_1,x_2,x_3]$.

The value $\mathrm{Res}_{\langle F \rangle}(P)$ of this linear form on a ternary sextic P is the image of P modulo the ideal $\langle F \rangle$. It can be computed via Gröbner basis normal form. Our map $F \mapsto \ell$ is birational because it has an explicit inverse: $F = \mathrm{kernel}(H_\ell)$. The inverse simply maps the rank 7 Hankel matrix representing ℓ to its kernel.

The degree is from [Harris-Tu 1994] using Cohen-Macaulayness.

A Dual Characterization

... of the non-discriminant component in $\partial\Sigma_{4,4}$:

$$
\begin{bmatrix}
a_{0004} & a_{0013} & a_{0022} & a_{0103} & a_{0112} & a_{0202} & a_{1003} & a_{1012} & a_{1102} & a_{2002} \\
a_{0013} & a_{0022} & a_{0031} & a_{0112} & a_{0121} & a_{0211} & a_{1012} & a_{1021} & a_{1111} & a_{2011} \\
a_{0022} & a_{0031} & a_{0040} & a_{0121} & a_{0130} & a_{0220} & a_{1021} & a_{1030} & a_{1120} & a_{2020} \\
a_{0103} & a_{0112} & a_{0121} & a_{0202} & a_{0211} & a_{0301} & a_{1102} & a_{1111} & a_{1201} & a_{2101} \\
a_{0112} & a_{0121} & a_{0130} & a_{0211} & a_{0220} & a_{0310} & a_{1111} & a_{1120} & a_{1210} & a_{2110} \\
a_{0202} & a_{0211} & a_{0220} & a_{0301} & a_{0310} & a_{0400} & a_{1201} & a_{1210} & a_{1300} & a_{2200} \\
a_{1003} & a_{1012} & a_{1021} & a_{1102} & a_{1111} & a_{1201} & a_{2002} & a_{2011} & a_{2101} & a_{3001} \\
a_{1012} & a_{1021} & a_{1030} & a_{1111} & a_{1120} & a_{1210} & a_{2011} & a_{2020} & a_{2110} & a_{3010} \\
a_{1102} & a_{1111} & a_{1120} & a_{1201} & a_{1210} & a_{1300} & a_{2101} & a_{2110} & a_{2200} & a_{3100} \\
a_{2002} & a_{2011} & a_{2020} & a_{2101} & a_{2110} & a_{2200} & a_{3001} & a_{3010} & a_{3100} & a_{4000}
\end{bmatrix}
$$

Theorem

The above Hankel matrices of rank ≤ 6 constitute a rational projective variety of dimension 24 and degree 28314. Its dual is a hypersurface, the Zariski closure of sums of 4 squares of quadrics.

Numerical Algebraic Geometry

Q: Can we trust Maulik and Pandharipande?

Numerical Algebraic Geometry

Q: Can we trust Maulik and Pandharipande?
A: It never hurts to double-check.

We independently verified the asserted degrees using **Bertini**.

Bertini is an amazing piece of *numerical* software for algebraic geometry (and its many applications), due to Daniel Bates, Jonathan Hauenstein, Andrew Sommese and Charles Wampler.

Try it tonight !

We computed the degrees of the irreducible variety of interest by intersecting with a generic linear space of complementary dimension, thus obtaining finitely many points over $\mathbb{C}$.

Bertini finds numerical approximations of these points.

Solution to the Pop Quiz

K3 stands for *Kummer, Kähler and Kodaira*.

Definition: K3 surfaces are complete smooth surfaces that have trivial canonical bundle and are not abelian surfaces.

Two standard models of algebraic K3 surfaces are

- smooth quartic surfaces in $\mathbb{P}^3$,
- double covers of $\mathbb{P}^2$ branched along a sextic curve.

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- double covers of $\mathbb{P}^2$ branched along a sextic curve.

Noether-Lefschetz Theorem:
General K3 surfaces in these families have Picard group $\mathbb{Z}$.

Max Noether (1882): Every irreducible curve on a general quartic surface S is the intersection of $S \subset \mathbb{P}^3$ with another surface in $\mathbb{P}^3$.

Noether-Lefschetz divisors correspond to exceptional K3 surfaces:

- S contains quartic elliptic curves, say $f = \det \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$,
- S is the general surface of degree $(3, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$.

Summary

Recent advances in convex optimization have led to a strong interest in understanding Hilbert's inclusion

$$\Sigma_{n,2d} \subset P_{n,2d}.$$

The varieties we wish to understand are:

- the Zariski closure of the extreme rays in $P_{n,2d} \backslash \Sigma_{n,2d}$,
- the algebraic boundary of $\partial\Sigma_{n,2d} \backslash \partial P_{n,2d}$,
- the projective duals to these varieties.

This talk: The smallest cases $\Sigma_{3,6}$ and $\Sigma_{4,4}$. We discovered

- a Severi variety and a variety of symmetroids,
- the two Noether-Lefschetz divisors on the previous slide,
- varieties defined by rank constraints on Hankel matrices.